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Spacing Distributions for Local Enstrophy Dissipation in Two-Dimensional Turbulence

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In the simulation of two-dimensional turbulence, the probability distribution of spacings between adjacent two points of iso local enstrophy dissipation (LED) is shown to follow an exponential type distribution. This implies that the number distribution of the points of the iso LED is given by Poisson distribution. With the results, some discussions are given, suggesting that the resulting Poisson distribution would lend some supports to the basic idea of the p-model that is constructed under the assumption that the probability distribution of the dissipation rate is given by the bimodal distribution.

Suppose a large number of points that are distributed randomly on a straight line. Some of adjacent points may keep their spacings separate. Some may hold their spacing closely. How can we know such the difference in the distributions ? The most simple idea would be to measure the spacings between two adjacent points. The spacings that follow an exponential distribution appear to represent an intermittent structure in turbulent flows. Can we apply the idea of the spacing distribution to the study of turbulent structure? Maybe, it can be of an interest to measure, for example, the spacings between adjacent two points marked with circles as shown in Fig.2.

If the probability distribution of the spacings is known, the number distribution of the k points to be found in a closed range, say, $B=(a,b)$, is calculated. For example, when the spacing distribution is exponential type distribution as[1]

$$P(s) = \exp(-\lambda s) \tag{1}$$

the number distribution of the k points to be found in the range $B=(a,b)$ is given by Poisson distribution as

$$P(k;B) = \frac{(\lambda B)^k}{k!} e^{-\lambda B} \tag{2}$$

When $k=0$, then eq.(1) is recovered from eq.(2). Corresponding to eq.(2), we can consider more general probability distribution functions, say, $E(k; B)$. When $k=0$, the function $E(k; B)$ gives the probability distribution function of spacing B. In the present letter, we have an interest in particular of studying the spacing distribution of the local enstrophy dissipation (LED) in two-dimensional turbulence

$$\sigma(x,y,t) = \sqrt{(\nabla \times \omega)^2} \equiv \sqrt{\left(\frac{\partial \omega}{\partial x}\right)^2 + \left(\frac{\partial \omega}{\partial y}\right)^2} \tag{3}$$

because the turbulence models such as p-model and β-model proposed by Meneveau and Sreenivassan[2,3], and by Frisch *et al*[4], respectively, for the study of the intermittent structures in turbulence are constructed on the base of the assumed probability distribution of the dissipation rate. It is expected that the study of the spacing distribution of the LED may lend some supports to the base of such the models.

In the study of the spacing distributions, we use the numerical data obtained from the equation for two-dimensional turbulent vorticities, $\omega(x,y,t)$:

$$\frac{\partial \omega\,(x,y,t)}{\partial t} + v \cdot \nabla \omega\,(x,y,t) = \nu\ \Delta \omega\,(x,y,t) \tag{4}$$

Then, the LED, $\sigma(x,y,t)$,is calculated from $\omega\,(x,y,t)$ using eq.(3).

Equation (4) is solved numerically with the pseudo-spectrum method with the spectral resolutions 256×256 [5]. The initial spectrum is

$$\omega\,(k,t=0) = \omega_0 \frac{k^2}{k_0^3} \exp\left[- \left(\frac{k}{k_0}\right)^2 \right] \tag{5}$$

and viscous coefficient $\nu = 1.5 \times 10^{-4}$, $\omega_0 = 0.1$ and $k_0 = 25$. Time marching is made with the second-order Runge-Kutta method. Time step is $\Delta t = 0.001$ to characteristic time.

First we check the simulation results, calculating the energy spectrum that is the average of $\frac{1}{2}v^2(k, t)$ over the wavenumbers k. The resulting energy spectrum shows -3 slope in an inertial range at high wavenumbers, and an inverse cascade toward low wavenumbers (Figures are not shown.), both of which are known as characteristic properties of two-dimensional turbulence. Further support to the simulation will be given by Brachet *et al.* who have shown that the LED in two-dimensional turbulence are spacially distributed, concentrating in strip-like regions (Fig. 1).

In Fig. 4, we show the spacing distributions of the LED. They are obtained from 256 × 2 one-dimensional cuts of the iso-LED lines that are horizontal slices through Fig.1: that is, to calculate the spacing distributions, we use the x-series and y-series of the cuts obtained by fixing the y-axis and the x-axis, respectively, at 1 to 256. (In Fig.2, we show an example of spacings to be measured.) We see that the resulting spacing distributions are the exponential type distributions with a peak in the leftist range of small spacings s. The peak part of the distribution becomes more peaked with increasing c, appearing like δ-function, and the slope of the exponential part decreases as c increases. Note that the distribution tends to take a definite form without depending on initial distributions because of the restrictions of the spectral resolutions at low wavenumbers.

