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VORTEX FILAMENT IN A TREE-MANIFOLD AND ITS CLASSICAL PARTITION FUNCTION

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1. Introduction

The kinematics of a very thin vortex tube in a three-dimensional fluid may be described by the filament equation in the local induction approximation. [1, 2] It is formulated as

$$\frac{\partial\gamma}{\partial t}=\frac{\partial\gamma}{\partial s}\times\frac{\partial^2\gamma}{\partial s^2}, \tag{1}$$

where $\gamma=\gamma\,(t,s)$ denotes the position of the vortex filament in R^3 with t and s being the time and the arc-length parameters respectively.

Hasimoto [3] introduced a map $h:\gamma\mapsto\psi=\kappa\,\exp[i\int^s\tau(u)du]$, in order to transform the filament equation into the nonlinear Schrödinger (NLS) equation for ψ. Here κ and τ respectively denote the curvature and the torsion along γ. Since the integrability of the NLS equation was well known, the filament equation was naturally expected to be integrable. Marsden and Weinstein [4] first described the filament equation as a Hamiltonian equation with the Hamiltonian simply being the length $\ell[\gamma]$ of the vortex filament. Later Langer and Perline [5] used this Hamiltonian structure to prove the existence of an infinite sequence of constants of motion in involution, and studied the evolution of the vortex filaments in connection with the solitons in the NLS equation.

With this concern in mind, we have investigated the filament equation in a curved three-manifold M. Although Langer and Perline have limited M to R^3, we find an analogous integrable hierarchy in the case of constant curvature. We further study the classical partition function for the vortex filaments

$$Z(\beta) = \int_\Gamma e^{-\beta\ell[\gamma]}\mathcal{D}\gamma. \tag{2}$$

It is not clear if the Duistermaat-Heckman formula [7] applies to this case, because our phase space Γ is neither finite dimensional nor compact, and furthermore because the Hamiltonian flow may not be periodic. But the perturbative calculation in our mode reveals that the loop corrections to the formula vanish up to the 3-loop in the case of constant curvature, and the 4-loop correction vanishes as well if the manifold is flat. We have not done the loop calculations beyond these orders, but it is plausible that all higher-loop corrections disappear.

2. Integrability

Let Γ be the space of vortex filaments with fixed end points p and q; Γ is the quotient space of $\{\gamma : [0,1] \to M \mid \gamma(0) = p, \gamma(1) = q\}$ with the reparametrization of γ. Hereafter γ

denotes the representative for which the parameter $x \in [0,1]$ is a multiple of the arc-length s, namely $\frac{ds}{dx} = \|\frac{d\gamma}{dx}\|$ is independent of x. Here the norm is defined by the inner product $(\ ,\ )$ on the tangent space $T_{\gamma(x)}M$. One can identify the tangent space $T_\gamma\Gamma$ with the subspace of $\Gamma(\gamma^*TM)$, and expand $X \in \Gamma(\gamma^*TM)$ in the Frenet-Serret frame along γ such that $X = f\,\mathsf{T} + g\,\mathsf{N} + h\,\mathsf{B}$, where T is the unit tangent vector to γ, N is the unit normal vector and B is the unit binormal vector. Let $\ell\,[\gamma]$ be the length of γ, so that $s = \ell\,[\gamma]\,x$. The Frenet-Serret equations are $\nabla_s\mathsf{T} = \kappa\,\mathsf{N}, \nabla_s\mathsf{N} = -\kappa\,\mathsf{T} + \tau\,\mathsf{B}, \nabla_s\mathsf{B} = -\tau\,\mathsf{N}$, with ∇ being the connection on γ^*TM induced by the Levi-Civita connection on TM. Let $\wp$ be the projection from $\Gamma(\gamma^*TM)$ to $T_\gamma\Gamma$, then one can show that the tangent component of $v = \wp(X) \in T_\gamma\Gamma$ satisfies

$$\frac{d}{dx}\,v_{\mathsf{T}} = \ell^{-1}(\nabla_x v, \frac{d\gamma}{dx}) + \ell\,\kappa\,v_{\mathsf{N}}, \tag{3}$$

and $(\nabla_x v, d\gamma/dx)$ is a constant. Fixing this constant by the boundary conditions $X(0) = X(1) = 0$, one obtains

$$\wp(X) = v = \ell\,(\int_0^x \kappa\,v_{\mathsf{N}} dx - x\int_0^1 \kappa\,v_{\mathsf{N}} dx)\,\mathsf{T} + v_{\mathsf{N}}\mathsf{N} + v_{\mathsf{B}}\mathsf{B}\ . \tag{4}$$

Geometrical structures on Γ were first studied by Marsden and Weinstein [4] for the vortex filament in R^3, and generalized to the loop space for a three-manifold M by Brylinski. [8] It is straightforward to find those for the vortex filament in M.

