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ラプラス作用素の指数型固有関数について

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1 Two Problems

Let $\tilde{\mathbb{E}} = \mathbb{C}_z^{n+1}$ be the $n+1$ dimensional complex vector space with the dot product $z \cdot \zeta = z_0\zeta_0 + \cdots + z_n\zeta_n$. $L(z)$ denotes the Lie norm and $L^*(\zeta)$ the dual Lie norm. $\tilde{B}(a) = \{z \in \tilde{\mathbb{E}}; L(z) < a\}$ and $\tilde{B}[a] = \{\zeta \in \tilde{\mathbb{E}}; L(z) \leq a\}$ are Lie balls of radius $a > 0$. We put

$$\mathcal{O}_{\Delta+\lambda^2}(\tilde{B}(a)) = \{f \in \mathcal{O}(\tilde{B}(a)); (\Delta_z + \lambda^2)f(z) = 0\},$$

where λ is a complex number and $\Delta_z = \partial^2/\partial z_0^2 + \cdots + \partial^2/\partial z_n^2$. Put

$$\mathcal{O}_{\Delta+\lambda^2}(\tilde{B}[a]) = \bigcup_{a'>a} \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}(a')).$$

For $\Lambda \in \mathcal{O}'_{\Delta+\lambda^2}(\tilde{B}[a])$ the spherical Fourier-Borel transform

$$\mathcal{F}_\lambda^S \Lambda(\zeta) = \langle \Lambda_z, \exp(iz\cdot\zeta)\rangle$$

is defined for $\zeta \in \tilde{S}_\lambda$, where $\tilde{S}_\lambda = \{\zeta \in \tilde{\mathbb{E}}; \zeta^2 = \lambda^2\}$ is the complex sphere with complex radius λ. We know that the spherical Fourier-Borel transformation

$$\mathcal{F}_\lambda^S : \mathcal{O}'_{\Delta+\lambda^2}(\tilde{B}[a]) \to \mathrm{Exp}(\tilde{S}_\lambda; (a))$$

is a topological linear isomorphism (Morimoto-Fujita [10]), where

$$\mathrm{Exp}(\tilde{S}_\lambda; (a)) = \{\phi \in \mathcal{O}(\tilde{S}_\lambda); \forall\epsilon > 0, \exists C_\epsilon \geq 0, |\phi(\zeta)| \leq C_\epsilon \exp((a+\epsilon)L^*(\zeta)) \text{ for } \zeta \in \tilde{S}_\lambda\}$$

We put

$$\mathrm{Exp}(\tilde{S}_\lambda; [a]) = \bigcup_{a'<a} \mathrm{Exp}(\tilde{S}_\lambda; (a')).$$

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For $T \in \mathrm{Exp}'(\tilde{S}_\lambda; [a])$ the Fourier-Borel transform

$$\mathcal{F}_\lambda T(z) = \langle T_\zeta, \exp(-iz\cdot\zeta)\rangle$$

is defined for $z \in \tilde{B}(a)$ and satisfies $(\Delta_z + \lambda^2)(\mathcal{F}_\lambda T)(z) = 0$. We know that the Fourier-Borel transformation

$$\mathcal{F}_\lambda : \mathrm{Exp}'(\tilde{S}_\lambda; [a]) \to \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}(a))$$

is a topological linear isomorphism (Wada-Morimoto [13]).

Problem 1 Construct two topological linear isomorphisms $\uparrow$ and $\downarrow$ so that the following diagram becomes commutative:

$$\begin{array}{ccccc} \mathcal{F}_\lambda^S & : & \mathcal{O}'_{\Delta+\lambda^2}(\tilde{B}[a]) & \tilde{\to} & \mathrm{Exp}(\tilde{S}_\lambda; (a)) \\ & & \uparrow & & \downarrow \\ & & \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}(a)) & \tilde{\leftarrow} & \mathrm{Exp}'(\tilde{S}_\lambda; [a]) \ : \ \mathcal{F}_\lambda \end{array} \qquad (1)$$

Let $\lambda \in \mathbb{C}$ and $\tilde{S}_\lambda = \{z \in \tilde{\mathbb{E}}; z^2 = \lambda^2\}$. For $r > |\lambda|$ we put

$$\tilde{S}_\lambda[r] = \tilde{S}_\lambda \cap \tilde{B}[r], \quad \tilde{S}_\lambda(r) = \tilde{S}_\lambda \cap \tilde{B}(r).$$

Note that $\tilde{S}_\lambda \cap \tilde{B}[|\lambda|] = \lambda S_1$ and $\tilde{S}_\lambda \cap \tilde{B}(|\lambda|) = \emptyset$, where S_1 is the real unit sphere.

