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On perturbation series of eigenvalues of anharmonic oscillators.

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0 Introduction.

Anharmonic oscillators have attracted much attention of physists, particularly because of their relevance to the ϕ^4-model in quantum field theory.

Here we will consider the asymptotic behavior of the perturbation series of eigenvalues of anharmonic oscillators. The purpose of this note is to prove the result which was firstly derived by Bender-Wu [**BW1,2**] in a rather heuristic manner (See also [**BB**] for some related topics).

The anharmonic oscillators are the following eigenvalue problems.

$$\left(-\frac{d^2}{dx^2}+\frac{1}{4}x^2(1+\lambda x^{2N})\right)\psi=E(\lambda)\psi \qquad (\lambda>0\ ,N=1,2,\cdots), \tag{1}$$

$$\lim_{|x|\to\infty}\psi(x)=0. \tag{2}$$

Because the spectrum of this equation (1) with the boundary condition (2) are discreet and nondegenerate, we can label $E(\lambda)$ by a nonnegative integer K :

$$\begin{array}{rcccccccccc} \text{eigenvalues} & : & E^0(\lambda) & < & E^1(\lambda) & < & \cdots & < & E^K(\lambda) & < & \cdots, \\ \text{corresponding eigenfunctions} & : & \psi^0(x,\lambda) & & \psi^1(x,\lambda) & & \cdots & & \psi^K(x,\lambda) & & \cdots. \end{array}$$

For $\lambda=0$, this equation becomes a well-known harmonic oscillator. Its eigenvalues are $E^K(0)=K+1/2$ and eigenfunctions are $\psi^K(x,0)=D_{K-1/2}(x)=e^{-x^2/4}H_K(x)$, where $D_\lambda(x)$ and $H_K(x)$ denote the Weber function and the Hermite function respectively.

For $\lambda\ll 1$, we can expand the eigenvalues $E^K(\lambda)$ and the eigenfunctions $\psi^K(x,\lambda)$ with respect to λ formally (and each term can be determined recursively):

$$E^K(\lambda)=K+\frac{1}{2}+\sum_{n=1}^{\infty}A_n^K\lambda^n,$$

$$\psi^K(x,\lambda)=\psi^K(x,0)+\sum_{n=1}^{\infty}\psi_n^K(x)\lambda^n.$$

Our purpose is to determine the asymptotic behavior of A_n^K for large n and for arbitrary N. The result is

$$A_n^K = \frac{(-1)^{n+1}N}{K!\sqrt{2\pi^3}} \, 4^{(K+\frac{1}{2})/N} \, (\frac{B(\frac{3}{2},\frac{1}{N})}{2N})^{-K-\frac{1}{2}-nN} \, \Gamma(K+\frac{1}{2}+nN) \, \left(1+O(\frac{1}{n})\right), (n \to \infty).$$

Here $B(x,y)$ denotes the Beta functions.

Here we will see that we can obtain the rigorous argument by using exact WKB method, supplementing [**BW1,2**] with some rigor.

In section 1, we review the analytic properties of $E^K(\lambda)$, which is necessary to determine the exact relation as above. By considering $E^K(\lambda)$ in a complex plane $\mathbf{C}_\lambda$, A_n^K can be represented by the difference of boundary values of $E^K(\lambda)$ on the cut. In section 2, we construct WKB solutions for anharmonic oscillators, and give the connection formula. In section 3, we derive the secular equations and calculate the difference of boundary values of $E^K(\lambda)$ on the cut. In section 4, we determine the asymptotic behavior of A_n^K. Appendix is a summary of the argument of Bender-Wu.

1 Analyticity of $E^K(\lambda)$.

In this section, we review the analytic properties of $E^K(\lambda)$, which are mainly obtained by B. Simon [**S**]. The next theorem is fundamental for our purpose.

Theorem 1.1 .

(i) Each $E^K(\lambda)$ has an analytic continuation with a $(N+2)$-nd branch point at $\lambda = 0$.

(ii) The location of singularities (we place the cut along $\{\lambda \leq 0\}$ in the λ-plane):

(a) The origin $\lambda = 0$ is the limit point of singularities.

(b) There are no singular points on the 1st sheet.

(c) For any θ ; $0 < \theta < \frac{N+2}{2}\pi$, there exist $R > 0$ such that $E^K(\lambda)$ is analytic in the region $\{\lambda \in \mathbf{C} \, ; \, |\lambda| > R, \ |\arg(\lambda)| < \theta\}$.

(iii) $E^K(\lambda) = O(|\lambda|^{\frac{1}{N+2}})$ $(\lambda \to \infty, \ \lambda \in \text{(any sheets)})$.

(iv) The perturbation series is an asymptotic expansion:

$$E^K(\lambda) \sim K + \frac{1}{2} + \sum A_n^K \lambda^n \ \ (\lambda \to 0, \ |\arg(\lambda)| < \theta),$$

where θ is an arbitrary number satisfying $0 < \theta < \frac{N+2}{2}\pi$.

