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Vanishing Theorems in Asymptotic Analysis III and Applications to Confluent Hypergeometric Differential Equations

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1 Intoduction

In this expository paper, I will explain the vanishing theorems in asymptotic analysis in the commutative case and the application to the confluent hypergeometric differential equations. At first, I will give you the short story about the vanishing theorems in asymptotic analysis in the commutative case. Secondly, I will talk about the application to inhomogeneous equations of which associated homogeneous equations are the confluent hypergeometric differential equations. We can analyze the structure of divergent solutions to the inhomogeneous equations. We will give you some numerical calculations for Whittaker equation and Weber equation by using 'Mathematica', which show us that we have good approximations. The author thanks to Dr. Boyd and Dr. Howls for giving some important comments for the calculation of error.

The calculations for Whittaker equation and Weber equation are written in [20].

2 Vanishing Theorems in Asymptotic Analysis in the Commutative Case and Deligne Isomorphism

We will explain vanishing theorems in asymptotic analysis here only the 1-dimensional and commutative case.

We identify the complex plane $\mathbf{C}$ with $S^1 \times \mathbf{R}^+$ by the natural polar mapping, where $S^1 = (\mathbf{R}/2\pi\mathbf{Z})$. For a connected open set c of S^1 and for any positive number $r \in \mathbf{R}^+$, we denote by $\mathcal{A}(c,r)$ the set of all functions holomorphic and asymptotically developable in the sector $S(c,r)$. If $r < r'$, then there exists a natural restriction mapping $i_{rr'}$ of $\mathcal{A}(c,r')$ into $\mathcal{A}(c,r)$, and $\{\mathcal{A}(c,r), i_{rr'}\}$ is an inductive system for any c. Denote by $\mathcal{A}(c)$ the direct limit. If $c \subset c'$, then there exists a natural restriction mapping $i_{cc'}$ of $\mathcal{A}(c')$ into $\mathcal{A}(c)$, and the inductive system $\{\mathcal{A}(c), i_{cc'}\}$ is a presheaf over S^1, which satisfies the sheaf condition. We denote the associated sheaf by $\mathcal{A}$ and call it the sheaf of germs of function class of functions holomorphic and asymptotically developable in a sector S including the direction ℓ. Then, a germ of holomorphic function is a global section of the sheaf $\mathcal{A}$ on S^1.

Let $\mathcal{A}_0$ be the sheaf of germs of function asymptotic to 0 over S^1. Then, by the theorem of Borel-Ritt, the following sequence of sheaves

$$0 \to \mathcal{A}_0 \to \mathcal{A} \to \underline{\hat{\mathcal{O}}} \to 0$$

is exact, where $\underline{\hat{\mathcal{O}}}$ is the constant sheaf of formal power series ring over S^1. Then, we obtain a long exact sequence

$$0 \to 0 \to \mathcal{O} \to \hat{\mathcal{O}} \to H^1(S^1, \mathcal{A}_0) \to H^1(S^1, \mathcal{A}) \to \cdots.$$

Sibuya [25], [26] and Malgrange [15] proved that

Theorem(commutative case).

$$\hat{\mathcal{O}}/\mathcal{O} = H^1(S^1, \mathcal{A}_0),$$

namely, the image of $H^1(S^1, \mathcal{A}_0)$ *in* $H^1(S^1, \mathcal{A}) = 0$. In other words, any 1-cocycle $\{f_{hh'}\}$ of open sectorial covering $\{c_h\}_{h\in H}$ of S^1 with coefficients in $\mathcal{A}_0$ is equal to 0 in $H^1(\{c_h\}_{h\in H}, \mathcal{A})$, which means that $f_{hh'} = f_h - f_{h'}$ for some $\{f_h\}$, f_h is asymptotic to some formal series.

Ramis and Malgrange [21] , [19] proved theorems analogous to the above in the framework of asymptotic theory with Gevrey order.

Consider a linear ordinary differential operator with coefficients in holomorphic functions at the origin in the Riemann Sphere:

$$Pu = (\sum_{i=0}^{m} a_i(x)(d/dx)^i)u.$$

where a_m is supposed not to be identically zero. Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ be the ring of convergent power-series and the ring of formal power-series in x, respectively. Then, we see the following isomorphism of linear spaces due to Deligne (cf. [21], etc.) :

$$H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0)) \simeq \mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O}),$$

where $\mathcal{A}_0$ is the sheaf of germs of functions asymptotically developable to the formal power-series 0 on the circle S^1, for, from the existence theorem of asymptotic solutions due to Hukuhara (cf. [24]) (and other many contributers), we have the short exact sequence

$$0 \to \mathcal{K}er(P : \mathcal{A}_0) \to \mathcal{A}_0 \xrightarrow{P} \mathcal{A}_0 \to 0,$$

from which, we get the exact sequence,

$$0 \to H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0)) \to H^1(S^1, \mathcal{A}_0)(= \hat{\mathcal{O}}/\mathcal{O}) \xrightarrow{P} H^1(S^1, \mathcal{A}_0)(= \hat{\mathcal{O}}/\mathcal{O}) \to 0.$$

The dimension is finite and is equal to

$$\begin{aligned} i_0(P) &= \sup\{i - v(a_i) : i = 0, ..., m\} - (m - v(a_m)) \\ &= (v(a_m) - m) - \inf\{v(a_i) - i : i = 0, ..., m\}, \end{aligned}$$

which is called the irregularity by Malgrange [14], [15], the invariant of Fuchs by Gérard-Levelt [3], [4] or the irregular index by Komatsu (in a private communication), where,

$$v(a) = \sup\{p : x^{-p}a(x) \quad is \quad holomorphic \quad at \quad the \quad origin.\}.$$

Remark 1: If we consider a linear ordinary differential operator with coefficients in holomorphic functions at the infinity in the Riemann Sphere and we do not use the variable $t = \frac{1}{x}$, the quantity is equal to

$$\begin{aligned} i_\infty(P) &= \sup\{v'(a_i) - i : i = 0, ..., m\} - (v'(a_m) - m) \\ &= (m - v'(a_m)) - \inf\{i - v'(a_i) : i = 0, ..., m\}, \end{aligned}$$

where

$$v'(a) = \sup\{p : x^{-p}a(x) \quad is \quad holomorphic \quad at \quad the \quad infinity.\}.$$

We now give you a new version of vanishing theorems in asymptotic analysis, which are regarded as theorems with refinements of Gevrey estimates ([13]).

