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Graphical illustration of "double Stokes phenomena"

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Abstract

We try to illustrate the formation of double Stokes phenomena or new Stokes lines by the hyperasymptotic method.

Stimulated by a series of talks by Dr Boyd and Dr Howls in this symposium, I would like to try to explain an example of double Stokes phenomena ([2]) in terms of hyperasymptotics. This is a revisit of the author's observation ([5], [6]).

Berk, Nevins and Roberts ([3]) gave an example of 3rd order differential equation with positive large parameter x:

(1) $$\psi'''(q) + 3x^2\psi'(q) - ix^3 q\psi(q) = 0,$$

which has so to call new Stokes lines. They analyzed formation of new Stokes lines in general setting. Aoki, Kawai and Takei ([1]) introduced a notion of a new turning point and proposed an Ansatz on the Stokes geometry. The author gave an illustration of the example (1) by saddle point method (the steepest descent method) and the Laplace integral representation of solutions (Balian - Bloch representation [7]). We give again an illustration of the same example but in a different view point.

As we did in [5], we use a basis of solutions by Laplace integral:

(2) $$\Psi^{(i)}(q) = \int_{\Gamma_i(q)} \exp\left[-x\left(-\frac{iu^4}{4} - \frac{3iu^2}{2} + uq\right)\right] du,$$

where $\Gamma_i(q)$ is the steepest descent contour passing through a saddle ponint $u_i(q)$ specified below.

The saddle points are given by the algebraic equation with parameter q:

$$u^3 + 3u + iq = 0. \tag{3}$$

This equation has no triple roots but a pair of double roots, if and only if $q = \pm 2$. The points $q = -2$ and $q = 2$ are simple turning points of the original differential equation (1). We cut the q-plane along the lines $(-\infty, -2]$ and $[2, +\infty)$ on the real axis and enumerate the three roots $u_i(q)$'s such that

$$u_1(0) = \sqrt{3}i,\ u_2(0) = 0,\ u_3(0) = -\sqrt{3}i. \tag{4}$$

We have

$$u_1(-2) = u_2(-2) = i, \quad u_3(-2) = -2i \tag{5}$$

and

$$u_2(2) = u_3(2) = -i, \quad u_1(2) = 2i. \tag{6}$$

We define the Stokes lines from the turnig point ± 2 by the loci of q satisfying

$$\Im \int_{-2}^{q} u_1(q')dq' = \Im \int_{-2}^{q} u_2(q')dq' \tag{7}$$

and

$$\Im \int_{2}^{q} u_2(q')dq' = \Im \int_{2}^{q} u_3(q')dq'. \tag{8}$$

Two pairs of Stokes lines emanating from ± 2 have intersection points q_+ and q_- on the imaginary axis. They are approximately $q_\pm = \pm 0.3193i$ (Figure 1). When $q = q_\pm$, the steepest path from $u_3(q)$ hits $u_2(q)$ and another steepest path from $u_2(q)$ hits $u_1(q)$ (Figure 2). This is an example of a "double Stokes phenomenon" mentioned in [2] p666.

Our aim is illustrating by the resurgent formula in hyperasymptotic analysis that we have a Stokes phenomenon when q crosses over the so called new Stokes line $q_- + i(-\infty, 0)$

and no Stokes phenomena when q crosses the line $q_- + i(0, 2|q_-|)$ near q_-.