i) Complex structure

For the tangent vector $v \in T_\gamma\Gamma$, J generates the 90-degree rotation

$$J(v) = -\wp(\mathsf{T} \times v)\,, \quad J^2 = -1. \tag{5}$$

ii) Riemann structure

The Riemann structure on Γ is simply defined by

$$\langle u, v\rangle_\Gamma = \ell \int_0^1 (u_\mathsf{N} v_\mathsf{N} + u_\mathsf{B} v_\mathsf{B})\, dx \tag{6}$$

for $u, v \in T_\gamma\Gamma$, and satisfies the hermitian condition

$$\langle u, v\rangle_\Gamma = \langle J(u), J(v)\rangle_\Gamma. \tag{7}$$

iii) Symplectic structure

The symplectic structure on Γ may be constructed from the volume form on M, and in the Frenet-Serret frame, it is:

$$\omega(u, v) = \ell \int_0^1 (u_\mathsf{N} v_\mathsf{B} - u_\mathsf{B} v_\mathsf{N})\, dx. \tag{8}$$

Constructed from the above two structures, it may also be:

$$\omega(u, v) = \langle u, J(v)\rangle_\Gamma. \tag{9}$$

Let $\ell : \Gamma \mapsto \mathsf{R}$ be a smooth Hamiltonian function. The Hamiltonian vector field X_0 has the form $X_0 = J(\operatorname{grad} \ell)$, and thus $X_0 = \kappa\,\mathsf{B}$. This yields a natural generalization of the filament equation in M [9]

$$\frac{\partial\gamma}{\partial t} = \kappa\,\mathsf{B} = \ell^{-3}\frac{\partial\gamma}{\partial x} \times \nabla_x \frac{\partial\gamma}{\partial x}. \tag{10}$$

Constructing an infinite sequence of vector fields recursively from X_0 such that

$$X_n = \mathcal{R}^n(X_0) \quad (n = 0, 1, 2, \ldots), \tag{11}$$

with the recursion operator being

$$\begin{aligned}
\mathcal{R}(v)_{\mathsf{T}} &= \ell \left(\int_0^x \kappa\, \mathcal{R}(v)_{\mathsf{N}} dx - x \int_0^1 \kappa\, \mathcal{R}(v)_{\mathsf{N}} dx \right), \\
\mathcal{R}(v)_{\mathsf{N}} &= \tau\, v_{\mathsf{N}} + \ell^{-1} \frac{d}{dx} v_{\mathsf{B}}, \\
\mathcal{R}(v)_{\mathsf{B}} &= -\frac{\ell}{2} \kappa \left(\int_0^x \kappa\, v_{\mathsf{N}} dx - \int_x^1 \kappa\, v_{\mathsf{N}} \right) - \ell^{-1} \frac{d}{dx} v_{\mathsf{N}} + \tau\, v_{\mathsf{B}},
\end{aligned} \tag{12}$$

we demand the following proposition. [1]

Prop. In the case of the constant curvature, X_n are the commuting vector fields, namely

$$[X_n, X_m] = 0 \quad (n, m = 0, 1, 2, \ldots). \tag{13}$$

Proof We can prove $[X_0, \mathcal{R}(Y)] = \mathcal{R}([X_0, Y])$ for any vector field Y, and $d\ell(X_n) = 0$ for any n. Tedious but straightforward calculation reads us to the "hereditary" property of $\mathcal{R}$ such that

$$[\mathcal{R}(X), \mathcal{R}(Y)] = \mathcal{R}([X, \mathcal{R}(Y)]) + \mathcal{R}([\mathcal{R}(X), Y]) - \mathcal{R}^2([X, Y]), \tag{14}$$

for any vector fields X and Y satisfying $d\ell(X) = d\ell(Y) = 0$. Combining these we can prove the proposition.

[1]Analogous statement holds for the closed vortex filament as is proved by Sasaki [6] in a different context.

3. Classical partition function

In this section we evaluate the classical partition function (2) with $\mathcal{D}\gamma$ being the symplectic volume form on Γ. The stationary phase method provides an asymptotic expansion for $Z(\beta)$ as $\beta \mapsto \infty$, such that

$$Z(\beta) = \sum_{\mathrm{grad}\,\ell[\gamma]=0} Z_{\mathrm{W}KB}[\gamma,\beta]\,(1 + \frac{a_1[\gamma]}{\beta} + \frac{a_2[\gamma]}{\beta^2} + \cdots). \tag{15}$$

Suppose that the Duistermaat-Heckman formula [7] holds, then we will get all the higher-loop corrections disappeared. This is the case if Γ is a compact symplectic manifold and ℓ is a periodic Hamiltonian with isolated critical points. In more general arguments presented in Audin, [10] the fixed points are not necessarily isolated, and it is not mandatory to consider the circle action alone according to the analogous results obtained for higher dimensional tori. For the infinite dimensional symplectic manifolds, the WKB exactness has not been proved rigorously, but a "proper" version of WKB approximation should yield a reliable result for a large class of integrable models. [11, 12, 13, 14] With this notion in mind, we present the explicit calculation of the asymptotic expansion (15). For simplicity, we will assume the followings:

(1) M is a three-manifold with a constant curvature c, so that the filament equation is integrable in the sense of Proposition.