In the following, we work at the infinity and for a real positive number R, real numbers a and b, we denote by $\mathcal{S}(R,\ a,\ b)$ the open sector at the infinity

$$\{\, z \;:\; |z| > R,\ a < \arg z < b\,\}.$$

Let $\{\mathcal{S}(R, a_\ell, b_\ell)\,(\ell = 1, \ldots, N)\}$ be an open sectorial covering of the annulus

$$\mathcal{D}(R, \infty) = \{\, z \,|\, +\infty > |z| > R\,\}.$$

We say that $\{\mathcal{S}(R, a_\ell, b_\ell)\,(\ell = 1, \ldots, N)\}$ is a good covering for $\sigma \geq 1$ when the following condition is satisfied:

$$a_{N+1} = a_1,\ a_\ell < b_{\ell-1} < a_{\ell+1} < b_\ell,\ \sigma(b_\ell - a_\ell) < \pi\,(\ell = 1, \ldots, N).$$

We set, for fixed a_ℓ, $b_\ell\,(\ell = 1, \ldots, N)$,

$$\mathcal{S}_{\ell-1,\ell}(R) = \mathcal{S}(R, a_{\ell-1}, b_{\ell-1}) \cap \mathcal{S}(R, a_\ell, b_\ell).$$

Theorem 1 *Let σ be a rational number ≥ 1 and*

$$\{\mathcal{S}(R, a_\ell, b_\ell)\,(\ell = 1, \ldots, N)\}$$

be a good open sectorial covering for σ *of* $\mathcal{D}(R,\infty)$. *For* $\ell = 1,\ldots,N$, *let* $U_{\ell-1,\ell}(z)$ *be an* $m \times n$ *matricial function defined in* $\mathcal{S}_{\ell-1,\ell}(R)$ *and, for some non-zero constant* $\kappa_{\ell-1,\ell}$ *with* $\arg\kappa_{\ell-1,\ell} = -\frac{a_\ell+b_{\ell-1}}{2\sigma}$, $\exp(-\kappa_{\ell-1,\ell}z^{\frac{1}{\sigma}})$ *is asymptotically developable to the formal power-series* 0 *and, for a complex number* $\mu_{\ell-1,\ell}$,

$$z^{\mu_{\ell-1,\ell}}\exp(\kappa_{\ell-1,\ell}z^{\frac{1}{\sigma}})U_{\ell-1,\ell}(z)$$

is asymptotically developable to a formal power-series matrix $\sum_{s=0}^{\infty} U_s^{\ell-1,\ell}z^{-s}$ *in the sector* $\mathcal{S}_{\ell-1,\ell}(R)$.

Then, there exist a positive number $R''(\geq R)$, *a formal power-series matrix* $\widehat{V}(z) = \sum_{r=0}^{\infty} T_r z^{-r}$ *and* $m \times n$ *matricial functions* V_ℓ *defined in* $\mathcal{S}_\ell(R'')$ $(\ell = 1,\ldots,N)$ *such that*

(i) *the relation*

$$U_{\ell-1,\ell}(z) = -V_{\ell-1}(z) + V_\ell(z)$$

holds for $z \in \mathcal{S}_{\ell-1,\ell}(R'') = \mathcal{S}(R'', a_{\ell-1}, b_{\ell-1}) \cap \mathcal{S}(R'', a_\ell, b_\ell)$.

(ii) V_ℓ *is aymptotically developable to the formal power-series matrix* $\widehat{V}(z)$ *in* $\mathcal{S}_\ell(R'')$, *and for any sufficiently large number* r,

$$\begin{aligned} T_r &= \sum_{(\ell-1,\ell)}\sum_{s=0}^{M-1} \sigma U_s^{\ell-1,\ell}(\kappa_{\ell-1,\ell})^{(s-r+\mu_{\ell-1,\ell})\sigma}\Gamma((r-s-\mu_{\ell-1,\ell})\sigma) \\ &+ O\{\Gamma((r-M-\Re\mu_{\ell-1,\ell})\sigma)\} \end{aligned}$$

provided $1 \leq M < r$.

We will use this new version to compute the coefficients of divergent solutions to the inhomogeneous equations of which associated homogeneous equations are the confluent hypergeometric differential equations.

3 Structure of Divergent Solutions to Inhomogeneous Equations Associated to the Confluent Hypergeometric Differential Equations

Consider the generalized confluent hypergeometric differential equation

$$P = \frac{d^2}{dz^2} + (A_0 + \frac{A_1}{z})\frac{d}{dz} + (B_0 + \frac{B_1}{z} + \frac{B_2}{z^2}),$$

where A_0, A_1, B_0, B_1 and B_2 are complex numbers. The value of irregularity in the sense of Malgrange may be equal to 0, 1 or 2 and the value of order may be equal to 0, $\frac{1}{2}$ or 1. Here, we will give bases of

$$H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0)) \simeq \mathrm{Ker}(P; \widehat{\mathcal{O}}/\mathcal{O}),$$

for Kummer, Whittaker and Weber differential equations.

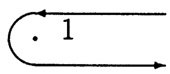
3.1 Confluent Hypergeometric(Kummer) Equation.

$$A_0 = -1,\ A_1 = c,\ B_0 = 0,\ B_1 = -a,\ B_2 = 0,\ k = 1,\ i_\infty(P) = 1.$$

Denote by $G_2(z)$ the confluent hypergeometric function, namely,

$$G_2(z) = \frac{2}{1-e^{2\pi i(\gamma-\alpha)}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_C e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

where, for $-\pi < \theta < \pi$, and $\frac{1}{2}\pi - \theta < \arg z < \frac{3}{2}\pi - \theta$, $C = C(1;\theta)$ is the path of integral on which $\arg(\zeta - 1)$ is taken to be initially θ and finally $\theta + 2\pi$, and so $G_2(z)$ is defined for $-\frac{1}{2}\pi < \arg z < -\frac{5}{2}\pi$, in particular, for $\theta = 0$ and $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, the path of integral is as follows,



and

$$G_2(z) = \frac{2}{1-e^{2\pi i(\gamma-\alpha)}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_1^{+\infty}(e^{\pi i(\gamma-\alpha-1)} - e^{-\pi i(\gamma-\alpha-1)})e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

$$G_2(z) = -2e^{-\pi i(\gamma-\alpha-1)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_1^{+\infty} e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

$$G_2(z) = -2e^{-\pi i(\gamma-\alpha-1)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^{-\infty} -e^{z(1-\zeta)}(1-\zeta)^{\alpha-1}\zeta^{\gamma-\alpha-1}d\zeta,$$

$$G_2(z) = -2e^{-\pi i(\gamma-\alpha)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}e^z\int_0^{-\infty} e^{-z\zeta}(1-\zeta)^{\alpha-1}\zeta^{\gamma-\alpha-1}d\zeta,$$

$$G_2(z) = -2e^{-\pi i(\gamma-\alpha)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}e^z z^{\alpha-\gamma}\int_0^{+\infty} e^{-t}(1-\frac{t}{z})^{\alpha-1}t^{\gamma-\alpha-1}dt,$$

by using the Newton's binomial expansion

$$(1 \pm \frac{t}{z})^{\alpha-1} = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-1-n)\Gamma(n+1)}(\pm\frac{t}{z})^n,$$

or

$$(1 \pm \frac{t}{z})^{\alpha-1} = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}(\pm\frac{t}{z})^n.$$

The asymptotic behaviours at the infinity for $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, is as follows (cf. [2] etc.) :

$$G_2(z) \approx -2e^{-\pi i(\gamma-\alpha)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}z^{-(\gamma-\alpha)}\exp(-(-z))\sum_{n=0}^{\infty}\frac{\Gamma(n+\gamma-\alpha)\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}z^{-n},$$

Therefore, we can choose a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$ in the following way: Put $U_1 = \{z \in \mathbf{C} : |z| > R, \frac{\pi}{2} < \arg z < \frac{5}{2}\pi\}$, and $U_2 = \{z \in \mathbf{C} : |z| > R, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\}$ for a positive real number R. Then, $\{U_1, U_2\}$ forms an open sectorial covering at $z = \infty$ and put

$$u_{12}(z) = u(z) \quad (\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi), \; u_{12}(z) = 0 \quad (\frac{3}{2}\pi < \arg z < \frac{5}{2}\pi).$$

