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Holonomic deformation of linear differential equations of the A_g type

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0 Introduction.

In this paper, we consider linear differential equations of the form:

$$\frac{d^2y}{dx^2}+p_1(x,t)\frac{dy}{dx}+p_2(x,t)y=0, \tag{0.1}$$

defined on the Riemann sphere $\mathbf{P}^1$, with the coefficients:

$$\begin{aligned} p_1(x,t)&=-2x^{g+1}-\sum_{j=1}^{g}jt_jx^{j-1}-\sum_{k=1}^{g}\frac{1}{x-\lambda_k}, \\ p_2(x,t)&=-(2\alpha+1)x^g-2\sum_{j=1}^{g}H_jx^{g-j}+\sum_{k=1}^{g}\frac{\mu_k}{x-\lambda_k}. \end{aligned} \tag{0.2}$$

The Riemann scheme of this equation reads:

$$\left(\begin{array}{ccccccccc} x=\lambda_k & & & & \overbrace{\qquad\qquad\qquad x=\infty \qquad\qquad\qquad} & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha+\frac{1}{2} \\ 2 & \frac{2}{g+2} & 0 & t_g & t_{g-1} & \cdots & t_1 & -\alpha+\frac{1}{2} \end{array}\right). \tag{0.3}$$

Here the symbol in (0.3) means that, at the irregular point $x=\infty$, the equation (0.1)-(0.2) admits a system of formal solutions of the form:

$$\begin{aligned} \hat{y}_1&=x^{-\alpha-\frac{1}{2}}(1+\sum_{i\geq 1}h_i^1x^{-i}), \\ \hat{y}_2&=x^{\alpha-\frac{1}{2}}\exp[\frac{2}{g+2}x^{g+2}+t_gx^g+\cdots+t_1x](1+\sum_{i\geq 1}h_i^2x^{-i}). \end{aligned} \tag{0.4}$$

Note that the Poincaré rank at $x=\infty$ of the linear equation (0.1)-(0.2) is $g+2$. The principal parts of these formal solutions are given by the primitive function of the polynomials representing the versal deformation of the simple singularity of the A_g type, so we call the linear equation (0.1)-(0.2) as the equation of the A_g*-type.*

When considering the holonomic deformation of equations of the A_1-type, we obtain the Hamiltonian structure:

$$(\lambda_1, \mu_1, H_1, t_1),$$

which determines the Hamiltonian system, equivalent to the second Painlevé equation, see [6]. C. H. LIN and Y. SIBUYA studied on the holonomic deformation of linear equations; having the irregular singularity at $x = \infty$ with higher Poincaré rank and admitting a non-logarithmic singular point $x = \lambda$, see [4].

When $g = 2$, it is known ([3]) that the holonomic deformation is governed by the Hamiltonian system with respect to the canonical variables:

$$(\lambda_1, \lambda_2, \mu_1, \mu_2, H_1, H_2, t_1, t_2).$$

On the other hand, in the case $g \geq 3$, the quantities $H = (H_1, \cdots, H_g)$ and $t = (t_1, \cdots, t_g)$ do not compose the Hamiltonian structure. In fact, in the case of $g = 3$, we have to determine the variables $s = (s_1, s_2, s_3)$ such that

$$s_1 = t_1 - \frac{3}{4}t_3^2, \quad s_2 = t_2, \quad s_3 = t_3, \tag{0.5}$$

and then obtain the Hamiltonian structure: (see [5])

$$(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, H_1, H_2, H_3, s_1, s_2, s_3).$$

As to general natural number g, in order to determine the Hamiltonian structure, we have tried to find the transformation of qualities $t = (t_1, \cdots, t_g)$ such as (0.5), but we didn't succeed. So we determine the following new qualities instead.

$$\overline{H}_j = 2\sum_{i=0}^{j-1} a_{i+1}^{(j)}(t)\left(H_{j-i} + T_{j-i}^*\right), \qquad (j = 1, \cdots, g), \tag{0.6}$$

where

$$\begin{aligned} T_j^* &= \frac{1}{4}(j-1)T_{g+2-j} + \frac{1}{8}\sum_{l=1}^{g} T_l T_{g+2-j-l} \quad (1 \leq j \leq g), \\ T_j &= jt_j \quad (1 \leq j \leq g), \qquad T_j = 0 \quad (j < 1) \quad \text{or} \quad (j > g), \end{aligned} \tag{0.7}$$

