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DIFFERENTIAL EQUATIONS OF THETA CONSTANTS OF GENUS TWO

YOUSUKE OHYAMA*

OSAKA UNIVERSITY

(阪大理 大山 陽介)

1. Introduction

We will study a system of differential equations satisfied by theta constants. In the one dimensional case, there are some classical work of Jacobi or Halphen. In 1881 Halphen studied the equation

$$\begin{cases} u_1' + u_2' = 2u_1u_2, \\ u_2' + u_3' = 2u_2u_3, \\ u_3' + u_1' = 2u_3u_1. \end{cases} \tag{1.1}$$

Halphen showed that (1.1) is satisfied by the logarithmic derivatives of null values of elliptic theta functions ([1], [2], [4]). The author found a Halphen-type equation which is satisfied by the logarithmic derivatives of modular forms with level three ([3]).

In this note we will consider the several dimensional case. Differential rerations between theta constants of genus two are studied by Tomae, Krause, Bolza and Wiltheiss in nineteen's century. The aim of this note is to find a holonomic equations which is satisfied by theta constants of genus two. The most of part of this work is due to M. Sato ([4]).

2. Definition

Let $z = {}^t\left(z_0, z_1, \cdots, z_{g-1}\right)$ be a g-dimensional complex vector, and

$$\tau = \begin{pmatrix} \tau_{00} & \tau_{01} & \cdots & \tau_{0,g-1} \\ \tau_{10} & \tau_{11} & \cdots & \tau_{1,g-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{g-1,0} & \tau_{g-1,1} & \cdots & \tau_{g-1,g-1} \end{pmatrix}$$

be a $g \times g$-matrix, where $\tau_{ij} = \tau_{ji}$ and $\Im\tau$ is positive definite. The theta function is defined by

$$\theta(z|\tau) = \sum_{\nu\in\mathbb{Z}^g} e^{2\pi i\langle\nu,z\rangle} e^{\pi i\langle\nu,\tau\nu\rangle}.$$

We set $N = 2^g$. For any g-dimensional Abelian variety the number of points of order two is N^2. For each point of order two, there is a theta function with characteristic.

* *E-mail address*:ohyama@mat.sci.osaka-u.ac.jp

Definition 2.1. A g-characteristic is a $2 \times g$ matrix of intergers, written

$$\lambda = \begin{bmatrix} \lambda' \\ \lambda'' \end{bmatrix} = \begin{bmatrix} \lambda'_0 & \cdots & \lambda'_{g-1} \\ \lambda''_0 & \cdots & \lambda''_{g-1} \end{bmatrix}.$$

The numerical character of a g-characteristic is $|\lambda| = (-1)^{\lambda' \cdot \lambda''}$. A g-characteristic is called even, resp. odd if and only if the numerical character is 1, resp. -1.

The bilinear character of a pair of g-characters $\lambda = \begin{bmatrix} \lambda' \\ \lambda'' \end{bmatrix}$ and $\mu = \begin{bmatrix} \mu' \\ \mu'' \end{bmatrix}$ is

$$|\lambda, \mu| = (-1)^{\lambda' \cdot \mu'' + \lambda'' \cdot \mu'}.$$

λ, μ are syzygetic or azygetic according to $|\lambda, \mu| = 1$ or -1.

A reduced characteristic is a characteristic each of whose elements is zero or one. The reduced characteristic is obtained from any characteristic by replacing each entry by its residye modulo 2, which is called reduced representative.

There are 2^{2g} reduced characteristics. The number of even functions is $2^{g-1}(2^g+1)$, and the number of odd functions is $2^{g-1}(2^g-1)$.

Definition 2.2. Theta function with characteristic $\lambda = \begin{bmatrix} \lambda' \\ \lambda'' \end{bmatrix}$ is

$$\theta[\lambda](z|\tau) = \theta \begin{bmatrix} \lambda' \\ \lambda'' \end{bmatrix} (z|\tau) = \sum_{\nu \in \mathbb{Z}^g} e^{2\pi i \left\langle \nu + \frac{\lambda'}{2}, z + \frac{\lambda''}{2} \right\rangle} e^{\pi i \left\langle \nu + \frac{\lambda'}{2}, \tau \left(\nu + \frac{\lambda'}{2} \right) \right\rangle}.$$

We will study null value of theta functions

$$\theta[\lambda](\tau) = \theta \begin{bmatrix} \lambda' \\ \lambda'' \end{bmatrix} (\tau) := \theta \begin{bmatrix} \lambda' \\ \lambda'' \end{bmatrix} (0|\tau).$$

The set G of reduced characteristics can be considered as a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$. We consider a special subgroup of G.