In this situation, the cohomology classes of $\{u_{12}\}$ forms a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$. By the original vanishing theorem due to [14] in asymptotic analysis, we have 0-cochains $\{u_1, u_2\}$ such that

$$u_{12}(z) = u_2(z) - u_1(z),$$

where $u_j(z)$ are defined in U_j for $j = 1, 2$ and asymptotically developable to a formal power-series $\hat{u} = \sum_{r=0}^{\infty} u_r z^{-r}$ at the first. The coefficient u_r is given by the following:

$$u_r = \frac{-1}{2\pi i}\int_0^{-\infty} z^{r-1} G_2(z)dz$$

$$u_r = \frac{-1}{2\pi i}\int_0^{-\infty} z^{r-1}(-2)e^{-\pi i(\gamma-\alpha-1)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_1^{+\infty} e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta dz,$$

$$u_r = \frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^{-\infty} z^{r-1}\int_1^{+\infty} e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta dz,$$

$$u_r = \frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^r\Gamma(r)\int_1^{+\infty} \zeta^{\alpha-r-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

$$u_r = \frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^r\Gamma(r)\int_0^{1} \zeta^{r-\gamma}(\zeta-1)^{\gamma-\alpha-1}d\zeta,$$

$$u_r = \frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^r\Gamma(r)\int_0^{1} \zeta^{r-\gamma}(-1)^{\gamma-\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

$$u_r = \frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^r\Gamma(r)(-1)^{\gamma-\alpha-1}\frac{\Gamma(r-\gamma+1)\Gamma(\gamma-\alpha)}{\Gamma(r-\alpha+1)},$$

$$u_r = \frac{1}{\pi i}\frac{\Gamma(\gamma)\Gamma(r-\gamma+1)}{\Gamma(\alpha)\Gamma(r-\alpha+1)}(-1)^r\Gamma(r).$$

By the vanishing theorem in asymptotic analysis with Gevrey estimates due to [21], we can assert secondly that $\hat{u}$ and $\hat{v}$ are formal power-series with Gevrey order $\sigma = 1$. Our new theorem [13] claims thirdly that we can have asymptotic estimates for the coefficients of $\hat{u}$, more precise than Gevrey estimates: for any sufficiently large number r,

$$u_r = \frac{e^{-\pi i(\gamma-\alpha)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\sum_{s=0}^{M-1}\frac{\Gamma(s+\gamma-\alpha)\Gamma(s+1-\alpha)}{\Gamma(1-\alpha)\Gamma(s+1)}(e^{\pi i})^{s-r+(\gamma-\alpha)}\Gamma(r-s-(\gamma-\alpha))$$

$$+O\{\Gamma(r-M-\Re(\gamma-\alpha))\}$$

$$u_r = \frac{1}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)}(-1)^r \sum_{s=0}^{M-1} \frac{\Gamma(s+\gamma-\alpha)\Gamma(s+1-\alpha)\Gamma(r-s-(\gamma-\alpha))}{\Gamma(\gamma-\alpha)\Gamma(1-\alpha)\Gamma(s+1)}(-1)^s$$
$$+O\{\Gamma(r-M-\Re(\gamma-\alpha))\}$$

provided $1 \leq M < r$.

In the intersection $U_1 \cap U_2$, $Pu_1(z) = Pu_2(z)$, which define holomorphic functions f at the infinity, and $P\hat{u} = f$, so the equivalence class of $\hat{u}$, forms a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$.

Of course, in this case, we can compute a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$ directly: for example, as a formal solution of the inhomogeneous linear ordinary differential equation $P\hat{w} = \frac{1-\gamma}{z^2}$, we have

$$\hat{w} = \sum_{r=0}^{\infty}(-1)^{r-1}\frac{\Gamma(r)\Gamma(r+1-\gamma)\Gamma(1-\alpha)}{\Gamma(1-\gamma)\Gamma(r+1-\alpha)}z^{-r}$$

and the equivalence class of $\hat{w}$ as a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$, of which coefficients admit asymptotic estimates by the result on Γ-function.

By a little more calculation, we find that $\hat{u}$ is equivalent to

$$\frac{-1}{\pi i}\frac{\Gamma(\gamma)\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(1-\alpha)}\hat{w} = \frac{-1}{\pi i}\frac{\sin \pi\alpha}{\sin \pi\gamma}\hat{w},$$

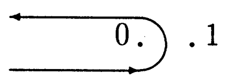
modulo $\mathcal{O}$. In this case, we have just the equality

$$\hat{u} = \frac{-1}{\pi i}\frac{\sin \pi\alpha}{\sin \pi\gamma}\hat{w}.$$

In the following subsections, we use another solution to Kummer differential equation:

$$G_1(\alpha;\gamma;z) = \frac{2}{1-e^{-2\pi i\alpha}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_C e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$
$$\left(-\tfrac{3}{2}\pi < \arg z < \tfrac{3}{2}\pi\right),$$

where, for $0 < \theta < 2\pi$ and $\frac{1}{2}\pi - \theta < \arg z < \frac{3}{2}\pi - \theta$, , $C = C(0;\theta)$ is the path of integral on which $\arg\zeta$ is taken to be initially θ and finally $\theta + 2\pi$, and so $G_1(z)$ is defined for $-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi$, in particular, for $\theta = \pi$ and $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$, the path of integral is as follows,



By the transformation $v = e^{\pi i}z\zeta$, we get the following:

$$G_1 = \frac{2}{1-e^{-2\pi i\alpha}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_C e^{-v}v^{\alpha-1}z^{-\alpha+1}(e^{\pi i})^{-\alpha+1}\left(1+\frac{v}{z}\right)^{\gamma-\alpha-1}\left(-\frac{1}{z}\right)dv,$$

$$\begin{aligned}
&= \frac{2e^{-\pi i\alpha}}{1-e^{2\pi i\alpha}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}z^{-\alpha}\int_C e^{-v}v^{\alpha-1}\left(1+\frac{v}{z}\right)^{\gamma-\alpha-1}dv,\\
&= \frac{2e^{\pi i\alpha}}{1-e^{2\pi i\alpha}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}z^{-\alpha}\left(e^{2\pi i\alpha}-1\right)\int_0^\infty e^{-v}v^{\alpha-1}\left(1+\frac{v}{z}\right)^{\gamma-\alpha-1}dv,\\
&= -2e^{\pi i\alpha}z^{-\alpha}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^\infty e^{-\zeta}\zeta^{\alpha-1}\left(1+\frac{\zeta}{z}\right)^{\gamma-\alpha-1}d\zeta.
\end{aligned}$$

By using the binomial series

$$(1\pm\frac{t}{z})^{\alpha-1}=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-1-n)\Gamma(n+1)}(\pm\frac{t}{z})^n,$$

or

$$(1\pm\frac{t}{z})^{\alpha-1}=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}(\pm\frac{t}{z})^n,$$

we have the following asymptotic expansion:

$$G_1(\alpha;\gamma;z)\approx -2e^{\pi i\alpha}z^{-\alpha}\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\sum_{s=0}^{\infty}\frac{(-1)^s\Gamma(\alpha+s)\Gamma(\alpha-\gamma+1+s)}{\Gamma(\alpha)\Gamma(\alpha-\gamma+1)\Gamma(s+1)}z^{-s}.$$