This theorem enables us to represent A_n^K through the difference of boundary values of $E^K(\lambda)$ on the cut, namely,

Proposition 1.2 .

$$A_n^K = \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{\Delta E^K(\lambda)}{\lambda^{n+1}} d\lambda \, , \tag{3}$$

where $\Delta E^K(\lambda) = E^K(\lambda + i0) - E^K(\lambda - i0)$.

Proof . Let $\tilde{E}^K(\lambda) = E^K(\lambda) - K - \frac{1}{2}$. Applying the Cauchy integral formula for $\tilde{E}^K(\lambda)/\lambda$ along the path C (see Figure 1), we obtain

$$\frac{\tilde{E}^K(t)}{t} = \frac{1}{2\pi i}\oint_C \frac{\tilde{E}^K(\lambda)}{\lambda(\lambda - t)} d\lambda \, .$$

Hence by letting the radius of C tend to infinity,

$$\tilde{E}^K(t) = \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{\Delta\tilde{E}^K(\lambda)}{\lambda}\frac{t}{t-\lambda} d\lambda.$$

Let

$$a_n^K = \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{\Delta E^K(\lambda)}{\lambda^{n+1}} d\lambda \, .$$

Then

$$\begin{aligned}
\tilde{E}^K(t) - \sum_{j=1}^{n_0} a_j^K t^j &= \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{\Delta E^K(\lambda)}{\lambda}\left(\frac{t}{t-\lambda} - \sum_{j=1}^{n_0}\left(\frac{t}{\lambda}\right)^j\right) d\lambda \\
&= \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{\Delta E^K(\lambda)}{\lambda - t}\left(\frac{t}{\lambda}\right)^{n_0+1} d\lambda \\
&= \left(\frac{1}{2\pi i}\int_{-\infty}^{0} \frac{\Delta E^K(\lambda)}{\lambda - t}\frac{d\lambda}{\lambda^{n_0+1}}\right) t^{n_0+1}.
\end{aligned}$$

The integral in the last equation converges because (see Theorem 1.1)

$$\begin{cases} \Delta E^K(\lambda) = O(\lambda^{\frac{1}{N+2}}) \ (\lambda \to \infty) \, , \\ \Delta E^K(\lambda) = O(\lambda^m) \ (\lambda \to 0, \ \ m = 1, 2, \cdots) \, . \end{cases}$$

This completes the proof of the proposition. $\square$

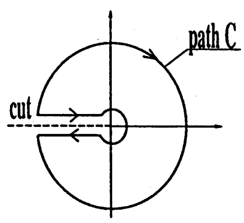

Figure 1: Path C of integration in the λ-plane.

Remark 1.3.
(i) In the above proposition, the path of integration is very close to the cut $\{\lambda < 0\}$ in the λ-plane. But we can change this cut to $\{-te^{i\epsilon}; t > 0\}$ for sufficiently small $\epsilon > 0$ (see Theorem 1.1). In this case, Proposition 1.2 also holds by changing the path of integration to $\{-te^{i\epsilon}; t > 0\}$. In the later(section 3), we use Proposition 1.2 in this form.
(ii) For large n, the dominant contribution comes from small λ due to the factor t^{-n} in the integrand, hence our problem is reduced to determining $\Delta E(\lambda)$ for small negative λ.

To determine $\Delta E^K(\lambda)$, we use the following proposition.

Proposition 1.4 .
For $\lambda \in \mathrm{C}$, each $E^K(\lambda)$ is the eigenvalue of the equation (1) with the following boundary condition (4):

$$\lim_{\substack{|x|\to\infty \\ x\in\Sigma_\pm(\theta)}} \psi(x) = 0, \tag{4}$$

where $\theta = \arg(\lambda)$ and

$$\Sigma_\pm(\theta) = \left\{ x \in \mathrm{C}; |\arg(\pm x) + \frac{\theta}{2(N+2)}| < \frac{\pi}{2(N+2)} \right\}.$$

Sketch of the proof . The asymptotic form of the solution of (1) with (4) is

$$\psi(x) \sim \exp\left(-\frac{\sqrt{\lambda}}{2(N+2)} x^{2(N+2)} \right) \quad (x \to \infty).$$

This is also true if we consider (1) in the complex domain. Hence we can change the boundary condition to $\psi(x) \to 0$ as $x \to \infty$ in the sector

$$\left\{ x \in \mathrm{C}; \mathrm{Re}\left(\sqrt{\lambda} x^{2(N+2)}\right) > 0 \right\} \ ,$$

that is $\Sigma_\pm(\theta)|_{\theta=0}$. What we have to do is to extend the above argument to $\lambda \in \mathrm{C}$.
□

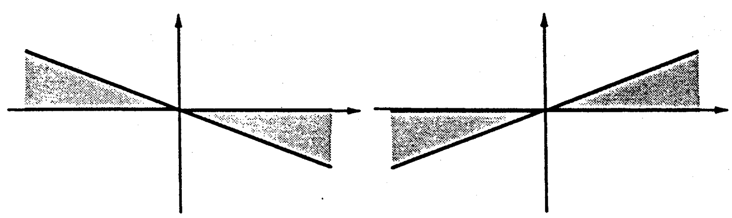

Figure 2: The sector $\Sigma_\pm(\pi)$ (left) and the sector $\Sigma_\pm(-\pi)$ (right).