Definition 2.3. A subgroup Γ of G is called syzygetic when its elemnts are mutually syzygetic. A maximal syzygetic subgroup is called a Göpel group.

It is evident by the definition thata Γ is a syzygetic subgroup if and only if the generators of Γ is mutually syzygetic. The number of generators of Γ is called degree of Γ. The degree of a Göpel group is g.

Proposition 2.1. *Given a syzygetic subgroup Γ, we take the coset decomposition of G*

$$\lambda_{(0)} + \Gamma, \lambda_{(1)} + \Gamma, \cdots \lambda_{(k)} + \Gamma.$$

If a coset has opposite character, it has as many characters of ones of the other character. If Γ has degree n, there are $2^{g-n-1}(2^{g-n}+1)$ coset whose elements are all even and $2^{g-n-1}(2^{g-n}-1)$ coset whose elements are all odd.

By Proposition 2.1, there exist one and only one coset whose elements are all even for any Göpel group Γ. We will call this coset as the Göpel system.

3. Differential Relations

We set

$$\partial_{ij} = \begin{cases} \dfrac{\partial}{\partial \tau_{ii}} & (i = j), \\ \dfrac{1}{2}\dfrac{\partial}{\partial \tau_{ij}} & (i \neq j). \end{cases}$$

We will fix a vector $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{g-1})$, and take a differential

$$\delta = \sum_{j,k} \alpha_j \alpha_k \partial_{jk}.$$

δ corresponds to an infinitesimal transformation on the Siegel upper half plane:

$$\begin{aligned} \tau \longrightarrow & \tau_0 + t\alpha \cdot {}^t\alpha \\ = & \tau_0 + t \begin{pmatrix} \alpha_0\alpha_0 & \alpha_0\alpha_1 & \cdots & \alpha_0\alpha_{g-1} \\ \alpha_1\alpha_0 & \alpha_1\alpha_1 & \cdots & \alpha_1\alpha_{g-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{g-1}\alpha_0 & \alpha_{g-1}\alpha_1 & \cdots & \alpha_{g-1}\alpha_{g-1} \end{pmatrix}. \end{aligned}$$

We notice that the rank of the matrix $\alpha \cdot {}^t\alpha$ is one. If we set

$$\partial = \sum_{j=0}^{g-1} \alpha_j \frac{\partial}{\partial z_j},$$

we have the heat equation

$$\partial^2 \theta(z|\tau) = 4\pi i \delta \theta(z|\tau). \tag{3.1}$$

We will study N special values of theta functions for $m \in (\mathbb{Z}/2\mathbb{Z})^g$:

$$\begin{aligned} \theta_m(\tau) := & \, \theta\left(\frac{m}{2}\middle|\tau\right) \\ = & \sum_{\nu \in \mathbb{Z}^g} (-)^{\langle \nu, m\rangle} e^{\pi i \langle \nu, \tau\nu\rangle}. \end{aligned}$$

$(\mathbb{Z}/2\mathbb{Z})^g$ is the simplest example of Göpel systems. We may take any Göpel system instead of $(\mathbb{Z}/2\mathbb{Z})^g$

For $\varepsilon = (\varepsilon_0, \cdots, \varepsilon_{g-1})$, $\quad \varepsilon_j = \pm 1$, we set

$$\varepsilon^m = \varepsilon_0^{m_0} \cdots \varepsilon_{g-1}^{m_{g-1}}.$$

If we take

$$A_\varepsilon = \sum_m \varepsilon^m \theta_m^2,$$

we get

$$A'_\varepsilon = \sum_m \varepsilon^m u_m \theta_m^2,$$

where $'$ is a differentiation with respect to δ. We may consider this equation as a definition of the new function A'_ε. In the following we will deduce nonlinear equations which are satisfied by theta constants

$$u_m = \delta\left(\log \theta_m^2\right) = \frac{2\theta'_m}{\theta_m}.$$

We will use generalized Hirota derivatives. Instead of defining Hirota derivatives, we only denote the notations which will be used in the followings.

$$
\begin{aligned}
D(f\otimes g) &= f'g - fg' \\
D^2(f\otimes g) &= f''g - 2f'g' + fg'' \\
D^2(f\otimes g\otimes h\otimes k) &= (f''ghk + fg''hk + fgh''k + fghk'') \\
&\quad - 2(f'g'hk + f'gh'k + f'ghk' + fg'h'k + fg'hk' + fgh'k').
\end{aligned}
$$

Theorem 3.1. *u_m, θ_m satisfy the following three systems of differential equations, which are equivalent to each other.*