3.2 Whittaker Differential Equation

Whittaker differential operator is as follows:

$$P=\frac{d^2}{dz^2}+\left(-\frac{1}{4}+\frac{\kappa}{z}+\frac{\frac{1}{4}-\mu^2}{z^2}\right),$$

The solutions to Whittaker equation $Pw=0$ are written as

$$Wh_i(\kappa;\mu;z)=e^{-\frac{z}{2}}z^{\mu+\frac{1}{2}}G_i\left(\frac{1}{2}+\mu-\kappa;2\mu+1;z\right),\ (i=1,2),$$

by using $G_i(z)$=$G_i(\alpha;\gamma;z)$ $(i=1,2)$ for Kummer differential equation with parameters,

$$\alpha=\frac{1}{2}+\mu-\kappa,\ \gamma=2\mu+1.$$

The integral representations and asymptotic behaviors are as follow:

$$Wh_1=-2e^{-\frac{z}{2}}z^{\kappa}e^{\pi i(\frac{1}{2}+\mu-\kappa)}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)}\int_0^\infty e^{-\zeta}\zeta^{\mu-\kappa-\frac{1}{2}}\left(1+\frac{\zeta}{z}\right)^{\mu+\kappa-\frac{1}{2}}d\zeta,$$

$$\left(-\tfrac{3}{2}\pi<\arg z<\tfrac{3}{2}\pi\right),$$

$$\begin{aligned}
Wh_1 \approx\ & -2e^{-\frac{z}{2}}z^{\kappa}e^{\pi i(\frac{1}{2}+\mu-\kappa)}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)}\\
&\times\sum_{s=0}^{\infty}\frac{(-1)^s\Gamma\left(\mu-\kappa+\frac{1}{2}+s\right)\Gamma\left(-\mu-\kappa+\frac{1}{2}+s\right)}{\Gamma\left(-\mu-\kappa+\frac{1}{2}\right)\Gamma(s+1)}z^{-s},
\end{aligned}$$

$$Wh_2 = -2e^{-\frac{z}{2}}z^{\mu+\frac{1}{2}}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)}\int_1^{+\infty} e^{z\zeta}\zeta^{\mu-\kappa-\frac{1}{2}}(1-\zeta)^{\mu+\kappa-\frac{1}{2}}d\zeta,$$

$$\left(-\tfrac{1}{2}\pi < \arg z < \tfrac{5}{2}\pi\right),$$

$$\begin{aligned} Wh_2 &\approx 2e^{\frac{z}{2}}z^{-\kappa}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \\ &\quad \times\sum_{s=0}^{\infty}\frac{\Gamma\left(\mu+\kappa+\frac{1}{2}+s\right)\Gamma\left(-\mu+\kappa+\frac{1}{2}+s\right)}{\Gamma\left(-\mu+\kappa+\frac{1}{2}\right)\Gamma(s+1)}z^{-s}. \end{aligned}$$

Therefore, we can make the basis of $H^1(S^1, \mathcal{K}er(P:\mathcal{A}_0))$ in the folloing manner. Put

$$U_1 := \{z\in \mathbf{C}:\ |z|>0,\ -\frac{3}{2}\pi < \arg\ z < \frac{1}{2}\pi\},$$

$$U_2 := \{z\in \mathbf{C}:\ |z|>0,\ -\frac{1}{2}\pi < \arg\ z < \frac{3}{2}\pi\},$$

then $\{U_1,\ U_2\}$ forms a sectorial covering of the punctured disc $\{z\ :\ |z|>0\}$ at the infinity. Put

$$u_{12}(z) = Wh_1\quad (-\frac{1}{2}\pi < \arg\ z < \frac{1}{2}\pi),\ u_{12}(z)=0\quad (\frac{1}{2}\pi < \arg\ z < \frac{3}{2}\pi),$$

$$v_{12}(z) = 0\quad (-\frac{1}{2}\pi < \arg\ z < \frac{1}{2}\pi),\ v_{12}(z)=Wh_2\quad (\frac{1}{2}\pi < \arg\ z < \frac{3}{2}\pi),$$

then the classes of the 1-st cohomology of $\{u_{12}\}$ and $\{v_{12}\}$ form the basis of $H^1(S^1, \mathcal{K}er(P:\mathcal{A}_0))$.

By the original vanishing theorem due to [14] in asymptotic analysis, we have 0-cochains $\{u_1,\ u_2\}$, $\{v_1,\ v_2\}$ such that

$$u_{12}(z) = u_2(z) - u_1(z),\ v_{12}(z) = v_2(z) - v_1(z),$$

where $u_j(z),\ v_j(z)$ are defined in U_j, $(j=1,2)$ and have asymptotic expansions $\hat{u} = \sum_{r=0}^{\infty} u_r z^{-r}$,$\hat{v} = \sum_{r=0}^{\infty} v_r z^{-r}$, respectively. Then, we have

$$Pu_{12}(z) = Pu_2(z) - Pu_1(z)$$

and $Pu_1(z) = Pu_2(z)$ for $z\in U_1\cap U_2$, from which we can define a function f in the punctured disc $\{z\ :\ |z|>0\}$ at the infinity in the following way:

$$f(z) = \begin{cases} Pu_1(z), & z\in U_1, \\ Pu_2(z), & z\in U_2, \end{cases}$$

The function f has only a removable singularity at the infinity because of the asymptoticity and is extended to the whole naeibourhood at the infinity by putting $f(\infty) = \lim_{z\to\infty} f(z)$. Therfore, $\hat{u} = \sum_{r=0}^{\infty} u_r z^{-r}$ is a divergent solution to the equation $P\hat{u} = f$. We have the same story for v_{12}.

From the contruction of the 0-cochains $\{u_1,\ u_2\}$, we can calculate the coefficient by using the following integral formula

$$u_r = \frac{1}{2\pi i}\int_0^\infty -\zeta^{r-1} Wh_1(\zeta)d\zeta,$$

and the integaral representation

$$Wh_1 = -2e^{-\frac{z}{2}} z^\kappa e^{\pi i\left(\frac{1}{2}+\mu-\kappa\right)} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \int_0^\infty e^{-\zeta}\zeta^{\mu-\kappa-\frac{1}{2}}\left(1+\frac{\zeta}{z}\right)^{\mu+\kappa-\frac{1}{2}} d\zeta,$$

:

$$\begin{aligned}
u_r &= \frac{1}{2\pi i}\int_0^\infty -z^{r-1} Wh_1(z)dz \\
&= \frac{1}{2\pi i}\int_0^\infty -z^{r-1}(-2)e^{-\frac{z}{2}} z^\kappa e^{\pi i\left(\frac{1}{2}+\mu-\kappa\right)} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \\
&\quad \times \int_0^\infty e^{-\zeta}\zeta^{\mu-\kappa-\frac{1}{2}}\left(1+\frac{\zeta}{z}\right)^{\mu+\kappa-\frac{1}{2}} d\zeta\, dz \\
&= \frac{e^{\pi i\left(\mu-\kappa+\frac{1}{2}\right)}}{\pi i} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \int_0^\infty z^{r-1+\kappa} e^{-\frac{z}{2}} \int_0^\infty e^{-\zeta}\zeta^{\mu-\kappa-\frac{1}{2}}\left(1+\frac{\zeta}{z}\right)^{\mu+\kappa-\frac{1}{2}} d\zeta\, dz \\
&= \frac{e^{\pi i\left(\mu-\kappa+\frac{1}{2}\right)}}{\pi i} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \int_0^\infty z^{r+\kappa-1} e^{-\frac{z}{2}} \\
&\quad \times \int_0^\infty e^{-zt} z^{\mu-\kappa-\frac{1}{2}} t^{\mu-\kappa-\frac{1}{2}} (1+t)^{\mu+\kappa-\frac{1}{2}}\, z\, dt\, dz \\
&= \frac{e^{\pi i\left(\mu-\kappa+\frac{1}{2}\right)}}{\pi i} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \int_0^\infty z^{r+\mu-\frac{1}{2}} e^{-\frac{z}{2}} \int_0^\infty e^{-zt} t^{\mu-\kappa-\frac{1}{2}} (1+t)^{\mu+\kappa-\frac{1}{2}}\, dt\, dz \\
&= \frac{e^{\pi i\left(\mu-\kappa+\frac{1}{2}\right)}}{\pi i} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \int_0^\infty t^{\mu-\kappa-\frac{1}{2}} (1+t)^{\mu+\kappa-\frac{1}{2}} \int_0^\infty z^{r+\mu-\frac{1}{2}} e^{-\left(\frac{1}{2}+t\right)z}\, dz\, dt \\
&= \frac{e^{\pi i\left(\mu-\kappa+\frac{1}{2}\right)}}{\pi i} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \int_0^\infty t^{\mu-\kappa-\frac{1}{2}} (1+t)^{\mu+\kappa-\frac{1}{2}} \left(\frac{1}{2}+t\right)^{-r-\mu-\frac{1}{2}} \\
&\quad \times \int_0^\infty s^{r+\mu-\frac{1}{2}} e^{-s}\, ds\, dt \\
&= \frac{e^{\pi i\left(\mu-\kappa+\frac{1}{2}\right)}}{\pi i} \frac{\Gamma(2\mu+1)}{\Gamma(\mu-\kappa+\frac{1}{2})\Gamma(\mu+\kappa+\frac{1}{2})} \\
&\quad \times \Gamma(r+\mu+\frac{1}{2}) \cdot 2^{r+\mu+\frac{1}{2}} \int_0^\infty t^{\mu-\kappa-\frac{1}{2}} (1+t)^{\mu+\kappa-\frac{1}{2}} (1+2t)^{-r-\mu-\frac{1}{2}} dt.
\end{aligned}$$

By using the following formula for Gauss hypergeometric function

$$F(a;b;c;1-z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty s^{b-1}(1+s)^{a-c}(1+sz)^{-a}\,ds$$

$$(\Re c>\Re b>0,\ |\arg z|<\pi)$$

with the parameters

$$a=r+\mu+\frac{1}{2},\ b=\mu-\kappa+\frac{1}{2},\ c=r-\kappa+1,\ z=2,$$

we get finally the exact value

$$\begin{aligned} u_r &= \frac{e^{\pi i\left(\mu-\kappa+\frac{1}{2}\right)}}{\pi i}\cdot 2^{r+\mu+\frac{1}{2}}\frac{\Gamma(2\mu+1)\Gamma\left(r+\mu+\frac{1}{2}\right)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)}\frac{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(r-\mu+\frac{1}{2}\right)}{\Gamma(r-\kappa+1)} \\ &\quad\times F\left(r+\mu+\frac{1}{2};\mu-\kappa+\frac{1}{2};r-\kappa+1;-1\right) \\ &= \frac{e^{\pi i\left(\mu-\kappa+\frac{1}{2}\right)}}{\pi i}\cdot 2^{r+\mu+\frac{1}{2}}\frac{\Gamma(2\mu+1)\Gamma\left(r+\mu+\frac{1}{2}\right)\Gamma\left(r-\mu+\frac{1}{2}\right)}{\Gamma\left(\mu+\kappa+\frac{1}{2}\right)\Gamma(r-\kappa+1)} \\ &\quad\times F\left(r+\mu+\frac{1}{2};\mu-\kappa+\frac{1}{2};r-\kappa+1;-1\right). \end{aligned}$$

We will derive the approximation formula for u_r. In the integral representation

$$u_r=\frac{1}{2\pi i}\int_0^\infty -\zeta^{r-1}Wh_1(\zeta)d\zeta,$$

we substitute the asymptotic representation

$$\begin{aligned} Wh_1(\zeta) &= -2e^{-\frac{\zeta}{2}}\zeta^\kappa e^{\pi i\left(\frac{1}{2}+\mu-\kappa\right)}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \\ &\quad\times\left(\sum_{s=0}^{M-1}\frac{(-1)^s\Gamma\left(\mu-\kappa+\frac{1}{2}+s\right)\Gamma\left(-\mu-\kappa+\frac{1}{2}+s\right)}{\Gamma\left(-\mu-\kappa+\frac{1}{2}\right)\Gamma(s+1)}\zeta^{-s}+O(\zeta^{-M})\right), \end{aligned}$$

then we get

$$\begin{aligned} u_r &= \frac{1}{2\pi i}\int_0^\infty \zeta^{r-1}\cdot 2e^{-\frac{\zeta}{2}}\zeta^\kappa e^{-\pi i\left(-\frac{1}{2}-\mu+\kappa\right)}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \\ &\quad\times\left(\sum_{s=0}^{M-1}\frac{(-1)^s\Gamma\left(\mu-\kappa+\frac{1}{2}+s\right)\Gamma\left(-\mu-\kappa+\frac{1}{2}+s\right)}{\Gamma\left(-\mu-\kappa+\frac{1}{2}\right)\Gamma(s+1)}\zeta^{-s}+O(\zeta^{-M})\right)d\zeta \\ &= -\frac{e^{-\pi i\left(-\frac{1}{2}-\mu+\kappa+1\right)}}{\pi i}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \\ &\quad\times\sum_{s=0}^{M-1}\frac{(-1)^s\Gamma\left(\mu-\kappa+\frac{1}{2}+s\right)\Gamma\left(-\mu-\kappa+\frac{1}{2}+s\right)}{\Gamma\left(-\mu-\kappa+\frac{1}{2}\right)\Gamma(s+1)}\int_0^\infty e^{-\frac{\zeta}{2}}\zeta^{r-1+\kappa-s}d\zeta+O(\zeta^{-M}). \end{aligned}$$

By using the following formula for Γ-function,

$$\int_0^\infty e^{-\frac{\zeta}{2}}\zeta^{r-1+\kappa-s}d\zeta$$
$$= 2^{r+\kappa-s}\int_0^\infty e^{-t}t^{r-1+\kappa-s}dt$$
$$= 2^{r+\kappa-s}\Gamma(r+\kappa-s),$$

we have

$$\begin{aligned} u_r \;=\; & -\frac{e^{-\pi i\left(\frac{1}{2}-\mu+\kappa\right)}}{\pi i}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \\ & \times\sum_{s=0}^{M-1}\frac{(-1)^s\Gamma\left(\mu-\kappa+\frac{1}{2}+s\right)\Gamma\left(-\mu-\kappa+\frac{1}{2}+s\right)}{\Gamma\left(-\mu-\kappa+\frac{1}{2}\right)\Gamma(s+1)}\cdot 2^{r+\kappa-s}\Gamma(r+\kappa-s) \\ & +O(\Gamma(r+\kappa-M)) \end{aligned}$$

for r enough large and $1 \le M < r$.