Figure 2 describes the sector $\Sigma_\pm(\theta)$ for $\theta = \pi, -\pi$.

2 WKB analysis for anharmonic oscillators.

In the previous section, our problem was reduced to the determination of the behavior of the eigenvalue of the equation (1) with the boundary condition (4) for $\lambda < 0, \lambda \to 0$. So, from now on, we treat λ as a small parameter.

2.1 WKB solutions for anharmonic oscillators.

First of all, we construct WKB solutions for anharmonic oscillators. We introduce the following scaling.

$$\begin{array}{rcl} \lambda & \longrightarrow & \eta^{-N} e^{i\theta} \ (\theta = \arg(\lambda)), \\ x & \longrightarrow & \sqrt{\eta}\, x. \end{array}$$

Because λ is a small parameter, η becomes a large parameter. The equation (1) becomes

$$\left(-\frac{d^2}{dx^2} + \eta^2(Q(x) - \eta^{-1}E)\right)\psi = 0\,, \tag{5}$$

$$Q(x) = \frac{1}{4}x^2(1 + e^{i\theta}x^{2N}).$$

We first consider the following transformation of unknown functions:

$$\psi(x) = \exp\int^x S dx. \tag{6}$$

Then the equation (5) becomes

$$S^2 + \frac{dS}{dx} = \eta^2\left(Q(x) - \eta^{-1}E\right).$$

We next expand S with respect to η:

$$S = \eta S_{-1} + S_0 + \eta^{-1}S_1 + \cdots\ .$$

Then each S_j satisfies

$${S_{-1}}^2 = Q(x), \tag{7}$$

$$2S_0S_{-1} + \frac{d}{dx}S_{-1} = -E, \tag{8}$$

$$2S_{n+1}S_{-1} + \sum_{\substack{i+j=n \\ i,j\geq 0}} S_iS_j + \frac{d}{dx}S_n = 0 \ \ (n = 0, 1, 2, \cdots). \tag{9}$$

Equations (8) and (9) determine a unique solution once the sign of S_{-1} is fixed; note that S_{-1} has th form

$$S_{-1} = \pm\sqrt{Q(x)}.$$

Hence there are two (formal) solutions of S;

$$S_\pm = \pm\eta\sqrt{Q(x)} + \left(\mp\frac{E}{2\sqrt{Q(x)}} - \frac{Q'(x)}{2Q(x)}\right) + O(\eta^{-1}). \tag{10}$$

(The signs are taken simultaneously.) We choose a branch of $\sqrt{Q(x)}$ so that

(a) We place a cut from each simple turning point to infinity without crossing any Stokes curve (cf. Figure 3 for N=1. See Definition 2.4 for the definitions of turning points and Stokes curves).

(b) $\sqrt{Q(x)} \sim \frac{1}{2}x \ (x \to 0)$.

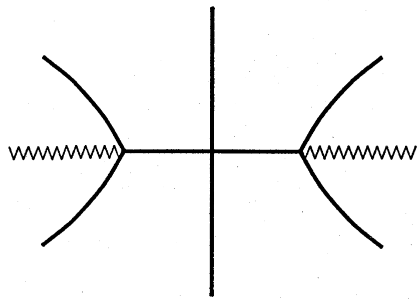

Figure 3: Wiggly lines indicate the cuts.

Definition 2.1.

$$S_{\text{odd}} = \frac{1}{2}(S_+ - S_-),\ S_{\text{even}} = \frac{1}{2}(S_+ + S_-).$$

From the equation (10), we easily obtain

$$S_{\text{odd}} = \eta\sqrt{Q(x)} - \frac{E}{2\sqrt{Q(x)}} + O(\eta^{-1})\,, \tag{11}$$

$$S_{\text{even}} = -\frac{Q'(x)}{2Q(x)} + O(\eta^{-1})\,. \tag{12}$$

Lemma 2.2.

$$S_{\text{even}} = -\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}} \ .$$

Proof. Because $S_\pm$ satisfies

$$S_\pm{}^2 + \frac{dS_\pm}{dx} = \eta^2\left(Q(x) - \eta^{-1}E\right),$$

we obtain

$$2S_{\text{odd}}S_{\text{even}} + \frac{d}{dx}S_{\text{odd}} = 0 \ .$$

This proves the lemma . $\square$

Up to the normalization constants, WKB solutions are, by definition,

$$\psi_\pm = \exp\left(\int^x S_\pm dx\right) ,$$

or, by Lemma 2.2,

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int^x S_{\mathrm{odd}}dx\right) .$$

In what follows we fix their normalization by the following.