We denote by $\mathcal{O}(\tilde{S}_\lambda(r))$ the space of holomorphic functions on $\tilde{S}_\lambda(r)$ and by $\mathcal{O}(\tilde{S}_\lambda[r])$ the space of germs of holomorphic functions on $\tilde{S}_\lambda[r]$. For $T \in \mathcal{O}'(\tilde{S}_\lambda[r])$ we define the Fourier-Borel transform $\mathcal{F}_\lambda T$ by

$$\mathcal{F}_\lambda T(\zeta) = \langle T_z, \exp(-iz\cdot\zeta)\rangle.$$

$\mathcal{F}_\lambda T$ is an entire function on $\tilde{\mathbb{E}}$ and satisfies $(\Delta_\zeta + \lambda^2)(\mathcal{F}_\lambda T)(\zeta) = 0$. We know that the Fourier-Borel transformation

$$\mathcal{F}_\lambda : \mathcal{O}'(\tilde{S}_\lambda[r]) \to \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; (r))$$

is a topological linear isomorphism (Wada-Morimoto [13]), where

$$\begin{aligned}\mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; (r)) = \{F \in \mathcal{O}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}); \forall\epsilon > 0, \exists C_\epsilon \geq 0,\\ |F(\zeta)| \leq C_\epsilon \exp((r+\epsilon)L^*(\zeta)) \text{ for } \zeta \in \tilde{\mathbb{E}}\}.\end{aligned}$$

For $r > |\lambda|$ we put

$$\mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; [r]) = \bigcup_{r'<r} \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; (r')).$$

Now for $\Lambda \in \mathrm{Exp}'_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; [r])$ the spherical Fourier-Borel transform is defined by

$$\mathcal{F}_\lambda^S \Lambda(z) = \langle \Lambda_\zeta, \exp(iz\cdot\zeta)\rangle$$

for $z \in \tilde{S}_\lambda(r)$. We know that the spherical Fourier-Borel transformation

$$\mathcal{F}_\lambda^S : \mathrm{Exp}'_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; [r]) \to \mathcal{O}(\tilde{S}_\lambda(r))$$

is a topological linear isomorphism (Fujita-Morimoto [3]).

Problem 2 Construct two topological linear isomorphisms $\uparrow$ and $\downarrow$ so that the following diagram becomes commutative:

$$\begin{array}{ccccc} \mathcal{F}_\lambda : & \mathcal{O}'(\tilde{S}_\lambda[r]) & \tilde{\rightarrow} & \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}};(r)) & \\ & \uparrow & & \downarrow & \\ & \mathcal{O}(\tilde{S}_\lambda(r)) & \tilde{\leftarrow} & \mathrm{Exp}'_{\Delta+\lambda^2}(\tilde{\mathbb{E}};[r]) & : \mathcal{F}^S_\lambda \end{array} \tag{2}$$

2 Case of harmonic functions

2.1 Resumé

In the case of $\lambda = 0$, Problems 1 and 2 were solved in Morimoto-Fujita [8]. (See also Morimoto-Fujita [9].) In this subsection we shall summarize our solutions. (See Morimoto-Fujita [7] for related topics.)

1) For $f \in \mathcal{O}_\Delta(\tilde{B}[a])$ and $g \in \mathcal{O}_\Delta(\tilde{B}(a))$ we can define the "symbolic integral form"

$$\int_{S_a} f(x)g(x)dS_a(x) = \int_{S_1} f(a\omega)g(a\omega)dS_1(\omega),$$

where S_a is the real sphere of radius a and $dS_a(x)$ is the normalized invariant measure on S_a. This symbolic integral form is a duality bilinear form on $\mathcal{O}_\Delta(\tilde{B}[a]) \times \mathcal{O}_\Delta(\tilde{B}(a))$ and defines the topological linear isomorphism $\uparrow$ in the diagram (1). The inverse mapping is called the Poisson transformation

$$\mathcal{P} : \mathcal{O}'_\Delta(\tilde{B}[a]) \to \mathcal{O}_\Delta(\tilde{B}(a)).$$

(The detailed account will be found in the following subsection.)