Where are then the two parts of the distributions come from ? The origins will be clarified with Fig.3 which shows the iso-LED lines for the LED intensity $c = 10$ and 20. They appear to be like the islands in the sea with the boundaries of iso-LED lines. This suggests that the exponential part of the distribution is consisted mostly of the spacings between two adjacent islands. (More strictly speaking, it is made of the spacings between two adjacent islands that are distributed along a straight line, say, x-axis.) The peak parts, meanwhile, are thought to result mainly from the spacings between two adjacent iso-LED lines that make the boundary of the islands.

The dependence of the mean spacing between adjacent two islands, $1/\lambda$, on the intensity c is obtained from Fig.5 as

$$\frac{1}{\lambda} \approx c^{1.6} \tag{6}$$

Then, we see that the mean spacing increases with c according to a power law. (We assume that the exponential part of the distribution is expressed with eq.(1) in which $1/\lambda$ is equal to the mean spacing.) Further, we see that the mean spacing between the islands is scaled with $x=\lambda s$, that can be understood from the fact that eq. (1) is normalized with λs. Therefore, with the scaling law and power law, it can be concluded that the turbulence has a self-similar structure about the mean spacing between the two adjacent islands for the iso LED lines when the intensity c is larger than 8 (see Fig.5).

Finally, we show an analogy between the p model and our result. As mentioned before, the p model is an intermittency model[3,4] constructed under the assumption that the probability distribution of the (total) dissipation rate, E_r, in a certain piece Ω is given by the bimodal distribution function as

$$E_r / E_L = p_1^m \, p_2^{(n-m)} \qquad (\text{where } m = 0, 1, \ldots, n) \tag{7}$$

where E_r is given using the (local) dissipation rate $\varepsilon(x)$ as

$$E_r = \int_{x \in \Omega} \varepsilon(x) \mathrm{d}^3 x . \tag{8}$$

and the distribution is given in terms of two δ-functions as

$$p(M) = 0.5\{\delta(M - 0.4) + \delta(M - 0.6)\}. \tag{9}$$

Then, corresponding to eq.(7), the probability distribution of the LED in a certain one-dimensional piece B will be given in terms of Poisson distribution. Remember that the spacings between the adjacent two islands for the iso LED lines follows the exponential type distribution as in eq.(1). In this case, as mentioned before, the number distribution of the k islands is given by Poisson distribution as in eq.(2). Since the turbulent dissipation is considered to occur mostly in the islands where the intensity of the LED is strong, then, in the piece B, we can estimate the probability distribution of the LED that is in a certain definite intensity c, though roughly, at the product between the intensity c and the island numbers as

$$E_c(B) \approx \overline{c} \frac{(\overline{\lambda} B)^k}{k!} e^{-\overline{\lambda} B} \tag{10}$$

where the bars over c and λ denote the average over the LED inside of the islands.

Further, if we recall that the probability distribution of islands of the LED in the different intensity c satisfies a scaling law, we can obtain the probability distribution of the total LED in the range B, that is taken into account all of the intensities c, barely replacing $\overline{c}$ in eq.(10) with a suitable function, say, $E_l(\overline{c})$:

$$E_T(B) \approx E_l(\bar{c})\frac{(\bar{\lambda} B)^k}{k!}e^{-\bar{\lambda} B}. \tag{11}$$

Note that in the limit of infinity trials, the bimodal distribution includes Poisson distribution as [1])

$$\frac{N!}{k!(N-k)!}\left(\frac{\lambda B}{N}\right)^k\left(1-\frac{\lambda B}{N}\right)^{N-k} \rightarrow \frac{(\lambda B)^k}{k!}e^{-\lambda B} \tag{12}$$

Thus, we can see that eq.(7) is equivalent to eq.(11).

The bimodal distribution can be a natural result from the nonlinear effect of the Navier-Stokes equation that makes turbulent motions most randomly. It seems that the p-model takes into account the nonlinear effects of the Navier-Stokes equation well. This can be a reason why the p-model is a good model for the study of intermittent structures in turbulence. Maybe, it can be of an interest to study further on this point,. Notice that further analysis is now under way and will be reported elsewhere together with more detailed simulation results.