Definition 2.3 (Normalization of WKB solutions).
Our choice of the normalization of WKB solutions is

$$\psi_\pm(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\eta\int_0^x \sqrt{Q(x)}dx\right)\exp\left(\pm\int_\infty^x (S_{\mathrm{odd}}-\eta S_{-1})dx\right) . \qquad (13)$$

(Note that $S_{-1} = S_{\mathrm{odd},-1} = \sqrt{Q(x)}$.)

Each integration in (13) is well-defined. This is why we choose the above normalization.

2.2 Connection formula for WKB solutions.

In this subsection we give the connection formula for WKB solutions.

Definition 2.4 (Turning point, Stokes curve.).
(i) A turning point is, by definition, a zero of $Q(x)$. We say it is a simple (double, resp.) turning point if it is a simple (double, resp.) zero.
(ii) A Stokes curve is the following integral curve:

$$\mathrm{Im}\int_a^x \sqrt{Q(x)}dx = 0 \ \text{ (where } a \text{ is a turning point)} .$$

As is easily seen, three Stoke curves emanate from a simple turning point, and four Stokes curves from a double turning point.
For anharmonic oscillators, turning points are

$$\begin{cases} \text{double} & : \ x = 0 , \\ \text{simple} & : \ x = \exp\left(\frac{i}{2N}(2k\pi-\theta)\right) \ \ (k = 0, 1, \cdots, 2N-1) . \end{cases}$$

Definition 2.5. Let L be a Stokes curve emanating from a turning point a.
If $\pm\int_a^x \sqrt{Q(x)}dx > 0$, we say that the WKB solution $\psi_\pm$ is dominant on L. (Similarly, if $\pm\int_a^x \sqrt{Q(x)}dx < 0$, we say that the WKB solution $\psi_\pm$ is subdominant on L.) Here the path of integration is along L.

We give an analytic meaning to WKB solutions by taking the Borel summation (cf. [**DDP**]). Namely, after expanding WKB solutions like

$$\psi_\pm = \exp\left(\pm\eta\int_0^x \sqrt{Q(x)}dx\right)\sum_{j=0}^\infty \psi_{\pm,j}(x)\eta^{-j-1/2},$$

we take the Borel summation with respect to η . Of course, whether it is Borel summable or not depends on x. Concerning this problem, we can establish the connection formula stated below. We can obtain these formulae by transforming the equation to canonical equations at each turning point (the canonical equation is the Airy equation for simple turning points, and the Weber equation for double turning points).

In the following, we assume that there are no Stokes curves which connect turning points.

Theorem 2.6. Any WKB solution is Borel summable except on Stokes curves.

Theorem 2.7.
Let L be a Stokes curve emanating from a *simple* turning point a, and let $\hat{\psi}_{\pm}$ denote the following normalized WKB solutions :

$$\hat{\psi}_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_a^x S_{\mathrm{odd}} dx\right) .$$

If we analytically continue the (Borel summed) WKB solutions $\hat{\psi}_{\pm}$ across L in counterclockwise manner with respect to the center a, we find that $\hat{\psi}_{\pm}$ obey the following formula.

$$\begin{cases} \hat{\psi}_+ & \rightarrow & \hat{\psi}_+ + i\,\hat{\psi}_- \\ \hat{\psi}_- & \rightarrow & \hat{\psi}_- \end{cases} \quad (\text{if } \hat{\psi}_+ \text{ is dominant on } L) .$$

$$\begin{cases} \hat{\psi}_+ & \rightarrow & \hat{\psi}_+ \\ \hat{\psi}_- & \rightarrow & \hat{\psi}_- + i\,\hat{\psi}_+ \end{cases} \quad (\text{if } \hat{\psi}_- \text{ is dominant on } L) .$$

Theorem 2.8. If we analytically continue the (Borel summed) WKB solutions $\psi_{\pm}$ across a Stokes curve emanating from the *origin* as described below, we find that $\psi_{\pm}$ obey the following formula (see Figure 4).

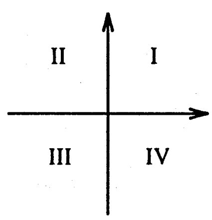

Figure 4: Regions I, II, III, IV.

$$\left\{\begin{array}{lcll}\text{Region IV} \to \text{Region I} & : & \psi_+ & \to \psi_+ + i\dfrac{C_-}{C_+}\dfrac{\sqrt{2\pi}}{\Gamma(-F+\frac{1}{2})}\eta^{-F}\psi_-, \quad \psi_- \to \psi_- . \\ \text{Region I} \to \text{Region II} & : & \psi_- & \to \psi_- + i\dfrac{C_+}{C_-}\dfrac{\sqrt{2\pi}}{\Gamma(F+\frac{1}{2})}e^{-i\pi F}\eta^{F}\psi_-, \quad \psi_+ \to \psi_+ . \\ \text{Region II} \to \text{Region III} & : & \psi_+ & \to \psi_+ + i\dfrac{C_-}{C_+}\dfrac{\sqrt{2\pi}}{\Gamma(-F+\frac{1}{2})}e^{2i\pi F}\eta^{-F}\psi_-, \quad \psi_- \to \psi_- .\end{array}\right.$$