For $r = 100, \mu = 9/10, \kappa = 2/10$, the exact value of u_{100} is ,

$$\begin{aligned} & -1.2017084886449290571790164028047654035799498501\times 10^{186} \\ + \; & 1.6540098374476188944723139433448383135996446978\times 10^{186}i, \end{aligned}$$

and the approximation with $M = 51$ is

$$\begin{aligned} & -1.2017084886449290571790164028047654035799498501 3\times 10^{186} \\ + \; & 1.6540098374476188944723139433448383135996446977 7\times 10^{186}i. \end{aligned}$$

In the same way, we have

$$v_r = \frac{1}{2\pi i}\int_0^{-\infty} -\zeta^{r-1}Wh_2(\zeta)d\zeta,$$

$$Wh_2 = -2e^{-\frac{z}{2}}z^{\mu+\frac{1}{2}}\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)}\int_1^{+\infty} e^{z\zeta}\zeta^{\mu-\kappa-\frac{1}{2}}(1-\zeta)^{\mu+\kappa-\frac{1}{2}}d\zeta,$$

$$F(a;b;c;z^{-1}) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_1^\infty (s-1)^{c-b-1}s^{a-c}(s-z^{-1})^{-a}ds,$$

$$(1+\Re a > \Re c > \Re b, \;|\arg(z-1)| < \pi)$$

$$a = r+\mu+\frac{1}{2},\; b = r-\mu+\frac{1}{2},\; c = r+\kappa+1,\; z = 2,$$

$$v_r = \frac{e^{\pi i(\kappa-r-1)}}{\pi i} \frac{\Gamma(2\mu+1)\Gamma\left(r+\mu+\frac{1}{2}\right)\Gamma\left(r-\mu+\frac{1}{2}\right)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma(r+\kappa+1)} F\left(r+\mu+\frac{1}{2}; r-\mu+\frac{1}{2}; r+\kappa+1; \frac{1}{2}\right)$$

$$\begin{aligned} v_r &= \frac{-e^{-\pi i(r-\kappa)}}{\pi i} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\Gamma\left(\mu+\kappa+\frac{1}{2}\right)} \\ &\times \sum_{s=0}^{M-1} \frac{(-1)^s \Gamma\left(\mu+\kappa+\frac{1}{2}+s\right)\Gamma\left(-\mu+\kappa+\frac{1}{2}+s\right)}{\Gamma\left(-\mu+\kappa+\frac{1}{2}\right)\Gamma(s+1)} \cdot 2^{r-\kappa-s}\Gamma(r-\kappa-s) \\ &+O(\Gamma(r-\kappa-M)) \end{aligned}$$

for r enough large and $1 \le M < r$.

3.3 Weber Differential Equation

Weber differential operator is as follows:

$$P = \frac{d^2}{dz^2} + (\lambda - z^2)$$

By the transformation

$$w(z) = e^{-\frac{z^2}{2}} v(z),$$

Weber differential equation $Pw = 0$ is changed into the following equation

$$\frac{d^2 v}{dz^2} - 2z\frac{dv}{dz} + 2\nu v = 0, \quad \nu = \frac{\lambda - 1}{2},$$

and, by using the transformation fo variable $\zeta = z^2$, we get

$$\zeta\frac{d^2 v}{d\zeta^2} + \left(\frac{1}{2} - \zeta\right)\frac{dv}{d\zeta} + \frac{\nu}{2}v = 0,$$

which is nothing but Kummer differential equation with the parameters

$$\alpha = -\frac{\nu}{2}, \; \gamma = \frac{1}{2}.$$

Therefore, by using $G_i(z)$=$G_i(\alpha;\gamma;z)$ $(i = 1, 2)$ for Kummer differential equation, we have the solutions

$$We_i(z) = e^{-\frac{z^2}{2}} G_i\left(-\frac{\nu}{2}; \frac{1}{2}; z^2\right), \quad (i = 1, 2),$$

to Weber differential equation $Pw = 0$.

The integral representations and asymptotic behaviors are as follow:

$$We_1 = -2e^{-\frac{z^2}{2}} z^{\nu} e^{-\frac{\pi i \nu}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \int_0^{\infty} e^{-\zeta}\zeta^{-\frac{\nu}{2}-1}\left(1+\frac{\zeta}{z^2}\right)^{\frac{\nu}{2}-\frac{1}{2}} d\zeta,$$

$$\left(-\tfrac{3}{4}\pi < \arg z < \tfrac{3}{4}\pi\right),$$

$$We_1 \approx -2e^{-\frac{z^2}{2}} z^{\nu} e^{-\frac{\pi i \nu}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \sum_{s=0}^{\infty} \frac{(-1)^s\Gamma(-\frac{\nu}{2}+s)\Gamma(-\frac{\nu}{2}+\frac{1}{2}+s)}{\Gamma(-\frac{\nu}{2}+\frac{1}{2})\Gamma(s+1)} z^{-2s},$$

$$We_2 = -2e^{-\frac{z^2}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \int_1^{\infty} e^{z^2\zeta}\zeta^{-\frac{\nu}{2}-1}(1-\zeta)^{\frac{\nu}{2}-\frac{1}{2}} d\zeta,$$

$$\left(-\tfrac{1}{4}\pi < \arg z < \tfrac{5}{4}\pi\right),$$

$$We_2 \approx 2e^{\frac{z^2}{2}} z^{-(1+\nu)} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \sum_{s=0}^{\infty} \frac{\Gamma(\frac{\nu}{2}+\frac{1}{2}+s)\Gamma(\frac{\nu}{2}+1+s)}{\Gamma(1+\frac{\nu}{2})\Gamma(s+1)} z^{-2s}.$$

Therefore, we can choose a basis of $H^1(S^1, \mathcal{K}er(P:\mathcal{A}_0))$ in the following way: Put

$$\begin{aligned}
U_1 &:= \{z \in \mathbf{C} : \ |z| > 0, \ -\frac{1}{4}\pi < \arg\ z < \frac{3}{4}\pi\},\\
U_2 &:= \{z \in \mathbf{C} : \ |z| > 0, \ \frac{1}{4}\pi < \arg\ z < \frac{5}{4}\pi\},\\
U_3 &:= \{z \in \mathbf{C} : \ |z| > 0, \ \frac{3}{4}\pi < \arg\ z < \frac{7}{4}\pi\},\\
U_4 &:= \{z \in \mathbf{C} : \ |z| > 0, \ \frac{5}{4}\pi < \arg\ z < \frac{9}{4}\pi\},
\end{aligned}$$

Then, $\{U_1,\ U_2,\ U_3,\ U_4\}$ forms an open sectorial covering of the punctured disc$\{z : |z| > 0\}$ t the infinity and put

$$\begin{aligned}
u_{12}^{(1)}(z) &= We_2,\ (\frac{1}{4}\pi < \arg\ z < \frac{3}{4}\pi),\ u_{23}^{(1)}(z) = u_{34}^{(1)}(z) = u_{41}^{(1)}(z) = 0,\\
u_{12}^{(2)}(z) &= 0,\ u_{23}^{(2)}(z) = We_1,\ (\frac{3}{4}\pi < \arg\ z < \frac{5}{4}\pi),\ u_{34}^{(2)}(z) = u_{41}^{(2)}(z) = 0,\\
u_{12}^{(3)}(z) &= u_{23}^{(3)}(z) = 0,\ u_{34}^{(3)}(z) = We_2,\ (\frac{5}{4}\pi < \arg\ z < \frac{7}{4}\pi),\ u_{41}^{(3)}(z) = 0,\\
u_{12}^{(4)}(z) &= u_{23}^{(4)}(z) = u_{34}^{(4)}(z) = 0,\ u_{41}^{(4)}(z) = We_1,\ (\frac{7}{4}\pi < \arg\ z < \frac{9}{4}\pi),
\end{aligned}$$

then, the classes of the 1-st cohomology of $\{u_{ij}^{(k)}\,;\ (i,j) = (1,2),(2,3),(3,4),(4,1)\}$, $(k = 1,2,3,4)$ form the basis of $H^1(S^1, \mathcal{K}er(P:\mathcal{A}_0))$.