2) For $\phi \in \mathcal{O}(\tilde{S}_0[r])$ and $\psi \in \mathcal{O}(\tilde{S}_0(r))$ we can define the "symbolic integral form"

$$\int_{M_r} \phi(\zeta)\psi(\overline{\zeta})dM_r(\zeta) = \int_{M_1} \phi(r\zeta')\psi(r\overline{\zeta'})dM_1(\zeta')$$

where $M_r = \partial\tilde{S}_0(r)$ and $dM_r(\zeta)$ the normalized invariant measure on M_r. This symbolic integral form is a duality bilinear form on $\mathcal{O}(\tilde{S}_0[r]) \times \mathcal{O}(\tilde{S}_0(r))$ and defines the topological linear isomorphism $\uparrow$ in the diagram (2). The inverse mapping is called the Cauchy transformation

$$\mathcal{C} : \mathcal{O}'(\tilde{S}_0[r]) \to \mathcal{O}(\tilde{S}_0(r)).$$

3) We define the measure $d\mu_a$ on the complex light cone

$$\tilde{S}_0 = \bigcup_{r>0} \partial\tilde{S}_0(r) = \bigcup_{r>0} M_r$$

by

$$\int_{\tilde{S}_0} \phi(z)d\mu_a(z) = \int_0^\infty \rho_a(r)dr \int_{M_r} \phi(z)dM_r(z) = \int_0^\infty \rho_a(r)dr \int_{M_1} \phi(rz')dM_1(z'),$$

where $\rho_a(r)$ is a weight function on $(0,\infty)$. For $\phi \in \mathrm{Exp}(\tilde{S}_0;[a])$ and $\phi \in \mathrm{Exp}(\tilde{S}_0;(a))$ we can define the "symbolic integral form"

$$\int_{\tilde{S}_0} \phi(z)\psi(\overline{z})d\mu_a(z).$$

This symbolic integral form is a duality bilinear form on

$$\mathrm{Exp}(\tilde{S}_0;[a]) \times \mathrm{Exp}(\tilde{S}_0;(a))$$

and defines the topological linear isomorphism $\downarrow$ in the diagram (1). The inverse mapping is called the F-Poisson transformation

$$\mathcal{M} \;:\; \mathrm{Exp}'(\tilde{S}_0;[a]) \to \mathrm{Exp}(\tilde{S}_0;(a)).$$

The weight function ρ_a can be described explicitly by the Ii-Wada function ρ_n (Ii [4], Wada [12]). This solves Problem 1 for $\lambda = 0$ and the diagram (1) becomes as follows:

$$\begin{array}{ccccc} \mathcal{F}_0^S \;:\; \mathcal{O}'_\Delta(\tilde{B}[a]) & \stackrel{\sim}{\to} & \mathrm{Exp}(\tilde{S}_0;(a)) & & \\ \mathcal{P}^{-1}\uparrow & & \downarrow_{\mathcal{M}^{-1}} & & \\ \mathcal{O}_\Delta(\tilde{B}(a)) & \stackrel{\sim}{\leftarrow} & \mathrm{Exp}'(\tilde{S}_0;[a]) & : & \mathcal{F}_0 \end{array} \tag{3}$$

4) We define the measure $d\mu^r$ on

$$\mathbb{E} = \mathbb{R}^{n+1} = \bigcup_{a>0} S_a$$

by

$$\int_{\mathbb{E}} f(x)d\mu^r(x) = \int_0^\infty \rho^r(a)da \int_{S_a} f(x)dS_a(x) = \int_0^\infty \rho^r(a)da \int_{S_1} f(a\omega)dS_1(\omega),$$

where $\rho^r(a)$ is a weight function on $(0,\infty)$ (Fujita [1], Morimoto-Fujita [8] and [9]). For $f \in \mathrm{Exp}_\Delta(\tilde{\mathbb{E}};[r])$ and $g \in \mathrm{Exp}_\Delta(\tilde{\mathbb{E}};(r))$ we define the "symbolic integral form"

$$\int_{\mathbb{E}} f(x)g(x)d\mu^r(x).$$

This symbolic integral form is a duality bilinear form on

$$\mathrm{Exp}_\Delta(\tilde{\mathbb{E}};[r]) \times \mathrm{Exp}_\Delta(\tilde{\mathbb{E}};(r))$$

and defines the topological linear isomorphism $\downarrow$ in the diagram (2). The inverse mapping is called the F-Cauchy transformation

$$\mathcal{E} \;:\; \mathrm{Exp}'(\tilde{\mathbb{E}};[r]) \to \mathrm{Exp}(\tilde{\mathbb{E}};(r)).$$