Here $C_\pm$ and F denote the following :

$$\begin{aligned} C_\pm &= (-4)^{\pm\frac{E}{2N}} + O(\eta^{-1}), \\ F &= \operatorname*{Res}_{x=0} S_{\text{odd}} = -E + O(\eta^{-1}). \end{aligned}$$

Remark 2.9.

(i) In Theorem 2.7, we use $\hat{\psi}_\pm$ instead of $\psi_\pm$. To obtain the connection formula for $\psi_\pm$, we simply use the following:

$$\psi_\pm = \exp\left(\pm\eta\int_0^a \sqrt{Q(x)}dx\right)\exp\left(\pm\int_\infty^a (S_{\text{odd}} - \eta S_{-1})dx\right)\hat{\psi}_\pm .$$

(ii) In Theorem 2.8, we state only the necessary formulae in the later use.

3 Derivation of the secular equation.

In this section, by using the connection formulae, we derive the secular equation, i.e., the equation that $E^K(\lambda)$ should satisfy, for $\arg(\lambda) = \pm(\pi - 0)$ (Note that our present aim is to determine the boundary values of $E^K(\lambda)$ at the cut in $\mathbf{C}_\lambda$).

As the Stokes curves are degenerate for $\arg(\lambda) = \pm\pi$, we rotate the cut slightly to use the results in section 2 (cf. Remark 1.3 (i)). Here we rotate the cut in counterclockwise manner to $\{\lambda = -t\,e^{-i\epsilon}; t \in \mathbf{R}_{\geq 0}\}$ (ϵ is a sufficiently small positive number), and seek for the value of $E^K(\lambda)$ for $\arg(\lambda) = \pm(\pi - 0) + \epsilon$. Note that the configuration of Stokes curves for $\arg(\lambda) = \pi + \epsilon$ is the same as that for $\arg(\lambda) = -\pi + \epsilon$ (cf. Figure 5 $\sim$ 7). The procedure how to derive the secular equation is as follows:

First, we should notice that ψ_- is subdominant in $\Sigma_\pm(\theta)$ for $\theta = \pm\pi + \epsilon$ for our choice of the branch of $\sqrt{Q(x)}$ (here we extend the definition of the subdominancy for "on Stokes curves" to "in some region"(see Definition 2.5), but there is no fear of confusions).
By the analytic continuation of ψ_- from Σ_+ to Σ_-, ψ_- becomes $K_+(\eta, E)\psi_+ + K_-(\eta, E)\psi_-$.

$$\begin{array}{ccc} \Sigma_+(\lambda) & & \Sigma_-(\lambda) \\ \psi_- & \underset{\substack{\text{analytic}\\ \text{continuation}}}{\Longrightarrow} & K_+(\eta,E)\psi_+ + K_-(\eta,E)\psi_- \end{array}$$

The existence of the eigenfunction forces $K_+(\eta, E)$ to be 0, which is the required secular equation. We describe its explicit form in the subsequent subsections.

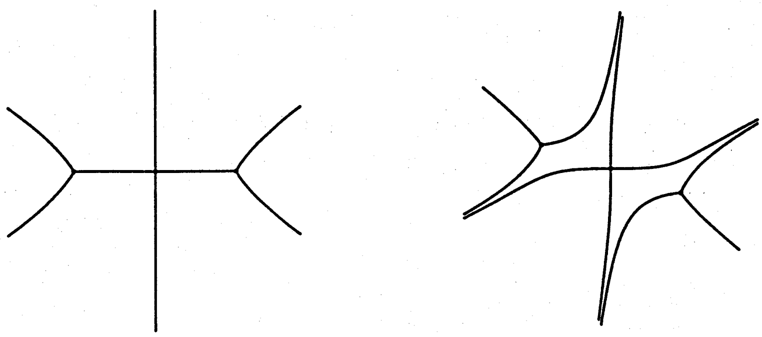

Figure 5: Stokes curves ($N = 1$) for $\theta = \pm\pi$ (left) and $\theta = \pm\pi + \epsilon$(right) .