References

1) Obata Nobuaki , Suri Kagaku **376**, (1994)13.

2) K.R. Sreenivasan and C.Meneveau, J. Fluid Mech. **11**, (1986)357.

3) C.Meneveau and K.R. Sreenivasan, J. Fluid Mech. **224**, (1991)429.

4) U.Frisch, P.L. Sulem, and M.Nelkin, J. Fluid Mech. 87, (1978)719.

5) M.E.Brachet, M.Meneguzzi, H.Politano, and P.L.Sulem, J.Fluid Mech. **194**, (1988)333.

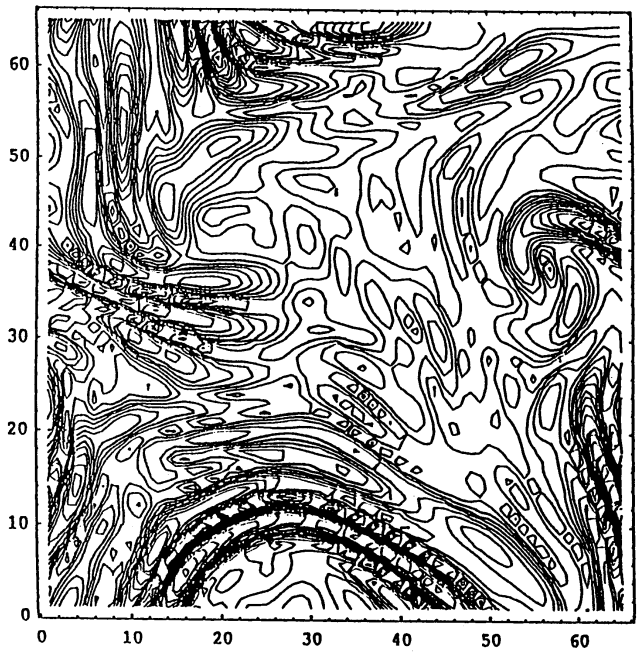

Fig.1 Iso local-enstrophy-dissipation (LED) lines, $\sigma(x, y, t) = c$, where

$$\sigma(x, y, t) = \sqrt{(\nabla \times \omega)^2} \equiv \sqrt{\left(\frac{\partial \omega}{\partial x}\right)^2 + \left(\frac{\partial \omega}{\partial y}\right)^2}.$$

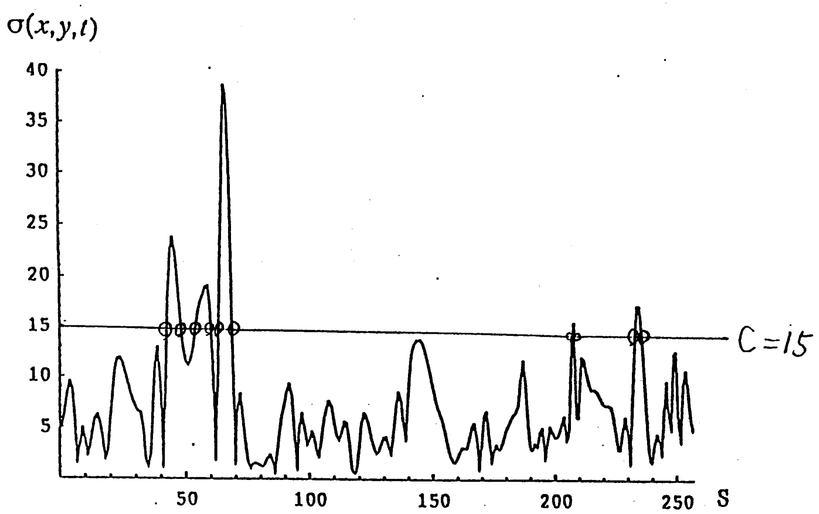

Fig.2 A horizontal slice through Fig.1. The spacings between the adjacent two points marked with circles are measured to calculate spacing distributions. The above example in the figure shows the points of the LED for the intensity c=15.