The weight function $\rho^r(a)$ can be explicitly represented by the Ii-Wada function. This solves Problem 2 for $\lambda = 0$ and the diagram (2) becomes as follows:

$$\begin{array}{ccccc} \mathcal{F}_0 \;:\; \mathcal{O}'(\tilde{S}_0[r]) & \stackrel{\sim}{\to} & \mathrm{Exp}_\Delta(\tilde{\mathbb{E}};(r)) & & \\ \mathcal{C}^{-1}\uparrow & & \downarrow_{\mathcal{E}^{-1}} & & \\ \mathcal{O}(\tilde{S}_0(r)) & \stackrel{\sim}{\leftarrow} & \mathrm{Exp}'_\Delta(\tilde{\mathbb{E}};[r]) & : & \mathcal{F}_0^S \end{array} \tag{4}$$

2.2 Symbolic integral form on S_a

Let $f \in \mathcal{O}_\Delta(\tilde{B}[a])$ and $g \in \mathcal{O}_\Delta(\tilde{B}(a))$. If $g \in \mathcal{O}_\Delta(\tilde{B}[a])$, then the integral

$$I(f,g) = \int_{S_a} f(x)g(x)dS_a(x) = \int_{S_1} f(a\omega)g(a\omega)dS_1(\omega)$$

is well-defined. Let

$$f(x) = \sum_{k=0}^{\infty} f_k(x), \quad g(x) = \sum_{k=0}^{\infty} g_k(x)$$

be the homogeneous harmonic expansion of f and g, where f_k and g_k are homogeneous harmonic polynomials of degree k. Then the orthogonality of spherical harmonics implies

$$\begin{aligned} I(f,g) &= \int_{S_1} \sum_{k=0}^{\infty} a^k f_k(\omega) \sum_{\ell=0}^{\infty} a^\ell g_\ell(\omega) dS_1(\omega) \\ &= \sum_{k=0}^{\infty} a^{2k} \int_{S_1} f_k(\omega) g_k(\omega) dS_1(\omega) = \sum_{k=0}^{\infty} a^{2k} (f_k, g_k)_{S_1}. \end{aligned}$$

Consider the general case; that is, $g \in \mathcal{O}_\Delta(\tilde{B}(a))$. Take $\epsilon > 0$ so small that $f((1+\epsilon)x)$ is defined for $x \in S_a$. Then we have

$$\begin{aligned} I_\epsilon(f,g) &= \int_{S_a} f((1+\epsilon)x) g(x/(1+\epsilon)) dS_a(x) \\ &= \int_{S_1} f((1+\epsilon)a\omega) g(a\omega/(1+\epsilon)) dS_1(\omega) \\ &= \int_{S_1} \sum_{k=0}^{\infty} (1+\epsilon)^k a^k f_k(\omega) \sum_{\ell=0}^{\infty} (1+\epsilon)^{-\ell} a^\ell g_\ell(\omega) dS_1(\omega) \\ &= \sum_{k=0}^{\infty} a^{2k} \int_{S_1} f_k(\omega) g_k(\omega) dS_1(\omega) \\ &= \sum_{k=0}^{\infty} a^{2k} (f_k, g_k)_{S_1}. \end{aligned}$$

This shows that $I_\epsilon(f,g)$ is defined for a sufficiently small $\epsilon > 0$ and independent of ϵ. Therefore, the bilinear form

$$(f,g)_{S_a} = \sum_{k=0}^{\infty} a^{2k} (f_k, g_k)_{S_1}$$

is well-defined for $f \in \mathcal{O}_\Delta(\tilde{B}[a])$ and $g \in \mathcal{O}_\Delta(\tilde{B}(a))$, and separately continuous. We call $(f,g)_{S_a}$ the symbolic integral form on S_a and sometimes write

$$(f,g)_{S_a} = \int_{S_a} f(x)g(x)dS_a(x).$$

For $g \in \mathcal{O}_\Delta(\tilde{B}(a))$ fixed, the mapping $T_g : f \mapsto (f, g)_{S_a}$ is a continuous linear functional on $\mathcal{O}_\Delta(\tilde{B}[a])$. We take the mapping $g \mapsto T_g$ as the mapping $\uparrow$ in the diagram (1).