Equation (I)

$$
\begin{cases}
\theta'_m = \dfrac{1}{2}u_m\theta_m \\
u'_m = \dfrac{1}{N}\dfrac{1}{\theta_m^2}\displaystyle\sum_\varepsilon \varepsilon^m \frac{(A'_\varepsilon)^2}{A_\varepsilon}
\end{cases}
$$

Equation (II)

$$
D^2\left(\left(\sum_m \varepsilon^m\theta_m\otimes\theta_m\right)^{\otimes 2}\right) = 0 \qquad \textit{for all } \varepsilon.
$$

Equation (III)

$$
\det\begin{pmatrix} A_\varepsilon & A'_\varepsilon \\ A'_\varepsilon & D^2\left(\sum_m \varepsilon^m\theta_m\otimes\theta_m\right)\end{pmatrix} = 0 \qquad \textit{for all } \varepsilon.
$$

Remark. Here we denote differential equations related to order two points. But in the case of special values of theta functions at general divided points, since there exist some algebraic relations, we may deduce differential equations.

In the rest of this section, we will show the proof of the Theorem 3.1. At first, we will show the three equations (I), (II) and (III) are equivalent.

It is easily verified that (II) and (III) are equivalent from the definition of Hirota derivatives. Since

$$
\begin{aligned}
D^2(f\otimes f) &= 2\left(f''f - f'^2\right) \\
D^2(f\otimes f\otimes g\otimes g) &= 2\left(f''fg^2 + f^2g''g\right) - 2\left(f'^2g^2 + f^2g'^2 + 4f'g'fg\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&D^2\left(\left(\sum_m \varepsilon^m\theta_m\otimes\theta_m\right)^{\otimes 2}\right) \\
&= 4\sum_{m,n}\left(\varepsilon^m\varepsilon^n\theta''_m\theta_m\theta_n^2\right) - 4\sum_{m,n}\varepsilon^m\varepsilon^n\left(\theta'^2_m\theta_n^2 + 2\theta'_m\theta'_n\theta_m\theta_n\right) \\
&= 4A_\varepsilon\sum_m\varepsilon^m\left(\theta''_m\theta_m - \theta'^2_m\right) - 8\sum_{m,n}\varepsilon^m\varepsilon^n\theta'_m\theta'_n\theta_m\theta_n \\
&= 2\left(A_\varepsilon\, D^2\left(\sum_m\varepsilon^m\theta_m\otimes\theta_m\right) - A'^2_\varepsilon\right),
\end{aligned}
$$

which is the determinant appeared in (III).

In the next step, we will deduce the equation (I) from (III).
By the formula

$$2\,(\log f)'' = 2\,\frac{f'' - f'^2}{f^2} = \frac{D^2\,(f \otimes f)}{f^2},$$

we have

$$D^2\left(\sum_m \varepsilon^m \theta_m \otimes \theta_m\right) = \sum_m \varepsilon^m {u_m}' \theta_m^2.$$

Therefore we can rewrite (III) as follows.

$$\sum_m \varepsilon^m {u_m}' \theta_m^2 = \frac{{A'_\varepsilon}^2}{A_\varepsilon}.$$

We will take a sum on ε:

$$\sum_\varepsilon \sum_m \varepsilon^{m-m_0} {u_m}' \theta_m^2 = \sum_\varepsilon \varepsilon^{m_0} \frac{{A'_\varepsilon}^2}{A_\varepsilon}.$$

Since

$$\sum_\varepsilon \varepsilon^m = \begin{cases} N & m = 0 \\ 0 & \text{otherwise}, \end{cases}$$

we obtain

$$N {u_m}' \theta_m^2 = \sum_\varepsilon \varepsilon^{m_0} \frac{{A'_\varepsilon}^2}{A_\varepsilon},$$

which is the second equation of (I). Since the first equation of (I) is evident from the definition of u_m, (I) is deduced from (III).

We can deduce (III) from (I) by the converse calculations above. Hence we can show the three equations are equivalent to each other.

Now, we will deduce the equation (III).
We will consider the theta function of degree two

$$\theta(z + z'|\tau)\theta(z - z'|\tau)$$

as a function of z. If we substitute

$$z \longmapsto z + n, \qquad z \longmapsto z + \tau n,$$

the automorphic factors are independent of z'. Therefore the dimension of the linear space

$$< \theta(z + z'_j|\tau)\theta(z - z'_j|\tau) >_j$$

is $2^g (= N)$. If we set

$$\Theta(z, z') := \theta(z + z'|\tau)\theta(z - z'|\tau),$$

the rank of the matrix

$$\begin{pmatrix} \Theta(z_0, z'_0) & \Theta(z_0, z'_1) & \dots \\ \Theta(z_1, z'_0) & \Theta(z_1, z'_1) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is at most N.