$a_{i+1}(t)$ is given by

$$(0.8) \qquad a_1(t) = \frac{1}{2}, \qquad a_2(t) = 0, \qquad a_{i+1}(t) = \sum_{m=1}^{[\frac{i}{2}]} M^{(m,2m-i)} \quad (i \geq 2).$$

$M^{(m,q)}$ are defined as the coefficients of the expansion of function,

$$(0.9) \qquad \sum_{q=n(1-g)}^{0} M^{(n,q)} x^{-q} = \frac{1}{2}\Big(-\frac{1}{2}\sum_{l=1}^{g} T_l x^{g-l}\Big)^n$$

Such that

$$(\lambda_1, \cdots, \lambda_g, \mu_1, \cdots, \mu_g, \overline{H}_1, \cdots, \overline{H}_g, t_1, \cdots, t_g)$$

is the Hamiltonian Structure of the equation of the A_g type.

We suppose throughout this paper that $2\alpha + 1$ is not an integer. This equation has an irregular singularity at $x = \infty$ of the Poincaré rank $g+2$ and g regular singular points $x = \lambda_k$ $(k = 1, \cdots, g)$. We also make the following assumption:

(A) *none of* $x = \lambda_k$ $(k = 1, \cdots, g)$ *is logarithmic singularity.*

Since exponents at each regular singular point, $x = \lambda_k$, are 0 and 2, we deduce from the assumption (A) that H_i $(i = 1, \cdots, g)$ are rational functions of $t = (t_1, \cdots, t_g)$, $\lambda = (\lambda_1, \cdots, \lambda_g)$ and $\mu = (\mu_1, \cdots, \mu_g)$. The explicit forms of H_i will be given in Section 2, they play important roles in our studies.

Now we state the Main Theorem:

Main Theorem. *The holonomic deformation of the linear ordinary differential equation* (0.1)-(0.2) *is governed by the completely integrable Hamiltonian system:*

$$(\overline{H}) \qquad \frac{\partial \lambda_k}{\partial t_j} = \frac{\partial \overline{H}_j}{\partial \mu_k}, \qquad \frac{\partial \mu_k}{\partial t_j} = -\frac{\partial \overline{H}_j}{\partial \lambda_k} \qquad (k, j = 1, \cdots, g),$$

with the Hamiltonian functions $\overline{H}_j$ *defined by* (0.6).

Since the completely integrable Hamiltonian system $(\overline{H})$ determines the holonomic deformation of linear equations of the A_g type, we call $(\overline{H})$ the *A_g-system.*

1 Holonomic deformation of linear equation of the second order.

In this section, we recall the theory of the holonomic deformation of linear differential equation of the form:

$$\frac{d^2y}{dx^2} + p_1(x,t)\frac{dy}{dx} + p_2(x,t)y = 0, \tag{1.1}$$

We make in the following of this section a review of known results, which are available for us to study the holonomic deformation of linear equation of the A_g-type.

Proposition 1.1. *The equation* (1.1) *has a fundamental system of solutions whose monodromy and Stokes multiplier are independent of t, if and only if there exist rational functions of x, $A_j(x)$, $B_j(x)$, such that the following system of partial differential equations is completely integrable:*

$$\begin{aligned} &\frac{\partial^2 y}{\partial x^2} + p_1(x,t)\frac{\partial y}{\partial x} + p_2(x,t)y = 0, \\ &\frac{\partial y}{\partial t_j} = B_j(x)y + A_j(x)\frac{\partial y}{\partial x}, \end{aligned} \qquad (j = 1, \cdots, g). \tag{1.2}$$

Proposition 1.2. *The conditions of the complete integrability of* (1.2) *are given by:*

$$\frac{\partial A_j}{\partial t_i} + A_j\frac{\partial A_i}{\partial x} = \frac{\partial A_i}{\partial t_j} + A_i\frac{\partial A_j}{\partial x}, \qquad (i, j = 1, \cdots, g), \tag{1.3}$$

$$\frac{\partial^3}{\partial x^3}A_j - 4P\frac{\partial}{\partial x}A_j - 2A_j\frac{\partial}{\partial x}P + 2\frac{\partial}{\partial t_j}P = 0 \qquad (j = 1, \cdots, g). \tag{1.4}$$

where

$$P(x,t) = -p_2(x,t) + \frac{1}{4}p_1^2(x,t) + \frac{1}{2}\frac{\partial}{\partial x}p_1(x,t). \tag{1.5}$$