We note that, if $a' > a$, $(f, g)_{S_a}$ is defined and separately continuous for $f \in \mathcal{O}_\Delta(\tilde{B}(a'))$ and $g \in \mathcal{O}_\Delta(\tilde{B}[a^2/a'])$.

Let $x = r\omega$, $r \geq 0$, $\omega \in S_1$. If $g \in \mathcal{O}_\Delta(\tilde{B}[a])$, we have

$$g(x) = g(r\omega) = \sum_{k=0}^{\infty} r^k g_k(\omega), \qquad (0 \leq r \leq a).$$

Because $g_k(\omega)$ is the k-spherical harmonic component of $a^{-k}g(a\omega)$, we have

$$g_k(\omega) = N(k) \int_{S_1} a^{-k} g(a\omega) P_k(\tau \cdot \omega) dS_1(\tau),$$

where P_k is the Legendre polynomial and $N(k)$ is the dimension of the space of k-spherical harmonics. Therefore, we have

$$\begin{aligned} g_k(x) &= r^k g_k(\omega) \quad (x = r\omega) \\ &= N(k) \int_{S_1} g(a\tau) a^{-2k} \tilde{P}_k(a\tau, r\omega) dS_1(\tau) \\ &= \int_{S_a} g(y) N(k) a^{-2k} \tilde{P}_k(y, x) dS_a(y), \end{aligned}$$

where $\tilde{P}_k(y, x) = (\sqrt{y^2})^k (\sqrt{x^2})^k P_k((y/\sqrt{y^2}) \cdot (x/\sqrt{x^2}))$. Finally we get

$$g(x) = \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} r^k g_k(\omega) = \int_{S_a} g(y) F_a(y, x) dS_a(y),$$

where

$$F_a(y, x) = \sum_{k=0}^{\infty} N(k) a^{-2k} \tilde{P}_k(y, x)$$

is the Poisson kernel. We know $F_a(y, x)$ is defined on

$$\{(y, x) \in \tilde{\mathbb{E}} \times \tilde{\mathbb{E}}; L(y)L(x) < a^2\}$$

and holomorphic in (y, x).

Suppose $g \in \mathcal{O}_\Delta(\tilde{B}(a))$. If $x \in \tilde{B}(a)$ is fixed, then the function $y \mapsto F_a(y, x)$ belongs to $\mathcal{O}_\Delta(\tilde{B}[a])$. By the symbolic integral form we have

$$g(x) = \int_{S_a} g(y) F_a(y, x) dS_a(y),$$

or, by means of the delta function,

$$\langle \delta_x, g \rangle = (g(y), F_a(y, x))_{y \in S_a}.$$

Suppose now $f \in \mathcal{O}_\Delta(\tilde{B}[a])$. Then there exists $a' > a$ such that $f \in \mathcal{O}_\Delta(\tilde{B}(a'))$. If $x \in \tilde{B}(a')$, then the function $y \mapsto F_a(x,y)$ belongs to $\mathcal{O}_\Delta(\tilde{B}[a^2/a'])$ and we have

$$f(x) = (f(y), F_a(y,x))_{y \in S_a} \text{ for } x \in \tilde{B}(a').$$

Let $T \in \mathcal{O}'_\Delta(\tilde{B}[a])$. If $y \in \tilde{B}(a)$, then the function $x \mapsto F_a(y,x)$ belongs to $\mathcal{O}_\Delta(\tilde{B}[a])$. Therefore, the Poisson transform $\tilde{T}_a$ of T is defined by

$$\tilde{T}_a(y) = \langle T_x, F_a(y,x)\rangle_x, \quad y \in \tilde{B}(a).$$

We have

$$\langle T, f\rangle = (f(y), \tilde{T}_a(y))_{y \in S_a}.$$

Thus the Poisson transformation $\mathcal{P} : T \to \tilde{T}_a$ is the inverse mapping of $g \mapsto T_g$ and establishes a topological linear isomorphism $\mathcal{O}'_\Delta(\tilde{B}[a]) \to \mathcal{O}_\Delta(\tilde{B}(a))$.

Similarly, it gives a topological linear isomorphism of $\mathcal{O}'_\Delta(\tilde{B}(a))$ onto $\mathcal{O}_\Delta(\tilde{B}[a])$.