For $m \in (\mathbb{Z}/2\mathbb{Z})^g$ we set

$$\Theta_m(z, z') := \theta_m(z + z'|\tau)\theta_m(z - z'|\tau).$$

We will take the $N \times N$-matrix

$$(\Theta_{m+m'}(z, z'))_{m,m' \in (\mathbb{Z}/2\mathbb{Z})^g}.$$

Since

$$\begin{aligned}\Theta_{m+m'}(z, z') &= \theta_{m+m'}(z + z')\theta_{m+m'}(z - z') \\ &= \theta(z + z' + \frac{m + m'}{2})\theta(z - z' + \frac{m + m'}{2}) \\ &= \Theta(z + \frac{m}{2}, z' + \frac{m'}{2}),\end{aligned}$$

the rank of the matrix $(\Theta_{m+m'}(z, z'))$ is at most N, even if we consider the more larger size of matrixes taking z, z' as many values.

Since $(\Theta_{m+m'}(z, z'))$ is a matrix related to the group $(\mathbb{Z}/2\mathbb{Z})^g$, we can diagonalize that by the $N \times N$-matrix

$$P = (\varepsilon^m)_{\varepsilon,m} = \left((-1)^{m+m'}\right)_{m,m' \in (\mathbb{Z}/2\mathbb{Z})^g}.$$

If we set

$$A_\varepsilon(z, z') = \sum_m \varepsilon^m \theta_m(z + z')\theta_m(z - z'),$$

we have

$$P\,(\Theta_{m+m'}(z, z'))\,P^{-1} = \operatorname{diag}\,(A_\varepsilon(z, z'))_\varepsilon.$$

For example, we consider the case $g = 2$. We have

$$(\Theta_{m+m'}) = \begin{pmatrix} \Theta_{0,0} & \Theta_{1,0} & \Theta_{0,1} & \Theta_{1,1} \\ \Theta_{1,0} & \Theta_{0,0} & \Theta_{1,1} & \Theta_{0,1} \\ \Theta_{0,1} & \Theta_{1,1} & \Theta_{0,0} & \Theta_{1,0} \\ \Theta_{1,1} & \Theta_{0,1} & \Theta_{1,0} & \Theta_{0,0} \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{4}P = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Therefore we obtain

$$(\Theta_{m+m'})\,P = \begin{pmatrix} A_{(0,0)} & A_{(1,0)} & A_{(0,1)} & A_{(1,1)} \\ A_{(0,0)} & -A_{(1,0)} & A_{(0,1)} & -A_{(1,1)} \\ A_{(0,0)} & A_{(1,0)} & -A_{(0,1)} & -A_{(1,1)} \\ A_{(0,0)} & -A_{(1,0)} & -A_{(0,1)} & A_{(1,1)} \end{pmatrix} = P\,\operatorname{diag}\,(A_\varepsilon)_\varepsilon.$$

Since the rank of the matrix $(\Theta_{m+m'}(z, z'))$ is at most N even if we change z, z', t, the rank of the matrix

$$\begin{pmatrix} A_\varepsilon(z_0, z_0') & A_\varepsilon(z_0, z_1') & \cdots \\ A_\varepsilon(z_1, z_0') & A_\varepsilon(z_1, z_1') & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

whose matrix elements are the diagonalization element of $(\Theta_{m+m'})$, is 1 (it is not 0!). If we take a limit $z_j \longrightarrow z_i$, the rank of the matrix

$$\begin{pmatrix} A_\varepsilon(z,z') & \partial' A_\varepsilon(z,z') & \partial'^2 A_\varepsilon(z,z') & \cdots \\ \partial A_\varepsilon(z,z') & \partial\partial' A_\varepsilon(z,z') & \partial\partial'^2 A_\varepsilon(z,z') & \cdots \\ \partial^2 A_\varepsilon(z,z') & \partial^2\partial' A_\varepsilon(z,z') & \partial^2\partial'^2 A_\varepsilon(z,z') & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is also 1. Especially we have

$$\det \begin{pmatrix} A_\varepsilon(z,z') & \partial'^2 A_\varepsilon(z,z') \\ \partial^2 A_\varepsilon(z,z') & \partial^2\partial'^2 A_\varepsilon(z,z') \end{pmatrix} = 0. \tag{3.2}$$