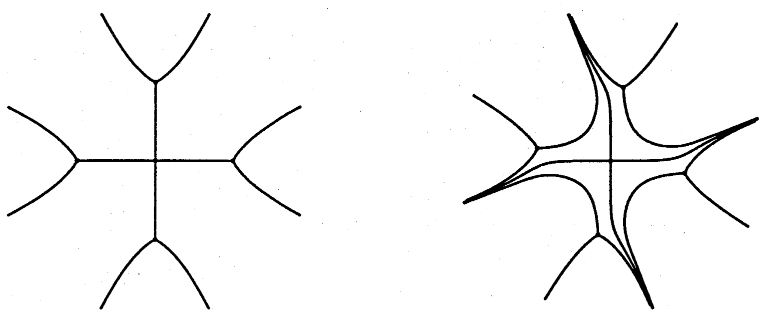

Figure 6: Stokes curves ($N = 2$) for $\theta = \pm\pi$ (left) and $\theta = \pm\pi + \epsilon$ (right) .

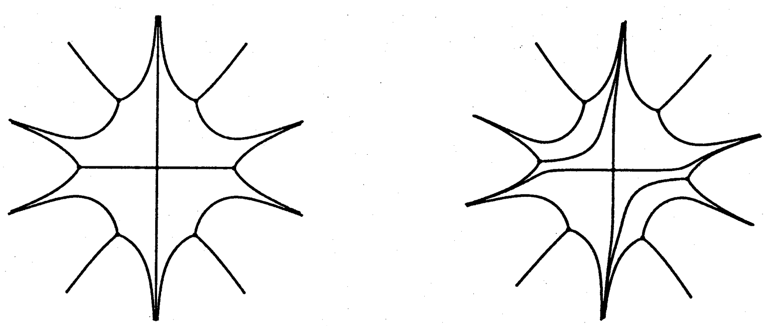

Figure 7: Stokes curves ($N = 3$) for $\theta = \pm\pi$ (left) and $\theta = \pm\pi + \epsilon$ (right) .

3.1 Eigenvalues for $\arg(\lambda) = \pi + \epsilon$.

Using the path of analytic continuation described in Figure 8, we obtain the following from Theorem 2.7 and 2.8:

$$\begin{aligned} K_+(\eta, E) &= i\,(B_2(\eta, E) \\ &\quad + A_-(\eta, E) - A_+(\eta, E) + B_1(\eta, E)B_2(\eta, E)A_+(\eta, E) \\ &\quad + B_1(\eta, E)A_+(\eta, E)A_-(\eta, E)) \ , \end{aligned}$$

where

$$\begin{aligned} A_\pm &= \exp(-2\eta \int_0^{\pm 1} S_{-1}dx)\exp(-2\int_\infty^{\pm 1}(S_{\rm odd} - \eta S_{-1})dx), \\ B_1 &= \frac{C_+}{C_-}\frac{\sqrt{2\pi}}{\Gamma(-F+\frac{1}{2})}\eta^{-F}, \\ B_2 &= \frac{C_-}{C_+}\frac{\sqrt{2\pi}}{\Gamma(F+\frac{1}{2})}e^{-i\pi}\eta^F. \end{aligned}$$

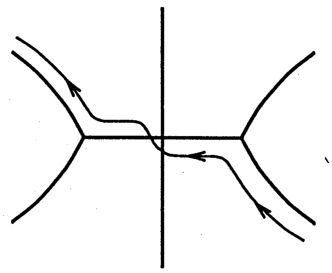

Figure 8: Path of analytic continuation for $\theta = \pi + \epsilon$.

If we let ω denote $2\int_0^1 S_{-1}dx = 2\int_0^{-1} S_{-1}dx = \frac{B(\frac{1}{N}, \frac{3}{2})}{2N}$, the above secular equation is a sum of terms of the order $O(1)$, $O(e^{-\eta\omega})$, and $O(e^{-2\eta\omega})$. Hence we try to seek the solution in the following form:

$$E = E_0(\eta) + E_1(\eta)e^{-\eta\omega} + E_2(\eta)e^{-2\eta\omega} + \cdots \ ,$$

$$E_j(\eta) = \sum_{n=0}^{\infty} E_{j,n}\eta^{-n} \ .$$

By substituting the above expansion, we obtain (up to $O(e^{-\eta\omega})$)

$$B_2(\eta, E_0) = 0 \ , \tag{14}$$

$$\frac{\partial B_2}{\partial E}(\eta, E_0)E_1 + \tilde{A}_-(\eta, E_0) - \tilde{A}_+(\eta, E_0) + B_1(\eta, E_0)B_2(\eta, E_0)\tilde{A}_+(\eta, E_0) = 0, \tag{15}$$

where

$$\tilde{A}_\pm = A_\pm e^{\eta\omega} = \exp\left(-2\int_\infty^{\pm 1}(S_{\rm odd} - \eta S_{-1})dx\right) \ .$$

As (14) is equivalent to $\frac{1}{\Gamma(F+\frac{1}{2})} = 0$, we find

$$E_0^K = K + \frac{1}{2} + O(\eta^{-1}) \ . \tag{16}$$

Hence, by using (14), (15) and (16), we find

$$E_1^K = \frac{2i}{\sqrt{2\pi}K!}\eta^{K+\frac{1}{2}}4^{\frac{K+\frac{1}{2}}{N}}(1+O(\eta^{-1})).$$

3.2 Eigenvalues for $\arg(\lambda) = -\pi + \epsilon$.

In this case the path of analytic continuation is described in Figure 9, and the resulting secular equation is

$$\frac{1}{\Gamma(F+\frac{1}{2})} = 0.$$

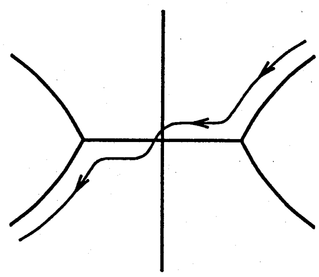

Figure 9: Path of analytic continuation for $\theta = -\pi + \epsilon$.