(2) Two points p and q on M are not conjugate. Consequently, the Hamiltonian ℓ is a Morse function on Γ, i.e., critical points are the geodesics on M connecting p and q,

and further the Hessian operator H_γ at each geodesic γ is a non-degenerate Jacobi operator

$$H_\gamma = -\nabla_x \nabla_x - c\,\ell[\gamma]^2. \tag{16}$$

Let us first expand the Hamiltonian ℓ around a geodesic γ such that

$$\ell\,[\gamma_s] = \ell\,[\gamma] \sum_{n=0}^{\infty} \frac{s^{2n}}{(2n)!} \int_0^1 W_{2n}(\xi) dx. \tag{17}$$

Here the integrand W_{2n} is given by the Bell Polynomial (see our paper [15] for details).

Now let us evaluate the WKB partition function

$$Z_{WKB}[\gamma, \beta] = e^{-\beta\ell[\gamma]} \int \mathcal{D}\xi \exp\left[-\frac{\beta\ell[\gamma]}{2} \int_0^1 W_2(\xi) dx\right], \tag{18}$$

$$\int_0^1 W_2(\xi) dx = \langle \xi, H_\gamma(\xi) \rangle_\Gamma. \tag{19}$$

Using the zeta-function reguralization technique, we can perform the infinite dimensional integral in (18), and obtain [16]

$$Z_{WKB}[\gamma, \beta] = \frac{1}{2}\, e^{-\beta\,\ell[\gamma]} \sqrt{\beta\,\ell[\gamma]} \left| \frac{\sqrt{c}\,\ell[\gamma]}{\sin(\sqrt{c}\,\ell[\gamma])} \right| e^{\mp \frac{\pi}{2} i\,\mu(\gamma)}. \tag{20}$$

Since the Morse index $\mu(\gamma)$ is an even integer, the last factor contains no ambiguities.

We now proceed to the higher-order calculation. It is convenient to choose an orthogonal frame $\{e_1, e_2\}$ along γ such that $\nabla_x e_i = 0, \quad (\mathsf{T}, e_i) = 0 \quad \text{for } i = 1, 2$. In this frame, the kernel of the Jacobi operator H_γ becomes diagonal, and both of the diagonal elements are identical to the Dirichlet Green function

$$G(x, x') = 2 \sum_{n=1}^{\infty} \frac{\sin(n\pi x)\sin(n\pi x')}{(n\pi)^2 - \lambda}, \tag{21}$$

with $\lambda = c\ell^2$. The 2-loop amplitude $a_1 = -\beta^2 \langle W_4/4! \rangle$ consists of four diagrams. Performing the x-integral first, and then making use of the analytic continuation method to evaluate the infinite n-summation, and multiplying the regularlized amplitudes with the weights of the diagrams, we find that the 2-loop amplitude vanishes. Beyond the 2-loop, however, we ought to generalize the analytic continuation method for a multiple infinite summation. One might think that applying the analytic continuation method directly to the Green function, we could regularize the Green function, and thereby making all the higher-loop amplitudes finite. This is certainly true, but regularizing the Green function in this way, we also eliminate the necessarily singularity at $x = x'$, and obtain non-vanishing 2-loop amplitude as a result. We can avoid this difficulty by treating $G(x, x')$ as a distribution w.r.t. x. We have examined this on the 2-loop and confirmed that the amplitude vanishes.

The 3-loop amplitude $a_2 = \beta^3 \langle \beta(W_4/4!)^2/2 - W_6/6! \rangle$ consists of 30 diagrams. Evaluating them in terms of the distribution method, we can determine 29 diagrams unambiguously. Yet, in one diagram, we encounter an ambiguous integral, which cannot be determined unless we specify the regularization of the delta function. If we were able to define the analytic continuation of the infinite double sum, such ambiguity would not appear; but we have no choice at our hand other than fixing the ambiguity by hand, and obtain the vanishing 3-loop amplitude as a result. If M is flat, namely $c = 0$, the weight of the ambiguous integral becomes zero, so that we don't have to impose the ad-hoc assumption in order to obtain the vanishing 3-loop amplitude. The perturbative calculation becomes rather simple for $c = 0$; for instance, even the 4-loop amplitude contains just 7 diagrams. No ambiguous integral appears in the 4-loop diagrams in this case, and the total 4-loop amplitude vanishes automatically.

Ambiguities appearing in higher loops are inevitable unless $c = 0$, because they relate to the regularization ambiguity of the integration measure $\mathcal{D}\gamma$, which has never been defined rigorously in the first place. Nevertheless our lower order calculations suggest that by regularizing $\mathcal{D}\gamma$ order by order, one can eliminate all the higher-loop corrections, and thereby preserving the Duistermaat-Heckman formula. The symplectic structure has been studied thoroughly in compact finite dimensional manifolds, but little is known for the infinite dimensional ones, which include most of the integrable hierarchies. This is exactly the place where the physical interests are, and the Duistermaat-Heckman formula would throw a new light over the integrable hierarchies as we have caught a glimpse of it here.

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