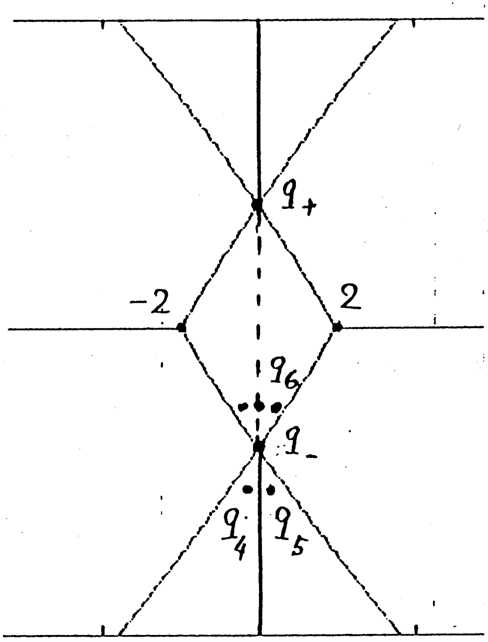

Figure 1: Stokes lines

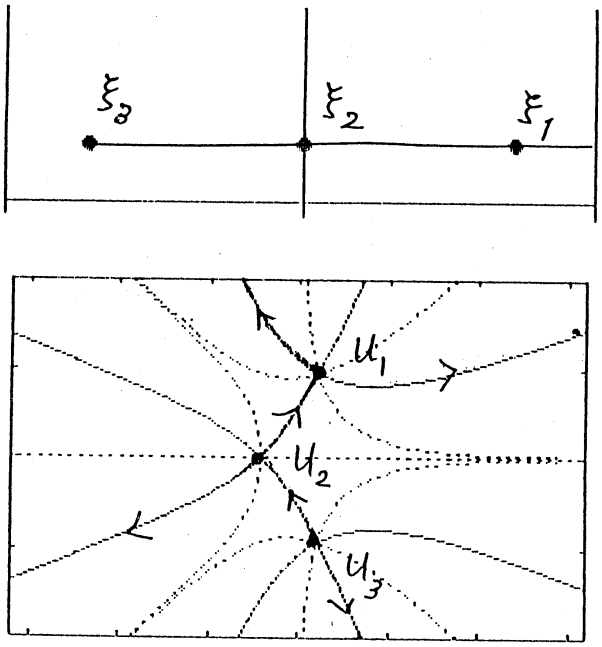

Figure 2: Steepest paths for $q = q_-$

We put

$$H(u, q) = -\frac{iu^4}{4} - \frac{3iu^2}{2} + uq,$$

and

$$\xi_i(q) = H(u_i(q), q).$$

We define $T^{(i)}(x, q)$ by

(9) $$\Psi^{(i)}(q) = x^{-1/2} \exp[-x\xi_i(q)] T^{(i)}(x, q)$$

namely,

(10) $$T^{(i)}(x, q) = x^{1/2} \int_{\Gamma_i(q)} du \exp\{-x[H(u, x) - \xi_i(q)]\}.$$

According to Berry and Howls ([2]), in virtue of an ingenious expression

(11) $$T^{(i)}(x, q) = \frac{1}{2\pi i} \int_0^\infty dv \frac{e^{-v}}{\sqrt{v}} \int_{\Gamma_i(q)} du \frac{[H(u, q) - \xi_i(q)]^{1/2}}{H(u, q) - \xi_i - v/x},$$

we have a resurgence relation for any positive integer N

$$T^{(i)}(x,q) = \sum_{r=0}^{N-1} \frac{T_r^{(i)}(q)}{x^r} +$$

(12)
$$\frac{1}{2\pi i}\sum_j \frac{(-1)^{\gamma_{ij}}}{(x\xi_{ij}(q))^N}\int_0^\infty dv \frac{v^{N-1}\exp(-v)}{1-\frac{v}{x\xi_{ij}(q)}} T^{(j)}\left(\frac{v}{\xi_{ij}(q)},q\right)$$

where

(13)
$$T_r^{(i)}(q) = \frac{\left(r-\frac{1}{2}\right)!}{2\pi i}\int^{(u_i(q)+)} du \frac{1}{[H(u,q)-\xi_i(q)]^{r+1/2}}.$$

In the formula (13), $(u_i(q)+)$ denotes a sufficiently small circle with positive direction around $u_i(q)$ in the complex u-plane. The summation $\sum_j$ is crucial in the resurgence relation. The sum is taken for the *adjacent* saddles $u_j(q)$ to the fixed saddle $u_i(q)$ ([2], [4]). γ_{ij} is an orientation factor 0 or 1 ([2] p662). $\xi_{ij}(q)$ is defined by

(14)
$$\xi_{ij}(q) = \xi_j(q) - \xi_i(q).$$

For simplicity, x has been restricted to a real large parameter. For the notion of adjacency, we have to consider in (10) all the steepest paths for the *complex* parameter x rotating its phase. A saddle $u_j(q)$ is called adjacent to $u_i(q)$, if there is a phase of x such that the corresponding steepest path from $u_i(q)$ hits $u_j(q)$.

In this example, the lines $\{q;\ \Im q = 0\}$ near q_- satisfies $\arg\xi_{31}(q) = 0$. We assume in general the saddle point $u_j(q)$ is adjacent to the saddle $u_i(q)$. By Cauchy's integral formula, when q crosses over the line $\{q;\ \arg\xi_{ij}(q) = 0\}$, we have a new term in the hyperasymptotic expansion (12), obtained by the residue calculus

$$\pm(-1)^{\gamma_{ij}}\exp\left(-x\xi_{ij}(q)\right)T^{(j)}(x,q).$$

The signature $\pm$ depends on the direction of q when q crosses the Stokes line.