Although the number of Stokes curves crossing the path of analytic continuation increases with N, the fact that ψ_- is subdominant on the newly relevant Stokes curves guarantees that the expression of $K_+(\eta, E)$ is irrelevant to N.

By solving this secular equation, we obtain

$$E^K = K + \frac{1}{2} + O(\eta^{-1}) \ .$$

4 Asymptotic behavior of A_n^K.

In this last section, we determine the expansion of eigenvalues for $\arg(\lambda) = \pm\pi + \epsilon$. The difference of these values is:

$$\Delta E^K = E_1^K e^{-\eta\omega} + \cdots \ , \tag{17}$$

where

$$E_1^K = \frac{2i}{\sqrt{2\pi}K!}\eta^{K+\frac{1}{2}}4^{\frac{E}{N}}(1+O(\eta^{-1})) \ . \tag{18}$$

By changing the variable in the integrand of (3), we have

$$\begin{aligned} A_n^K &= \frac{1}{2\pi i}\int_{-\infty}^{0}\frac{\Delta E^K(\lambda)}{\lambda^{n+1}}d\lambda \\ &= \frac{(-1)^{n+1}N}{2\pi i}\int_0^\infty \Delta E^K(\eta)\eta^{nN-1}d\eta \ . \end{aligned} \tag{19}$$

Substituting (17) and (18) to (19), we finally obtain

$$A_n^K = \frac{(-1)^{n+1}N}{K!\sqrt{2\pi^3}}\ 4^{(K+\frac{1}{2})/N}\ \omega^{-K-\frac{1}{2}-nN}\ \Gamma(K+\frac{1}{2}+nN)\ \Big(1+O(\frac{1}{n})\Big)\,,$$

$$(n\to\infty).$$

Appendix.

As this note was motivated by **[BW1,2]**, in this Appendix we give its summary as we understand; we hope its ingenious ideas indicated below will help the reader's understanding of our reasoning.

Because the reflection principle implies

$$E^K(\overline{\lambda}) = \overline{E^K(\lambda)}\,,$$

the equation (3) becomes

$$A_n^K = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}(E^K(\lambda))}{\lambda^{n+1}}d\lambda\ .$$

Hence what should be done is to determine Im $E^K(\lambda)$ for $\lambda = -\epsilon$ ($\epsilon > 0$, small). Then they start their reasoning with the following Ansatz:

$$E = E_{(0)} + E_{(1)}\,,$$

$$\begin{cases} E_{(0)} &= K + \dfrac{1}{2} + \sum\limits_{n=1}^{\infty} E_{(0),n}\epsilon^n \ \ (E_{(0),n}\in \mathbf{R})\,, \\ E_{(1)} &: \ \text{exponentially small relative to} \ \ E_{(0)}\ . \end{cases}$$

Let $V(x) = \frac{1}{4}x^2(1-\epsilon x^{2N})$, and let x_0, x_1 $(0 < x_0 < x_1)$ denote the zeros of $V(x)$ on $\mathbf{R}_{\geq 0}$. Notice that x_1 is very large ($x_0 = O(1)$, $x_1 = O(\epsilon^{-1/2})$). We define Regions A, B, C, D, E as in Figure 10. We employ different approximations in each region.

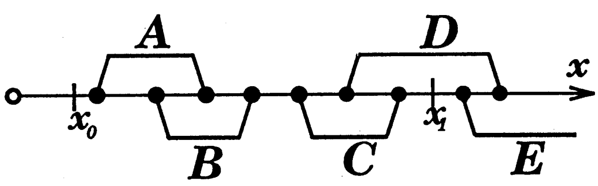

Figure 10: Regions A,B,C,D,E.