If we make the change of the unknown function:

$$y = \Phi(x)z, \qquad \Phi(x) = \exp(-\frac{1}{2}\int^x p_1(x,t)\,dx), \tag{1.6}$$

then (1.1) is transformed into an equation of the form:

$$\frac{d^2 z}{dx^2} = P(x,t)z, \tag{1.7}$$

where $P(x,t)$ is the function given by (1.5). It follows that:
Proposition 1.3. *The holonomic deformation of* (1.1) *is reduced to that of* (1.7) Finally, the holonomic deformation of (1.1) is reduced to the existence of rational functions $A_j(x)$ $(j = 1, \cdots, g)$, satisfying the system (1.3)-(1.4) of partial differential equations. We will call (1.2) *the extended system* of (1.1) and the functions $A_j(x)$ *the deformation functions.*

2 Deformation functions $A_j(x)$.

In the following of this paper, we consider the holonomic deformation of linear equations of the form:

$$\frac{d^2 y}{dx^2} + p_1(x,t)\frac{dy}{dx} + p_2(x,t)y = 0, \tag{2.1}$$

$$\begin{aligned} p_1(x,t) &= -2x^{g+1} - \sum_{(j)} j t_j x^{j-1} - \sum_{(k)} \frac{1}{x-\lambda_k}, \\ p_2(x,t) &= -(2\alpha+1)x^g - 2\sum_{(j)} H_j x^{g-j} + \sum_{(k)} \frac{\mu_k}{x-\lambda_k}, \end{aligned} \tag{2.2}$$

For the limiting pages, here we only can give the results, omit their proofs.

Firstly we determine the deformation functions.

Proposition 2.1. *For $j = 1, \cdots, g$, the deformation functions $A_j(x)$ are given as follows:*

$$A_j(x) = \frac{\overline{Q}_j(x)}{\Lambda(x)}, \tag{2.3}$$

where $\Lambda(x) = \prod_{j=1}^{g}(x-\lambda_j)$, and $\overline{Q}_j(x)$ is a polynomial of degree $j-1$.
The explicit form of $\overline{Q}_j(x)$ will be given by proposition 3.2. In order to prove this proposition, we need following lemmata.

Lemma 2.1. *As function of x, $A_j(x)$ is holomorphic on $\mathbf{C}\backslash\{\lambda_1, \cdots, \lambda_g\}$.*

Lemma 2.2. *For $k = 1, \cdots, g$; $x = \lambda_k$ is a pole of the first order of $A_j(x)$.*
Lemma 2.3. *$A_j(x)$ admits a zero of order $g + 1 - j$ at $x = \infty$.*
proposition 2.1 is an immediate consequence of lemmata 2.1, 2.2 and 2.3.

To give the explicit form of $\overline{Q}_j(x)$, we prove following lemma:
Lemma 2.4. *For $i \geq 3$, $a_i(t)$ defined by (0.8) satisfies*

$$a_1(t) = \frac{1}{2}, \quad a_2(t) = 0, \quad a_i(t) = -\frac{1}{2}\sum_{m=1}^{i-2} T_{g+m+2-i}\, a_m \quad (i \geq 3). \tag{2.4}$$

Proposition 2.2. *If differential equation (0.1)-(0.2) admits the holonomic deformation, then the deformation functions $A_j(x) = \frac{\overline{Q}_j(x)}{\Lambda(x)}$ $(j = 1, \cdots, g)$ are determined as*

$$\overline{Q}_j(x) = 2\sum_{i=0}^{j-1} a_{i+1}(t) Q_{j-i}(x), \qquad (j = 1, \cdots, g) \tag{2.5}$$

where

$$Q_j(x) = -\frac{1}{2}\sum_{n=0}^{j-1} \sigma_n x^{j-1-n}.$$

$$\sigma_n = (-1)^{n+1} e_n, \qquad (n = 0, 1, \cdots, g), \tag{2.6}$$

and e_n denotes the n-th elementary polynomial of the g variables, $\lambda_1, \cdots, \lambda_g$, in particular we define $e_0 = 1$.
Remark 2.1. *From the proof of proposition 2.2, we know when $\overline{Q}_j(x)$ is of the form of (2.5), then the degree of $\Delta_j(x)$ is at most $g - 1$.*
Proposition 2.3. $Q_j(x)$ and $\overline{Q}_j(x)$ have the following properties:

$$Q_j(\lambda_k) = \frac{1}{2}N^{j,k}, \qquad \overline{Q}_j(\lambda_k) = \frac{1}{2}\overline{N}^{j,k}. \tag{2.7}$$