By the original vanishing theorem due to [14] in asymptotic analysis, we have 0-cochains $\{u_1^{(k)}, u_2^{(k)}, u_3^{(k)}, u_4^{(k)}\}$ $(k = 1, 2, 3, 4)$ such that

$$\begin{aligned} u_{12}^{(k)}(z) &= -u_1^{(k)}(z) + u_2^{(k)}(z), \\ u_{23}^{(k)}(z) &= -u_2^{(k)}(z) + u_3^{(k)}(z), \\ u_{34}^{(k)}(z) &= -u_3^{(k)}(z) + u_4^{(k)}(z), \\ u_{41}^{(k)}(z) &= -u_4^{(k)}(z) + u_1^{(k)}(z), \end{aligned}$$

where $u_j^{(k)}(z)$ are defined in U_j, $(j = 1, 2, 3, 4)$ and have asymptotic expansions $\hat{u}^{(k)} = \sum_{r=0}^{\infty} u_r^{(k)} z^{-r}$, $(k = 1, 2, 3, 4)$, respectively. Then, we have

$$Pu_{ij}^{(k)}(z) = Pu_j^{(k)}(z) - Pu_i^{(k)}(z),$$

and $Pu_j^{(k)}(z) = Pu_i^{(k)}(z)$ for $z \in U_i \cap U_j$, from which we can define functions $f^{(k)}$ $(k = 1, 2, 3, 4)$ in the punctured disc$\{z : |z| > 0\}$ at the infinity in the following way:

$$f^{(k)}(z) = Pu_i^{(k)}(z), \quad z \in U_i, \quad (i = 1, 2, 3, 4).$$

The function $f^{(k)}$ have only a removable singularity at the infinity because of the asymptoticity and is extended to the whole naeibourhood at the infinity by putting $f^{(k)}(\infty) = \lim_{z\to\infty} f^{(k)}(z)$. Therefore, $\hat{u}^{(k)} = \sum_{r=0}^{\infty} u_r^{(k)} z^{-r}$ is a divergent solution to the equation $P\hat{u}^{(k)} = f^{(k)}$ for each $k = 1, 2, 3, 4$ and $< [\hat{u}^{(1)}], [\hat{u}^{(2)}], [\hat{u}^{(3)}], [\hat{u}^{(4)}] >$ form the basis of $\mathrm{Ker}\left(\frac{d^2}{dz^2} + (\lambda - z^2) : \hat{\mathcal{O}}/\mathcal{O} \to \hat{\mathcal{O}}/\mathcal{O}\right)$.

From the contruction of the 0-cochains, we can calculate the coefficient by using the following integral formula

$$u_r^{(1)} = \frac{1}{2\pi i} \int_0^{i\infty} -\zeta^{r-1} We_2(\zeta) d\zeta.$$

Substituting the integral representation

$$We_2 = -2e^{-\frac{z^2}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2} + \frac{\nu}{2})} \int_1^{\infty} e^{z^2\zeta} \zeta^{-\frac{\nu}{2}-1} (1 - \zeta)^{\frac{\nu}{2}-\frac{1}{2}} d\zeta,$$

into the above expression, we get

$$\begin{aligned} u_r^{(1)} &= \frac{1}{2\pi i} \int_0^{i\infty} -z^{r-1} We_2(z) dz \\ &= \frac{1}{\pi i} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2} + \frac{\nu}{2})} \int_0^{i\infty} z^{r-1} e^{-\frac{z^2}{2}} \int_1^{\infty} e^{z^2\zeta} \zeta^{-\frac{\nu}{2}-1} (1 - \zeta)^{\frac{\nu}{2}-\frac{1}{2}} d\zeta \, dz \\ &= \frac{1}{\pi i} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2} + \frac{\nu}{2})} \int_1^{\infty} \zeta^{-\frac{\nu}{2}-1} (1 - \zeta)^{\frac{\nu}{2}-\frac{1}{2}} \int_0^{i\infty} z^{r-1} e^{(\zeta-\frac{1}{2})z^2} dz \, d\zeta \\ &= \frac{1}{\pi i} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2} + \frac{\nu}{2})} \cdot \frac{i^r}{2} \int_1^{\infty} \zeta^{-\frac{\nu}{2}-1} (1 - \zeta)^{\frac{\nu}{2}-\frac{1}{2}} \left(\zeta - \frac{1}{2}\right)^{-\frac{r}{2}} \int_0^{\infty} t^{\frac{r}{2}-1} e^{-t} dt \, d\zeta \\ &= \frac{1}{\pi i} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2} + \frac{\nu}{2})} \cdot \frac{e^{\frac{\pi i}{2} r}}{2} \Gamma(\frac{r}{2}) e^{-\pi i(\frac{\nu}{2}-\frac{1}{2})} \int_1^{\infty} (\zeta - 1)^{\frac{\nu}{2}-\frac{1}{2}} \zeta^{-\frac{\nu}{2}-1} \left(\zeta - \frac{1}{2}\right)^{-\frac{r}{2}} d\zeta, \end{aligned}$$

and using the following formula for Gauss hypergeometric function

$$F(a;b;c;1-z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty s^{b-1}(1+s)^{a-c}(1+sz)^{-a}\, ds$$

$$(\Re c > \Re b > 0,\ |\arg z| < \pi)$$

with the parameters

$$a = \frac{r}{2},\ b = \frac{r}{2} + \frac{1}{2},\ c = \frac{r}{2} + \frac{\nu}{2} + 1,\ z = 2,$$

we get finally the exact value

$$u_r^{(1)} = \frac{e^{\frac{\pi i}{2}(r-\nu+1)}}{2\pi i} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})\Gamma(\frac{r}{2}+\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{r}{2}+\frac{\nu}{2}+1)} F\left(\frac{r}{2}; \frac{r}{2} + \frac{1}{2}; \frac{r}{2} + \frac{\nu}{2} + 1; \frac{1}{2}\right).$$