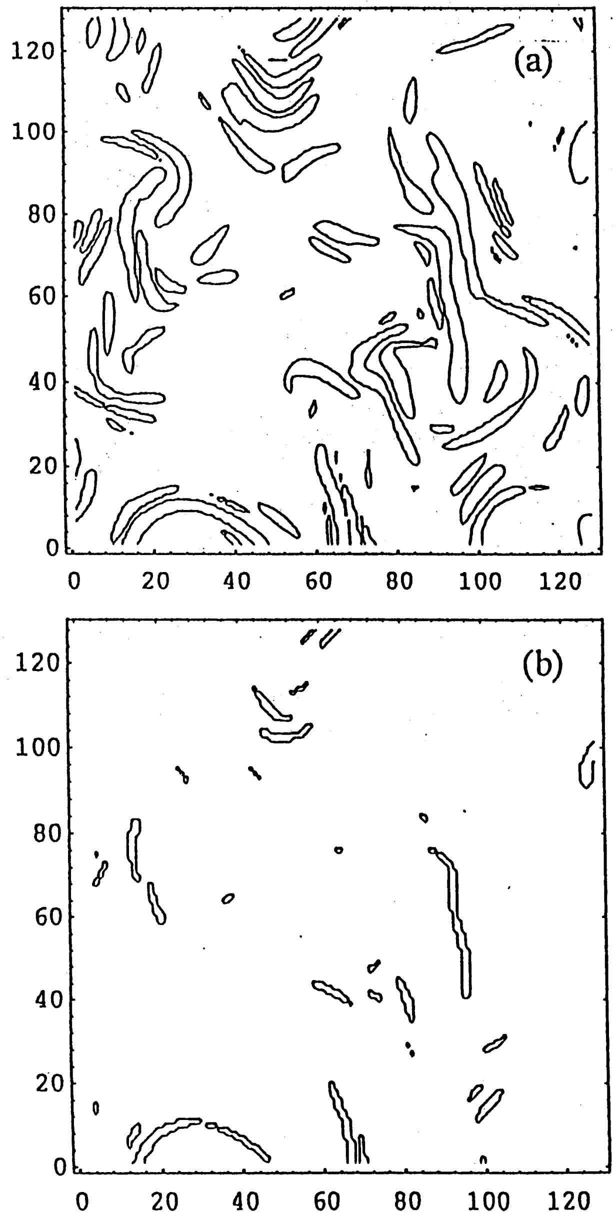

Fig.3 Iso local-enstrophy-dissipation (LED) lines for the intensities (a) c=10, (b) c=18.