3 Equation of the SL-type.

In this section, we will investigate the equation of the SL-type:

$$\begin{aligned} &\frac{d^2 z}{dx^2} = P(x,t)z, \\ &P(x,t) = -p_2(x,t) + \frac{1}{4}p_1^2(x,t) + \frac{1}{2}\frac{\partial}{\partial x}p_1(x,t). \end{aligned} \tag{3.1}$$

By using (0.2), we see that $P(x,t)$ can be written in the following form:

$$
\begin{aligned}
P(x,t) &= x^{2g+2} + \sum_{i=0}^{g} F_i x^{g+i} + 2\sum_{(j)} K_j x^{g-j} \\
&\quad - \sum_{(k)} \frac{\nu_k}{x-\lambda_k} + \frac{3}{4}\sum_{(k)} \frac{1}{(x-\lambda_k)^2},
\end{aligned}
\tag{3.2}
$$

where we denote by $\Sigma_{(k)}$ the sum for $k = 1, \cdots, g$. And we have:

$$
F_j = \frac{1}{4}\sum_{i=2+j}^{g} T_i T_{g+2+j-i} + T_j + 2\alpha\delta_{j0} \qquad (0 \le j \le g). \tag{3.3}
$$

$$
K_j = H_j + T_j^* + \frac{1}{2}\sum_{(k)} \lambda_k^j + \frac{1}{4}\sum_{(k)}\sum_{m=1}^{j-2} T_{m+g+2-j}\lambda_k^m \quad (1 \le j \le g). \tag{3.4}
$$

$$
\nu_k = \mu_k - \frac{1}{2}\Big(\sum_{(l)}^{(k)} \frac{1}{\lambda_k - \lambda_l} + \sum_{(i)} T_i \lambda_k^{i-1} + 2\lambda_k^{g+1}\Big) \qquad (1 \le k \le g). \tag{3.5}
$$

Here we denote by $\sum_{(l)}^{(k)}$ the sum for $l = 1, \cdots, g$ except for $l = k$. Let $e_j^{(k)}$ be the j-th elementary symmetric polynomial of $g-1$ variables, λ_l ($l = 1, \cdots, g$, $\neq k$), in particular, we put $e_0^{(k)} = 1$. Moreover, we define $\sigma_j^{(k)} = (-1)^{j+1} e_j^{(k)}$. For the simplicity of presentation, we put:

$$
N_k = \frac{1}{\Lambda'(\lambda_k)}, \quad N^{j,K} = -\sigma_{j-1}^{(k)}, \quad k, j = 1, \cdots, g, \tag{3.6}
$$

where $\Lambda(x) = \prod_{i=1}^{g}(x - \lambda_i)$, and $\Lambda'(x) = \frac{d}{dx}\Lambda(x)$.

We have following two propositions. Since the proofs are almost same, we omit them (see [5]).

Proposition 3.1. *In the linear equation* (0.1)-(0.2) H_j ($j = 1, \cdots, g$) *are given by:*

$$
H_j = \frac{1}{2}\sum_{(k)} [N_k N^{j,K} \mu_k^2 - U_{jk}\mu_k - N_k N^{j,k}(2\alpha + 1)\lambda_k^g], \tag{3.7}
$$

where

$$U_{jk} = N_k N^{j,k}(2\lambda_k^{g+1} + \sum_{(l)} T_l \lambda_k^{l-1}) - \sum_{(l)}^{(k)} \frac{N_k N^{j,k} + N_l N^{j,l}}{\lambda_l - \lambda_k}.$$

