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Semicontinuous solutions of Hamilton-Jacobi equations with degeneracy

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1 Introduction

In this note, we consider the Hamilton-Jacobi equation with a degenerate (i.e. non-negative) coefficient $f:\mathbf{R}^{n}\to[0,\infty)$:

$$(HJ)\qquad \begin{cases} f(x)u_{t}(x,t)+H(Du(x,t))=0 & \text{for } (x,t)\in \mathbf{R}^{n}\times(0,T)\\ u(x,T)=u_{0}(x) & \text{for } x\in \mathbf{R}^{n}.\end{cases}$$

We will suppose that H is negative (except at $p=0$) and concave. Typically, we treat the case when $H(p)=-|p|$ and $f(x)=|x|^{a}$ for $a\geq 1$;

$$|x|^{a}u_{t}-|Du|=0$$

In this case, Siconolfi [4] presented a formula for it and showed that the solution is unique among solutions with some geometric property. To obtain the uniqueness without this geometric requirement, Ishii and Ramaswamy [3] proposed a bit stronger definition of solutions than that of the standard one.

On the other hand, in view of control theory, it is quite natural to expect that the solutions loose continuity (at least) when u_{0} is discontinuous or the zero set of f is a little "bigger". We will see such examples in the next section.

In order to deal with semicontinuous solutions, we follow the idea of Barron and Jensen [2]. (See also [1].) In fact, they presented a new definition of solutions under which we can show the uniqueness of semicontinuous solutions of (HJ) (with nondegenerate f). We remind that the comparison theorem in [3] yields the continuity of solutions. Thus, the definition in [3] is not suitable to our aim here.

Therefore, we will adapt a combination of the definitions of Barron-Jensen and Ishii-Ramaswamy for our definition.

Following [3], we will utilize the set of functions:

$$C^1_- = \left\{ \phi \in W^{1,\infty}_{loc}(\mathbf{R}^n \times (0,T)) \;\middle|\; \begin{array}{c} D^-\phi(x,t) \neq \emptyset \text{ for all } (x,t) \\ \in \mathbf{R}^n \times (0,T), \text{ and the mapping} \\ (x,t) \to D^-\phi(x,t) \text{ is graph closed.} \end{array} \right\}$$

Throughout this note, we shall suppose the following hypotheses:

$$(A) \quad \left\{ \begin{array}{rl} (i) & p \to H(p) \text{ concave}, \\ (ii) & \exists h \in C([0,\infty);[0,\infty)); \text{ increasing such that } h(0)=0, \\ & \text{and } H(p) \leq -h(|p|) \text{ for } p \in \mathbf{R}^n, \\ (iii) & f \in W^{1,\infty}_{loc}(\mathbf{R}^n;[0,\infty)) \text{ such that } K \equiv \{x \in \mathbf{R}^n \mid f(x)=0\} \\ & \text{has no interior point}, \\ (iv) & \exists R, \theta > 0 \text{ such that } f(x) \geq \theta \text{ for } |x| \geq R. \end{array} \right.$$

Definition: A lower semicontinuous function $u : \mathbf{R}^n \times (0,T] \to \mathbf{R}$ is called a (viscosity) solution of (HJ) in the BJ-IR sense if, for any $\phi \in C^1_-$, $(x,t) \in \arg\min(u-\phi)$ yields

$$\left\{ \begin{array}{ll} f(x)\phi_t(x,t) + H(D\phi(x,t)) \geq 0 \geq F(x,t;\phi) & \text{provided } \phi \in C^1, \\ 0 \geq F(x,t;\phi) & \text{otherwise}, \end{array} \right.$$

where

$$F(x,t;\phi) \equiv \lim_{\epsilon \to 0} \inf_{y \in B_\epsilon(x)\setminus K, |s-t|<\epsilon} \sup_{(p,q)\in D^-\phi(y,s)} \left(q + \frac{H(p)}{f(y)} \right).$$

Remark: Under the assumption (A), we can show the equivalence of definitions between ours and Ishii-Ramaswamy's in [3] whenever we consider continuous viscosity solutions. See [5] for it.

On the other hand, it is possible to obtain the comparison result below assuming that f is continuous instead of the (locally) Lipschitz continuity. However, since we will give a simplified proof of the comparison result below, we suppose (A) throughout this paper.

2 Examples of semicontinuous solutions

It is not trivial to check that the value functions in the following examples are the solutions in the BJ-RI sense. To this end, we refer to the existence result in [5].