Since each function $\theta_m(z)$ is even,

$$\begin{aligned} \partial^2\theta_m(z+z')\theta_m(z-z')|_{z=0,z'=0} &= 2\ddot{\theta}_m\theta_m, \\ \partial'^2\theta_m(z+z')\theta_m(z-z')|_{z=0,z'=0} &= 2\ddot{\theta}_m\theta_m, \\ \partial^2\partial'^2\theta_m(z+z')\theta_m(z-z')|_{z=0,z'=0} &= 2\ddot{\ddot{\theta}}_m\theta_m - 2\ddot{\theta}_m^2, \end{aligned}$$

where $\dot{}$ is a derivatives with respect to ∂. By (3.1),

$$\begin{aligned} \ddot{\theta}_m\theta_m &= 4\pi i\theta'_m\theta_m, \\ \ddot{\ddot{\theta}}_m\theta_m - (\ddot{\theta}_m)^2 &= (4\pi i)^2\left(\theta''_m\theta_m - \theta'^2_m\right). \end{aligned}$$

Therefore we get

$$\begin{aligned} \partial'^2 A_\varepsilon(z,z')|_{z=0,z'=0} &= 4\pi i A'_\varepsilon \\ \partial^2 A_\varepsilon(z,z')|_{z=0,z'=0} &= 4\pi i A'_\varepsilon \\ \partial^2\partial'^2 A_\varepsilon(z,z')|_{z=0,z'=0} &= (4\pi i)^2 D^2\left(\sum_m \varepsilon^m\theta_m\otimes\theta_m\right). \end{aligned}$$

Substituting the above into (3.2), we obtain

$$\det\begin{pmatrix} A_\varepsilon & 4\pi i A'_\varepsilon \\ 4\pi i A'_\varepsilon & (4\pi i)^2 D^2\left(\sum_m \varepsilon^m\theta_m\otimes\theta_m\right)\end{pmatrix} = 0,$$

which is just the same as (III).

4. The Case of Genus Two

In this section we will consider the equations in Theorem 3.1 when $g = 2$. When $g = 2$,

$$(\mathbb{Z}/2\mathbb{Z})^2 = \{00, 01, 10, 11\}$$

We will denote

$$u_0 = u_{00}, \quad u_1 = u_{01}, \quad u_2 = u_{10}, \quad u_3 = u_{11}.$$

Then Equation (II) is equivalent to the following equation.

$$\sum_{k=0}^{3}\left(\delta u_k + \frac{1}{2}u_k^2\right) = \sum_{j<k} u_j u_k. \tag{4.1}$$

(4.1) is represented by logarithmic derivatives of theta constants.

It is easily shown that there are fifteen Göpel group when $g = 2$. For each Göpel group, there is one and only one Göpel system. If we take a Göpel system instead of $(\mathbb{Z}/2\mathbb{Z})^g$, we obtain the same type of the equation as (4.1). Since there are ten even theta functions when $g = 2$, we have fifteen differential equations between ten functions.

Proposition 4.1. *The fifteen differential equations between logarithmic derivatives of even theta constants are linearly independent. Essentially there are ten nonlinear differential eqautions and five algebraic equations. Moreover there is one algebraic relation between these five algebraic equations.*

By Proposition 4.1, we get a differential ring which is generated by ten elements. From now on will write these generators u_m for $m = 0, 1, ..., 9$. The algebraic dimension of this ring is six. Thus we have a Halphen-type system of nonlinear differential equations whose solutions are given by logarithmic derivatives of theta constants.

We will study generic solutions of this Halphen-type equations. Let c be a complex number and B be a symmetric matrix whose size is two. We set

$$\begin{pmatrix} \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \end{pmatrix} = \begin{pmatrix} 1 + c\tau_{22} & -c\tau_{12} \\ -c\tau_{12} & 1 + c\tau_{11} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

$$\tilde{\delta} = \tilde{\alpha}_1^2 \partial_{11} + \tilde{\alpha}_1 \tilde{\alpha}_2 \partial_{12} + \tilde{\alpha}_2^2 \partial_{22}.$$

Proposition 4.2. *If $u_m(\tau)$ $(m = 0, 1, ..., 9)$ is a solution of the Halphen-type equations for $g = 2$, then functions*

$$\tilde{u}_m = \frac{\tilde{\delta} u_m \left((\tau + B)(1 + c(\tau + B))^{-1} \right)}{\det(1 + c(\tau + B))^2} - \frac{c(\alpha_1^2 + \alpha_2^2) + (\alpha_1 \tilde{\alpha}_1 + \alpha_2 \tilde{\alpha}_2)}{\det(1 + c(\tau + B))}$$

give a solution of the Halphen-type equations for any c and B. Here we should take $\tilde{\delta}$ instead of δ.

By Proposition 4.2, we obtain six parameter family of solutions of the Halphen-type equations.

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