We will derive the approximation formula for $u_r^{(1)}$. In the integral representation

$$u_r^{(1)} = \frac{1}{2\pi i} \int_0^{i\infty} -\zeta^{r-1} We_2(\zeta) d\zeta,$$

we substitute the asymptotic representation

$$We_2(\zeta) = 2e^{\frac{\zeta^2}{2}} \zeta^{-(1+\nu)} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \left(\sum_{s=0}^{M-1} \frac{\Gamma(\frac{\nu}{2}+\frac{1}{2}+s)\Gamma(\frac{\nu}{2}+1+s)}{\Gamma(1+\frac{\nu}{2})\Gamma(s+1)} \zeta^{-2s} + O(\zeta^{-2M}) \right),$$

then we get

$$\begin{aligned} u_r^{(1)} &= \frac{-1}{2\pi i} \int_0^{i\infty} \zeta^{r-1} \cdot 2e^{\frac{\zeta^2}{2}} \zeta^{-(1+\nu)} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \\ &\quad \times \left(\sum_{s=0}^{M-1} \frac{\Gamma(\frac{\nu}{2}+\frac{1}{2}+s)\Gamma(\frac{\nu}{2}+1+s)}{\Gamma(1+\frac{\nu}{2})\Gamma(s+1)} \zeta^{-2s} + O(\zeta^{-2M}) \right) d\zeta \\ &= \frac{-1}{\pi i} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \sum_{s=0}^{M-1} \frac{\Gamma(\frac{\nu}{2}+\frac{1}{2}+s)\Gamma(\frac{\nu}{2}+1+s)}{\Gamma(1+\frac{\nu}{2})\Gamma(s+1)} \int_0^{i\infty} e^{\frac{\zeta^2}{2}} \zeta^{r-2-\nu-2s} d\zeta \\ &\quad + O(\zeta^{-2M}). \end{aligned}$$

By using the following formula for Γ-function,

$$\begin{aligned} &\int_0^{i\infty} e^{\frac{\zeta^2}{2}} \zeta^{r-2-\nu-2s} d\zeta \\ &= e^{\frac{\pi i}{2}(r-2-\nu-2s+1)} \int_0^\infty e^{-\frac{t^2}{2}} t^{r-2-\nu-2s} dt \\ &= e^{\frac{\pi i}{2}(r-1-\nu-2s)} \cdot 2^{(\frac{r}{2}-\frac{\nu}{2}-1-s-\frac{1}{2})} \int_0^\infty e^{-x} x^{(\frac{r}{2}-\frac{\nu}{2}-1-s-\frac{1}{2})} dx \\ &= 2^{(\frac{r}{2}-\frac{\nu}{2}-s-\frac{3}{2})} e^{\frac{\pi i}{2}(r-1-\nu-2s)} \Gamma(\frac{r}{2}-\frac{\nu}{2}-s-\frac{1}{2}), \end{aligned}$$

we have

$$
\begin{aligned}
u_r^{(1)} &= \frac{-e^{\frac{\pi i}{2}(r-1-\nu)}}{\pi i}\frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \\
&\quad \times \sum_{s=0}^{M-1}\frac{(-1)^s\Gamma(\frac{\nu}{2}+\frac{1}{2}+s)\Gamma(\frac{\nu}{2}+1+s)}{\Gamma(1+\frac{\nu}{2})\Gamma(s+1)}\cdot 2^{(\frac{r}{2}-\frac{\nu}{2}-s-\frac{3}{2})}\Gamma(\frac{r}{2}-\frac{\nu}{2}-s-\frac{1}{2}) \\
&\quad +O\left(\Gamma(\frac{r}{2}-\frac{\nu}{2}-M-\frac{1}{2})\right)
\end{aligned}
$$

for r enough large and $1 \le M < r$.

For $r = 100, \nu = -4/10$, the exact value of u_{100} is ,

$$
\begin{aligned}
& 8.5646396461302660900700060507391099218168876369 \times 10^{75} \\
+ \; & 6.222574939954422998270833037517777508311639165 \times 10^{75} i,
\end{aligned}
$$

and the approximation with $M = 34$ is

$$
\begin{aligned}
& 8.56463964613026609007000349163717984541887526359 \times 10^{75} \\
+ \; & 6.22257493995442299827083117822139180720714067915 \times 10^{75} i.
\end{aligned}
$$

Repeating the above procedures, we have the following results:

$$
u_r^{(2)} = \frac{2^{\frac{r}{2}-1}e^{\pi i(-\frac{\nu}{2}+r)}}{\pi i}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})\Gamma(\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2})\Gamma(\frac{r}{2}-\frac{\nu}{2}+\frac{1}{2})}F\left(\frac{r}{2};-\frac{\nu}{2};\frac{r}{2}-\frac{\nu}{2}+\frac{1}{2};-1\right),
$$

$$
\begin{aligned}
u_r^{(2)} &= \frac{e^{\pi i(r+\frac{\nu}{2})}}{\pi i}\frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \\
&\quad \times \sum_{s=0}^{M-1}\frac{(-1)^{-s}\Gamma(-\frac{\nu}{2}+s)\Gamma(-\frac{\nu}{2}+\frac{1}{2}+s)}{\Gamma(-\frac{\nu}{2}+\frac{1}{2})\Gamma(s+1)}\cdot 2^{\frac{r}{2}+\frac{\nu}{2}-s-1}\Gamma(\frac{r}{2}+\frac{\nu}{2}-s)+O\left(\Gamma(\frac{r}{2}+\frac{\nu}{2}-M)\right)
\end{aligned}
$$

for r enough large and $1 \le M < r$,

$$
u_r^{(3)} = \frac{e^{\frac{\pi i}{2}(\nu-1-r)}}{2\pi i}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})\Gamma(\frac{r}{2}+\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{r}{2}+\frac{\nu}{2}+1)}F\left(\frac{r}{2};\frac{r}{2}+\frac{1}{2};\frac{r}{2}+\frac{\nu}{2}+1;\frac{1}{2}\right),
$$

$$
\begin{aligned}
u_r^{(3)} &= \frac{-e^{-\frac{\pi i}{2}(r-1-\nu)}}{\pi i}\frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \\
&\quad \times \sum_{s=0}^{M-1}\frac{(-1)^s\Gamma(\frac{\nu}{2}+\frac{1}{2}+s)\Gamma(\frac{\nu}{2}+1+s)}{\Gamma(1+\frac{\nu}{2})\Gamma(s+1)}\cdot 2^{(\frac{r}{2}-\frac{\nu}{2}-s-\frac{3}{2})}\Gamma(\frac{r}{2}-\frac{\nu}{2}-s-\frac{1}{2}) \\
&\quad +O\left(\Gamma(\frac{r}{2}-\frac{\nu}{2}-M-\frac{1}{2})\right)
\end{aligned}
$$

for r enough large and $1 \leq M < r$,

$$u_r^{(4)} = \frac{2^{\frac{r}{2}-1}e^{-\frac{\pi i \nu}{2}}}{\pi i} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})\Gamma(\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2})\Gamma(\frac{r}{2}-\frac{\nu}{2}+\frac{1}{2})} F\left(\frac{r}{2}; -\frac{\nu}{2}; \frac{r}{2}-\frac{\nu}{2}+\frac{1}{2}; -1\right),$$

$$\begin{aligned} u_r^{(4)} &= \frac{e^{-\frac{\pi i \nu}{2}}}{\pi i} \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}+\frac{\nu}{2})} \\ &\times \sum_{s=0}^{M-1} \frac{(-1)^s \Gamma(-\frac{\nu}{2}+s)\Gamma(-\frac{\nu}{2}+\frac{1}{2}+s)}{\Gamma(-\frac{\nu}{2}+\frac{1}{2})\Gamma(s+1)} \cdot 2^{\frac{r}{2}+\frac{\nu}{2}-s-1}\Gamma(\frac{r}{2}+\frac{\nu}{2}-s) + O\left(\Gamma(\frac{r}{2}+\frac{\nu}{2}-M)\right) \end{aligned}$$

for r enough large and $1 \leq M < r$.

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