2.1 Semicontinuous terminal data

Let us consider the simple problem in one space dimension: Fix $a > 0$.

$$\begin{cases} |x|u_t - |u_x| = 0 & \text{in } \mathbf{R} \times (0,T), \\ u(x,T) = u_0(x) \equiv \begin{cases} 0 & \text{for } x \leq -a, \\ 1 & \text{for } x > -a. \end{cases} \end{cases}$$

We can expect the formal value function associated with this PDE is the unique solution. To introduce it, let us consider the state ODEs: Fix $(x,t) \in \mathbf{R}^n \times (0,T)$ and $\alpha \in \mathcal{A}_t \equiv \{\alpha : [t,T] \to \{-1,1\};\ \text{measurable}\}$.

$$\begin{cases} X'(s) = \alpha(t) & \text{for } s \in (t,T), \\ X(t) = x, \\ Y'(s) = |X(s)| & \text{for } s \in (t,T), \\ Y(t) = t. \end{cases}$$

We denote by $(X(\cdot;x,t,\alpha), Y(\cdot;x,t,\alpha))$ the unique solution of the above.

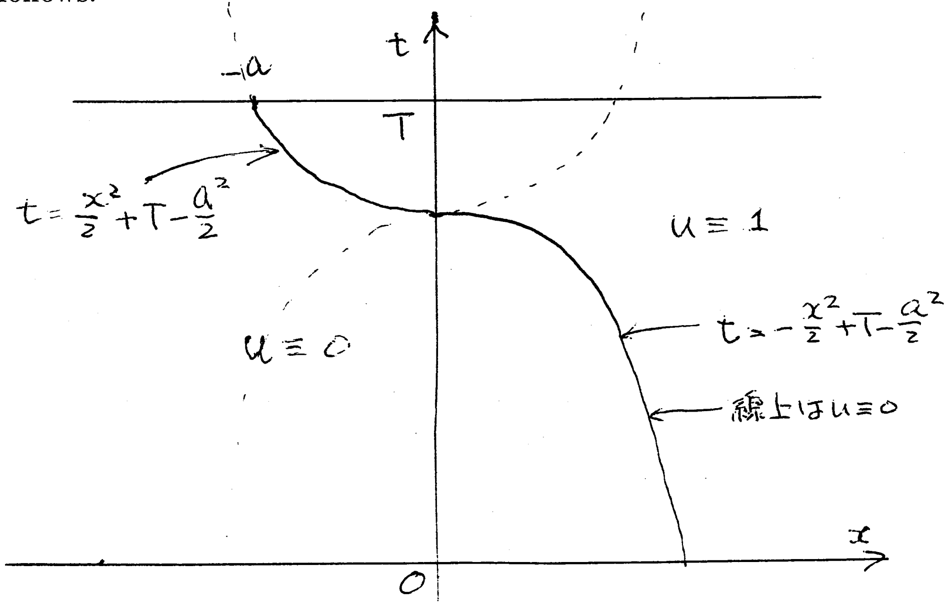
The value function can be represented by

$$V(x,t) = \inf_{\alpha \in \mathcal{A}_t} u_0(X(\tau^{x,t,\alpha};x,t,\alpha)),$$

where

$$\tau^{x,t,\alpha} = \inf\{s > t \mid Y(s;x,t,\alpha) > T\}.$$

Since we are minimizing "$u_0(\cdot)$" with respect to the control α, we have to choose $\alpha \equiv -1$ (because u_0 is nonincreasing). Thus, we can calculate V as follows:



2.2 A large degenerate set

Let us consider the case when

$$f(x) = \begin{cases} x - 1 & \text{for } x \geq 1, \\ 0 & \text{for } |x| < 1, \\ -x - 1 & \text{for } x \leq -1. \end{cases}$$

With this f, let us consider the PDE: Fix $a \in (0, 1)$.

$$\begin{cases} f(x)u_t - |u_x| = 0 \quad \text{in } \mathbf{R} \times (0, T), \\ u(x, T) = u_0(x) \equiv \begin{cases} 1 & \text{for } x \geq a, \\ x/a & \text{for } |x| < a, \\ -1 & \text{for } x \leq -a. \end{cases} \end{cases}$$