In Region A: We approximate the equation by the Weber equation,

$$\left(-\frac{d^2}{dx^2}+\frac{1}{4}x^2-E\right)\psi_A=0.$$

We suppose the solution to be symmetric or antisymmetric. Then

$$\psi_A(x)=\frac{1}{2}\left[D_{E-1/2}(x)+(-1)^K D_{E-1/2}(-x)\right]$$

(the factor $\frac{1}{2}$ is for the normalization of the solution). If we expand ψ_A with respect to $E_{(1)}$, we have

$$\psi_A(x)=\psi_A\,|_{E_{(1)}=0}+E_{(1)}\left(\frac{\partial\psi_A}{\partial E_{(1)}}\,|_{E_{(1)}=0}\right)+\cdots .$$

Let $\psi_{A(0)}$ (resp., $\psi_{A(1)}$) denote $\psi_A\,|_{E_{(1)}=0}$ (resp., $E_{(1)}\left(\frac{\partial\psi_A}{\partial E_{(1)}}\,|_{E_{(1)}=0}\right)$). Then the following relations hold:

$$\left\{\begin{array}{rcl}\psi_{A(0)}(x) & = & \frac{1}{2}\left[D_{E_{(0)}-1/2}(x)+(-1)^K D_{E_{(0)}-1/2}(-x)\right] \quad , \\ & \sim & D_K(x) \\ \psi_{A(1)}(x) & : & \text{exponentially small relative to } \psi_{A(0)}(x) \quad .\end{array}\right.$$

In Region B and Region C: We approximate the solution by WKB solutions,

$$\psi_{B,\pm}(x)=V^{-1/4}\exp(\pm\int_{x_0}^{x}\sqrt{V}dx),$$

$$\psi_{C,\pm}(x)=V^{-1/4}\exp(\pm\int_{x_1}^{x}\sqrt{V}dx).$$

In Region D: We approximate the equation by the Airy equation.
In Region E: Using the boundary condition (4), we approximate the solution by the following WKB solution:

$$\psi_D(x)=V^{-1/4}\exp(-\int_{x_1}^{x}\sqrt{V}dx).$$

So far, we construct the approximate solutions. We next try to match these solutions.

For example we consider the matching from Region A to Region B in the following way. In Region B , the WKB solution is

$$\psi_B(x)=B\,\psi_{B,-}(x)+B'\,\psi_{B,+}(x)$$

with some constants B and B' . As we approximate $V(x)$ by $\frac{1}{4}x^2-E$ in the intersection of Region A and Region B, we have

$$\left\{\begin{array}{rcl}V(x)^{-1/4} & \sim & \sqrt{\frac{2}{x}} \quad , \\ \int_{x_0}^{x}\sqrt{V}dx & \sim & \frac{1}{4}(x^2-{x_0}^2\log x+\frac{{x_0}^2}{2}(\log\frac{{x_0}^2}{4}-1)) \quad .\end{array}\right.$$

Hence

$$B\,\psi_{B-}(x) \sim B\sqrt{2}\exp(-\frac{{x_0}^2}{8}(\log\frac{{x_0}^2}{4}-1))x^K\exp(-\frac{1}{4}x^2)\,. \tag{20}$$

On the other hand, the asymptotic form of $\psi_{A(0)}(x)$ is

$$\begin{aligned}\psi_{A(0)}(x) &\sim D_K(x)\\ &\sim x^K e^{-x^2/4}\ (x\to\infty).\end{aligned} \tag{21}$$

It follows from (20) and (21) that the constant B must be

$$B = \frac{1}{\sqrt{2}}\exp(\frac{{x_0}^2}{8}(\log\frac{{x_0}^2}{4}-1))\,.$$

By repeating these procedures, we can determine constants B, B', C, C', E (in Region D, the argument is essentially the same as the determination of the connection formula for simple turning points). The following table indicates the order of the determination of constants.

$$\begin{array}{ccccccccc}
\text{Region A} & & \text{Region B} & & \text{Region C} & \text{Region D} & \text{Region E}\\
\text{(Weber)} & & \text{(WKB)} & & \text{(WKB)} & \text{(Airy)} & \text{(WKB)}\\
\psi_{A(0)}(x) & \Rightarrow & B\,\psi_{B,-}(x) & \Rightarrow & C\,\psi_{C,-}(x) & \Longrightarrow & \\
+ & & + & & + & & E\psi_D(x)\\
\psi_{A(1)}(x) & \Leftarrow & B'\,\psi_{B,+}(x) & \Leftarrow & C'\,\psi_{C,+}(x) & \Longleftarrow &
\end{array}$$

In particular, since $B = e^{-\Omega}\,C$ and $B' = e^{-\Omega}\,C'$ hold with Ω denoting $\int_{x_0}^{x_1}\sqrt{V(x)}dx$, we have

$$\begin{cases} B &= O(1)\,,\\ C, D, C' &= O(e^{-\Omega})\,,\\ B' &= O(e^{-2\Omega})\,.\end{cases}$$

Hence we find

$$E_{(1)} = O(e^{-2\Omega})\ .$$

Thus $E_{(1)}$ is exponentially small; this fact is consistent with our assumption. Detailed calculation gives

$$E_{(1)} = \frac{i}{\sqrt{2\pi}K!}\left(\frac{\epsilon}{4}\right)^{(K+1/2)/N}\exp\left(-\frac{\epsilon^{-\frac{1}{N}}}{2N}B(\frac{1}{N},\frac{3}{2})\right)\,.$$

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