3 General cases (first solutions)

We are investigating Problems 1 and 2 for general λ in Morimoto-Fujita [10], Fujita [2], Fujita-Morimoto [3] and Morimoto-Fujita [11]. In this section we will survey our results obtained in Morimoto-Fujita [10] and [11].

1) Let Σ_a be the Shilov boundary of the Lie ball $\tilde{B}[a]$. We know $\Sigma_a = \{e^{i\theta}x; \theta \in \mathbb{R}, x \in S_a\}$. For $f \in \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}[a])$ and $g \in \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}(a))$ we can define the "symbolic integral form" on Σ_a by

$$\int_{\Sigma_a} f(z)g(\overline{z})d\Sigma_a(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_{S_a} f(e^{i\theta}x)g(e^{-i\theta}x)dS_a(x),$$

where $d\Sigma_a(z)$ is the normalized invariant measure on Σ_a. If $\lambda = 0$, then the integral over Σ_a reduces to the integral over S_a. The topological linear isomorphism $\uparrow$ in the diagram (1) is defined by the symbolic integral form over the Shilov boundary. The inverse mapping is called the λ-Poisson transformation

$$\mathcal{P}^\lambda \ : \ \mathcal{O}'_{\Delta+\lambda^2}(\tilde{B}[a]) \to \mathcal{O}'_{\Delta+\lambda^2}(\tilde{B}(a)).$$

2) Put $\tilde{S}_{\lambda,r} = \partial\tilde{S}_\lambda(r)$ and denote by $d\tilde{S}_{\lambda,r}$ the normalized invariant measure on it. For $\phi \in \mathcal{O}(\tilde{S}_\lambda[r])$ and $\psi \in \mathcal{O}(\tilde{S}_\lambda(r))$ we can define the "symbolic integral form"

$$\int_{\tilde{S}_{\lambda,r}} \phi(z)\psi(\overline{z})d\tilde{S}_{\lambda,r}(z).$$

This symbolic integral form is a duality bilinear form on $\mathcal{O}(\tilde{S}_\lambda[r]) \times \mathcal{O}(\tilde{S}_\lambda(r))$ and defines the topological linear isomorphism $\uparrow$ in the diagram (2). The inverse mapping is called the λ-Cauchy transformations

$$\mathcal{C}^\lambda \ : \ \mathcal{O}'(\tilde{S}_\lambda[r]) \to \mathcal{O}(\tilde{S}_\lambda(r)).$$

3) We define the measure $d\mu_{\lambda,a}(z)$ on the complex sphere

$$\tilde{S}_\lambda = \bigcup_{r>|\lambda|} \partial\tilde{S}_\lambda[r] = \bigcup_{r>|\lambda|} \tilde{S}_{\lambda,r}$$

by

$$\int_{\tilde{S}_\lambda} \phi(z) d\mu_{\lambda,a}(z) = \int_{|\lambda|}^{\infty} \rho_{\lambda,a}(r) dr \int_{\tilde{S}_{\lambda,r}} \phi(z) d\tilde{S}_{\lambda,r}(z),$$

where $\rho_{\lambda,a}(r)$ is a weight function on $(|\lambda|, \infty)$.

For $\phi \in \mathrm{Exp}(\tilde{S}_\lambda; [a])$ and $\psi \in \mathrm{Exp}(\tilde{S}_\lambda; (a))$ we can define the "symbolic integral form"

$$\int_{\tilde{S}_\lambda} \phi(z)\psi(\overline{z}) d\mu_{\lambda,a}(z).$$

This symbolic integral is a duality bilinear form on $\mathrm{Exp}(\tilde{S}_\lambda; [a]) \times \mathrm{Exp}(\tilde{S}_\lambda; (a))$ and defines the topological linear isomorphism $\downarrow$ in the diagram (1) The inverse mapping is called the λ-F-Poisson transformation

$$\mathcal{M}^\lambda : \mathrm{Exp}'(\tilde{S}_\lambda; [a]) \to \mathrm{Exp}(\tilde{S}_\lambda; (a)).$$

We do not know the exact form of the weight function $\rho_{\lambda,a}(r)$. This solves Problem 1 for the general case (Morimoto-Fujita [10]).