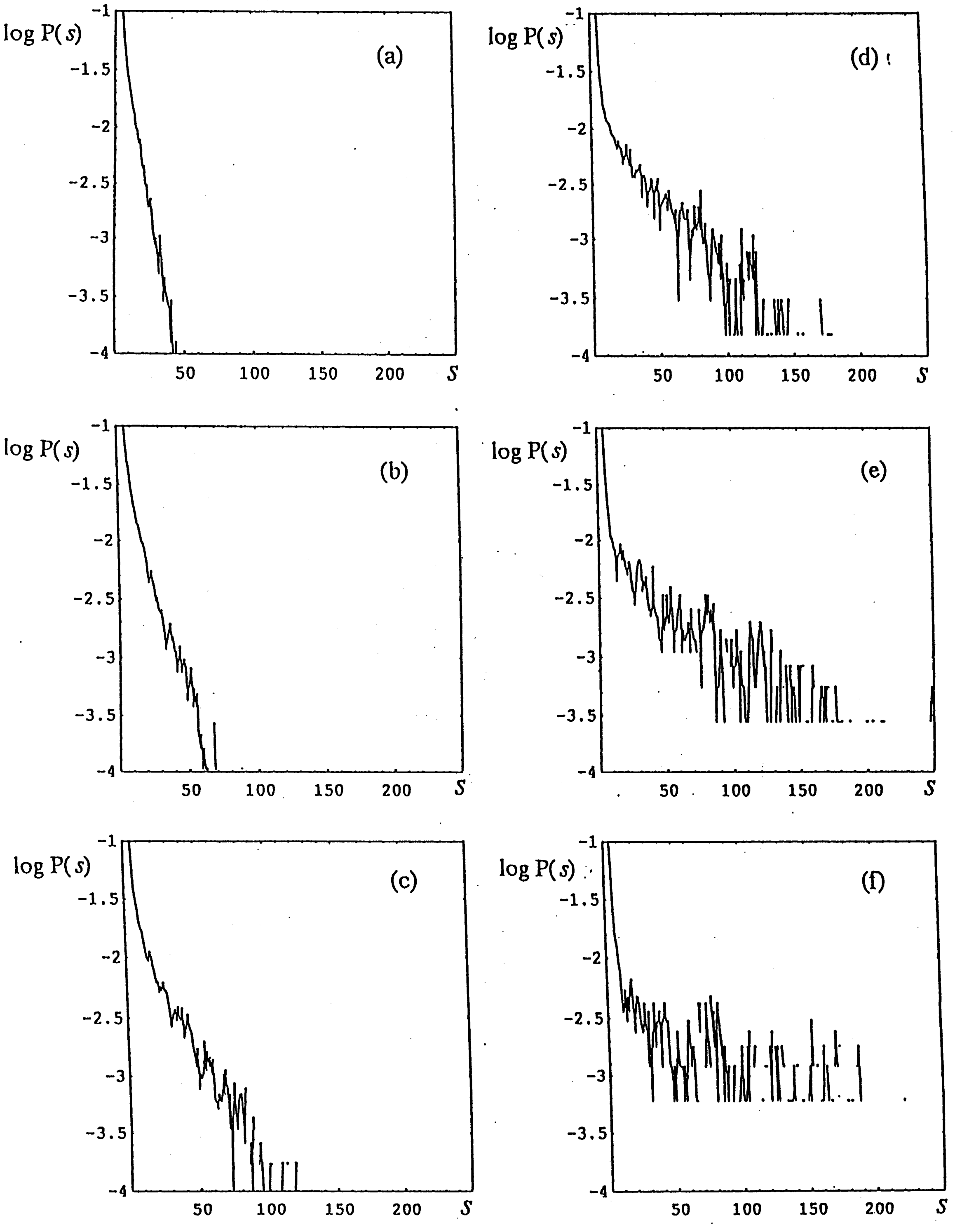

Fig.4 Spacing distributions for the iso local-enstrophy-dissipation (LED) lines in the intensities: (a) $c = 5$; (b) 8; (c) 11; (d) 14; (e) 17 ; (f) 20.

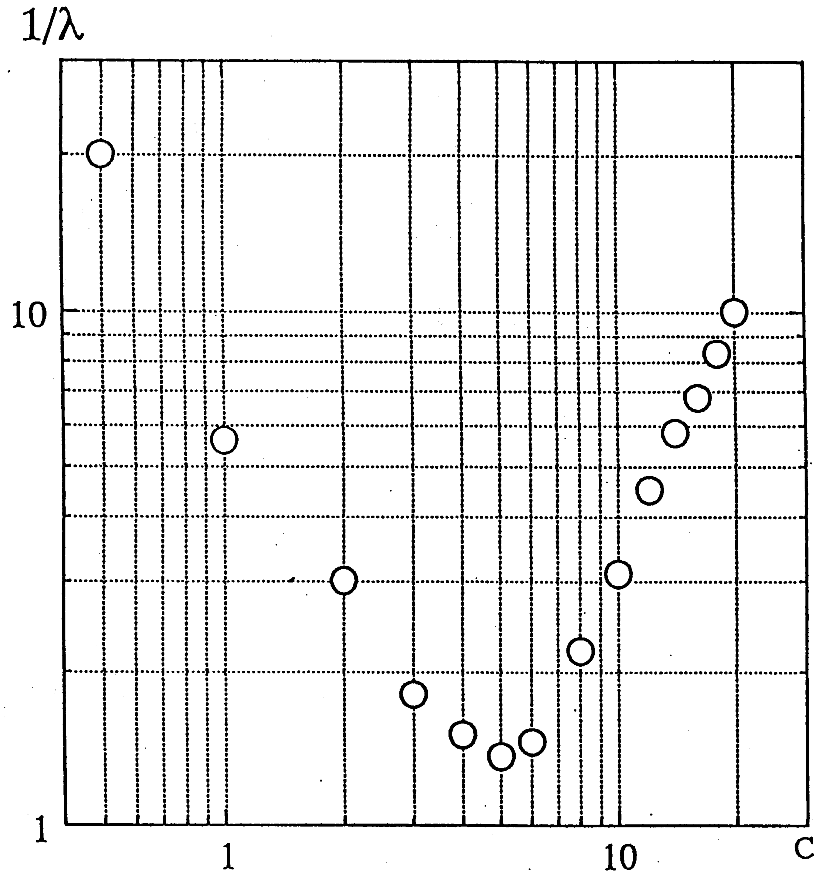

Fig.5 The mean spacing between the islands in the iso LED lines, *1/λ*, v.s. the intensity of the LED, *c*.