Proposition 3.2. *In the linear equation* (3.1)-(3.2), K_j $(j = 1, \cdots, g)$ *are written as follows:*

$$K_j = \frac{1}{2}\sum_{(k)}\Big(N_k N^{j,k}\nu_k^2 - \sum_{(l)}^{(k)} \frac{N_l N^{j,l}}{\lambda_k - \lambda_l}\nu_k - N_k N^{j,k} V_k\Big), \tag{3.8}$$

where $V_k = \lambda_k^{2g+2} + \lambda_k^g \sum_{i=0}^{g} F_i \lambda_k^i + \frac{3}{4}\sum_{(l)}^{(k)} \frac{1}{(\lambda_k - \lambda_l)^2}$.
For the $M^{(m,q)}$ given by (0.9), we have following lemma.
Lemma 3.1. *For arbitrary natural numbers* m *and* n*; ,nonnegative integer* q *satisfying* $1 - g \leq -q \leq 0$*, we have:*

$$M^{(m+n,q)} = \sum_{r=q}^{0} M^{(m,r)}\, M^{(n,q-r)}. \tag{3.9}$$

4 the canonical transformation.

Using H_j and K_j given by (2.7) and (2.8) respectively, we define $\overline{H}_j$ and $\overline{K}_j$ $j = 1, \cdots, g$ as follows:

$$\overline{H}_j = 2\sum_{i=0}^{j-1} a_{i+1}(t)\Big(H_{j-i} + T_{j-i}^*\Big) \quad (j = 1, \cdots, g), \tag{4.1}$$

$$\overline{K}_j = 2\sum_{i=0}^{j-1} a_{i+1}(t) K_{j-i} \quad (j = 1, \cdots, g), \tag{4.2}$$

Note that (4.1) is nothing but (0.6). Combining (2.7) with (4.1) and (2.8) with (4.2), we obtain

$$\overline{H}_j = \frac{1}{2}\sum_{(k)}[N_k \overline{N}^{j,k}\mu_k^2 - \overline{U}_{jk}\mu_k - N_k \overline{N}^{j,k}(2\alpha + 1)\lambda_k^g + \overline{T}_j^*], \tag{4.3}$$

$$\overline{K}_j = \frac{1}{2}\sum_{(k)}(N_k\overline{N}^{j,k}\nu_k^2 - \sum_{(l)}^{(k)}\frac{N_l\overline{N}^{j,l}}{\lambda_k - \lambda_l}\nu_k - N_k\overline{N}^{j,k}V_k), \tag{4.4}$$

where

$$\overline{N}^{j,k} = 2\sum_{i=0}^{j-1} a_{i+1}(t)N^{j-i,k}, \tag{4.5}$$

$$\overline{U}_{j,k} = 2\sum_{i=0}^{j-1} a_{i+1}(t)U_{j-i,k}, \qquad \overline{T}_j^* = 2\sum_{i=0}^{j-1} a_{i+1}(t)T_{j-i}^*.$$

We have

Lemma 4.1. $\overline{K}_j$ and $\overline{H}_j$ have the following relation:

$$\overline{K}_j = \overline{H}_j + \frac{1}{2}\sum_{(k)}\lambda_k^j. \tag{4.6}$$

Proposition 4.1. *The transformation defined by (2.5) and (4.6)*

$$(\lambda, \mu, \overline{H}, t) \to (\lambda, \nu, \overline{K}, t)$$

is canonical, where $\lambda = (\lambda_1, \cdots, \lambda_g)$, $\mu = (\mu_1, \cdots, \mu_g)$, $\overline{H} = (\overline{H}_1, \cdots, \overline{H}_g)$, $\nu = (\nu_1, \cdots, \nu_g)$, $\overline{K} = (\overline{K}_1, \cdots, \overline{K}_g)$ *and* $t = (t_1, \cdots, t_g)$.

5 the A_g-system.

In this section, we will prove Main Theorem. By means of propositions 1.2, 1.3 and 2.3, it suffices to establish the following theorem:

Theorem 5.1. *The conditions* (1.3), (1.4) *of the complete integrability are equivalent to the following completely integrable Hamiltonian system:*

$$(\overline{K}) \qquad \frac{\partial \overline{K}_j}{\partial \nu_k} = \frac{\partial \lambda_k}{\partial t_j}, \quad \frac{\partial \overline{K}_j}{\partial \lambda_k} = -\frac{\partial \nu_k}{\partial t_j}, \qquad (j, k = 1, \cdots, g).$$

Lemma 5.1. *The equation* (1.4) *induces the system* $(\overline{K})$.

Lemma 5.2. *The equations* (1.4) *is derived from the system* $(\overline{K})$.

Lemma 5.3. *The equation* (1.3) *is derived from the system* $(\overline{K})$.

Lemma 5.4. *system* $(\overline{K})$ *is complete integrable.*

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