$$\begin{array}{cccc} \mathcal{F}^S_\lambda : \mathcal{O}'_{\Delta+\lambda^2}(\tilde{B}[a]) & \stackrel{\sim}{\rightarrow} & \mathrm{Exp}(\tilde{S}_\lambda; (a)) & \\ (\mathcal{P}^\lambda)^{-1} \uparrow & & \downarrow (\mathcal{M}^\lambda)^{-1} & \\ \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}(a)) & \stackrel{\sim}{\leftarrow} & \mathrm{Exp}'(\tilde{S}_\lambda; [a]) & : \mathcal{F}_\lambda \end{array} \tag{5}$$

4) We define the measure $d\mu^{\lambda,r}$ on

$$\Sigma = \bigcup_{a>0} \Sigma_a$$

by

$$\int_\Sigma f(z) d\mu^{\lambda,r}(z) = \int_0^\infty \rho^{\lambda,r}(a) da \int_{\Sigma_a} f(z) d\Sigma_a(z), = \int_0^\infty \rho^{\lambda,r}(a) da \int_{\Sigma_1} f(az') d\Sigma_1(z'),$$

where $\rho^{\lambda,r}(a)$ is a weight function on $[0, \infty]$.

For $f \in \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; [r])$ and $g \in \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; (r))$ we can define the "symbolic integral form"

$$\int_\Sigma f(z) g(\overline{z}) d\mu^{\lambda,r}(z).$$

This symbolic integral form is a duality bilinear form on

$$\mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; [r]) \times \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; (r))$$

and defines the topological linear isomorphism $\uparrow$ in the diagram (2). The inverse mapping is called the λ-F-Cauchy transformation

$$\mathcal{E}^\lambda : \mathrm{Exp}'_{\Delta+\lambda^2}(\tilde{\mathbb{E}};[r]) \to \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}};(r)).$$

We do not know the exact form of the weight function $\rho^{\lambda,r}(a)$. This solves Problem 2 for the general case (Morimoto-Fujita [11]).

$$\begin{array}{rcccl} \mathcal{F}_\lambda : & \mathcal{O}'(\tilde{S}_\lambda[r]) & \stackrel{\sim}{\to} & \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}};(r)) & \\ & {\scriptstyle (\mathcal{C}^\lambda)^{-1}}\uparrow & & \downarrow{\scriptstyle (\mathcal{E}^\lambda)^{-1}} & \\ & \mathcal{O}(\tilde{S}_\lambda(r)) & \stackrel{\sim}{\leftarrow} & \mathrm{Exp}'_{\Delta+\lambda^2}(\tilde{\mathbb{E}};[r]) & : \mathcal{F}^S_\lambda \end{array} \tag{6}$$

4 General case (Second solutions)

In this section we survey the results obtained in Fujita-Morimoto [3]. (See also Fujita [2].) This method is interesting because we can prove the spherical Fourier-Borel transformations $\mathcal{F}^S_\lambda$ are topological linear isomorphisms using the conical case described in §2.

4.1 Second solution of Problem 1

We know that the restriction mappings

$$\beta : \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}(a)) \to \mathcal{O}(\tilde{S}_0(a)), \quad \beta : \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}[a]) \to \mathcal{O}(\tilde{S}_0[a])$$

are topological linear isomorphisms (Wada [12]). We know that the restriction mappings

$$\alpha : \mathrm{Exp}_\Delta(\tilde{\mathbb{E}};(a)) \to \mathrm{Exp}(\tilde{S}_\lambda;(a)), \quad \alpha : \mathrm{Exp}_\Delta(\tilde{\mathbb{E}};[a]) \to \mathrm{Exp}(\tilde{S}_\lambda;[a])$$

are also topological linear isomorphisms (Morimoto [5], Wada-Morimoto [13]).

Consider the following diagram which contains the diagram (1).

$$\begin{array}{rcccl} \mathcal{F}^S_\lambda : & \mathcal{O}'_{\Delta+\lambda^2}(\tilde{B}[a]) & \to & \mathrm{Exp}(\tilde{S}_\lambda;(a)) & \\ & {\scriptstyle \beta^*}\uparrow & & \downarrow{\scriptstyle \alpha^{-1}} & \\ \mathcal{F}_0 : & \mathcal{O}'(\tilde{S}_0[a]) & \to & \mathrm{Exp}_\Delta(\tilde{\mathbb{E}};(a)) & \\ & {\scriptstyle \mathcal{C}^{-1}}\uparrow & & \downarrow{\scriptstyle \mathcal{E}^{-1}} & \\ & \mathcal{O}(\tilde{S}_0(a)) & \leftarrow & \mathrm{Exp}'_\Delta(\tilde{\mathbb{E}};[a]) & : \mathcal{F}^S_0 \\ & {\scriptstyle \beta}\uparrow & & \downarrow{\scriptstyle (\alpha^*)^{-1}} & \\ & \mathcal{O}_{\Delta+\lambda^2}(\tilde{B}(a)) & \leftarrow & \mathrm{Exp}'(\tilde{S}_\lambda;[a]) & : \mathcal{F}_\lambda \end{array} \tag{7}$$