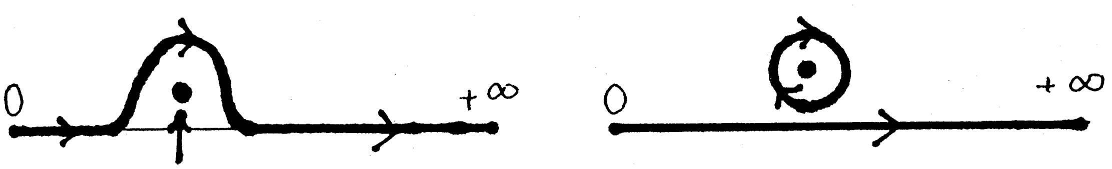

Figure 3: Deformation of integration contour.

Thus, in the hyperasymptotic theory, it is crucial to determine whether a saddle point is adjacent to another or not. In this report, the author did it numerically using computer graphics implemented by Visual Basic.

We follow the notations in [5], where figures of steepest paths through saddle points and location of ξ_i's are given. In order to determine the adjacency, we need steepest paths for complex parameter x with given phase.

We put $q_- = -3.193i$, $q_4 = q_- + \exp(17\pi i/12)$, $q_5 = q_- + \exp(19\pi i/12)$, $q_6 = q_- + \exp(\pi i/2)$, $q_6' = q_- + \exp(7\pi i/12)$, $q_6'' = q_- + \exp(5\pi i/12)$.

By drawing steepest paths for $\arg x = -\arg \xi_{ij}$, we determine whether $u_j(q_k)$ is adjacent to $u_i(q_k)$. In the following relations of saddle points, the arrow denotes adjacency. $a \to b$ means the saddle b is adjacent to a. We omit q_k of $u_j(q_k)$.

When $q = q_4$ or q_5,

$$u_1 \to u_2 \text{ and } u_1 \to u_3; \quad u_2 \to u_3 \text{ and } u_2 \to u_1; \quad u_3 \to u_1 \text{ and } u_3 \to u_2.$$

When $q = q_6, q_6', q_6''$,

$$u_1 \to u_2; \quad u_2 \to u_1 \text{ and } u_2 \to u_3; \quad u_3 \to u_2.$$

We observe from these data that there is a Stokes phenomenon for $\Psi^{(3)}(q)$ when q crosses the new Stokes line $q_- + (-\infty, 0)i$ from q_4 to q_5. On the other hand, there are no Stokes phenomena when q crosses the line $q_- + (0, 2|q_-|)i$ from q_6' to q_6'', since u_1 is not adjacent to u_3. These facts are explained in [5] by the Riemann sheet structure of the functions $\xi_i(q)$'s.

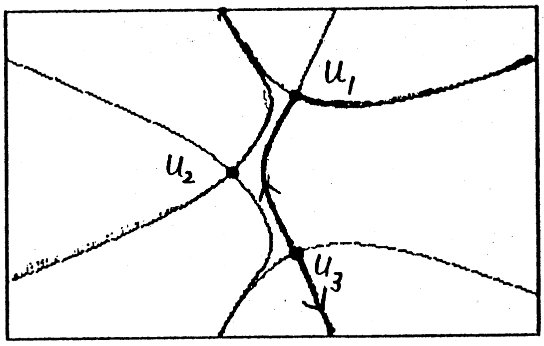

Figure 4: $q = q_4$, $\arg x = -\arg \xi_{31}(q_4)$

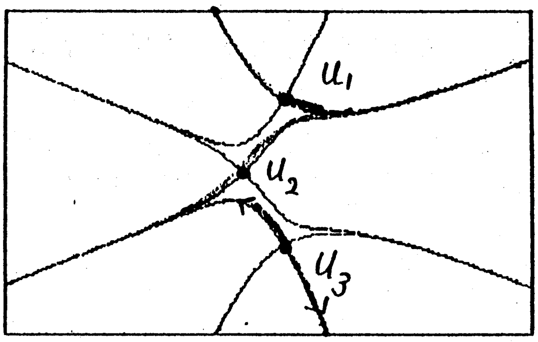

Figure 5: $q = q_6$, $\arg x = -\arg \xi_{31}(q_6)$

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