Note that the middle sub-diagram (the second and the third rows) is the solution diagram (4) of Problem 2 ($\lambda = 0$). Because the diagram is commutative and the Fourier-Borel transformation $\mathcal{F}_\lambda$ in the fourth row is a topological linear isomorphism (Wada-Morimoto [13]), we can conclude that the first row is a topological linear isomorphism. This proof is different from that given in Morimoto-Fujita [10].

Note The sub-diagram composed of the first and the fourth rows gives the second solution to Problem 1. Note that, even if $\lambda = 0$, this diagram is different from the solution diagram (5) of Problem 1. The topological linear isomorphism $\beta^* \circ \mathcal{C}^{-1} \circ \beta$ is given by the symbolic integral form on $M_a = \partial \tilde{S}_0(a)$, while the topological linear isomorphism $\mathcal{P}^{-1}$ is given by the symbolic integral form on the real sphere S_a.

4.2 Second solution of Problem 2

We know that the restriction mappings

$$\alpha : \mathcal{O}_\Delta(\tilde{B}(r)) \to \mathcal{O}(\tilde{S}_\lambda(r)), \quad \alpha : \mathcal{O}_\Delta(\tilde{B}[r]) \to \mathcal{O}(\tilde{S}_\lambda[r])$$

are topological linear isomorphisms (Morimoto [5]). We know that the restriction mappings

$$\beta : \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; (r)) \to \mathrm{Exp}(\tilde{S}_0; (r)), \quad \beta : \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; [r]) \to \mathrm{Exp}(\tilde{S}_0; [r])$$

are also topological linear isomorphisms (Wada [12]).

Consider the following diagram which contains the diagram (2).

$$\begin{array}{rcccl} \mathcal{F}_\lambda \ : & \mathcal{O}'(\tilde{S}_\lambda[r]) & \to & \mathrm{Exp}_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; (r)) & \\ & {\scriptstyle (\alpha^*)^{-1}} \uparrow & & \downarrow_{\beta} & \\ \mathcal{F}_0^S \ : & \mathcal{O}'_\Delta(\tilde{B}[r]) & \to & \mathrm{Exp}(\tilde{S}_0; (r)) & \\ & {\scriptstyle \mathcal{P}^{-1}} \uparrow & & \downarrow_{\mathcal{M}^{-1}} & \\ & \mathcal{O}_\Delta(\tilde{B}(r)) & \leftarrow & \mathrm{Exp}'(\tilde{S}_0; [r]) & : \ \mathcal{F}_0 \\ & {\scriptstyle \alpha^{-1}} \uparrow & & \downarrow_{\beta^*} & \\ & \mathcal{O}(\tilde{S}_\lambda(r)) & \leftarrow & \mathrm{Exp}'_{\Delta+\lambda^2}(\tilde{\mathbb{E}}; [r]) & : \ \mathcal{F}_\lambda^S \end{array} \qquad (8)$$

The middle sub-diagram (the second and the third rows) is the solution diagram (3) of Problem 1 ($\lambda = 0$). Because the diagram is commutative and the Fourier-Borel transformation $\mathcal{F}_\lambda$ in the first row is a topological linear isomorphism (Wada-Morimoto [13]), the spherical Fourier-Borel transformation $\mathcal{F}_\lambda^S$ in the forth row is a topological linear isomorphism.

Note The sub-diagram composed of the first and the fourth rows gives the second solution to Problem 2. Note that, even if $\lambda = 0$, this diagram is different from the solution diagram (4) of Problem 2. The topological linear isomorphism $(\alpha^*)^{-1} \circ \mathcal{P}^{-1} \circ \alpha^{-1}$ is given by the symbolic integral form on the real sphere S_r, while the topological linear isomorphism $\mathcal{C}^{-1}$ is given by the symbolic integral form on $M_r = \partial \tilde{S}_0(r)$.

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