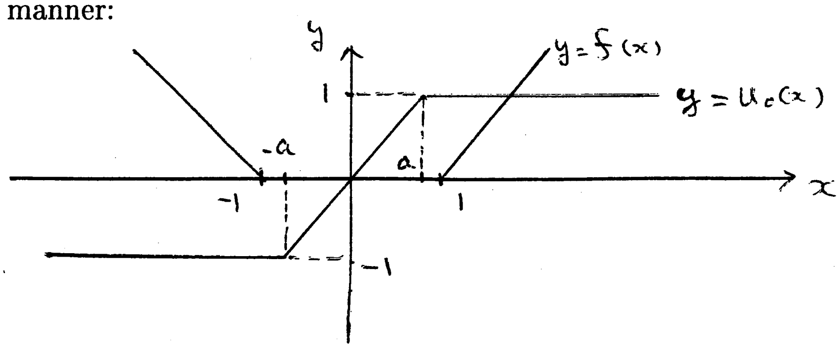
The associated state ODE is as follows: Fix $(x, t) \in \mathbf{R}^n \times (0, T)$ and an

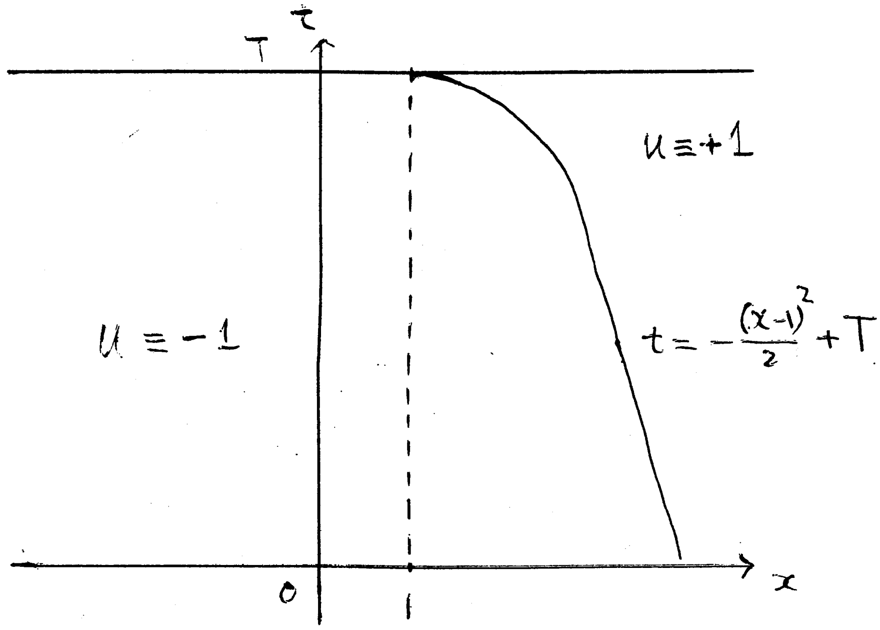
$\alpha \in \mathcal{A}_t$.

$$\begin{cases} X'(s) = \alpha(t) & s \in (t, T), \\ X(t) = x, \\ Y'(s) = f(X(s)) & s \in (t, T), \\ Y(t) = t. \end{cases}$$

The value function can be represented by the same formula as in the previous example.

By the same reason as before, we can calculate the value function in the following manner:





We notice that $(A)-(iii)$ does not hold in this example (i.e. $K^o \neq \emptyset$). However, in the same spirit as in this example, we can construct an example where (A) holds and the solution is discontinuous. See [5] for this.

3 A comparison result and a sketch of its proof

Our main theorem is as follows:

Comparison Principle: Let bounded, lower semicontinuous functions u and v be solutions of (HJ) and suppose that

$$u(x,T) \geq v(x,T) = \lim_{\epsilon \to 0} \inf_{(y,t)\in B_\epsilon(x)\times(T-\epsilon,T)} v(y,t).$$

Then, $u \geq v$ in $\mathbf{R}^n \times (0,T]$.

The strategy of our proof is first to approximate the function by "inf" and "sup" convolutions and then, to use the Barron-Jensen's Lemma ([2]):

Remark: The strange equality in the above assumption is essential because, for any solution u, $\hat{u}(x,t) \equiv C$ for $t=T$, $\equiv u(x,t)$ otherwise, is also a solution for any $C \in \mathbf{R}$.

Lemma: Let $W \in C(\mathbf{R}^n \times (0,T))$ satisfy $W \leq W(\hat{x},\hat{t})$ in $\mathbf{R}^n \times (0,T)$ for $\exists(\hat{x},\hat{t}) \in \mathbf{R}^n \times (0,T)$. Then, $\exists \omega_j \in C([0,\infty))$ $(j=1,2)$ with $\omega_j(0)=0$ satisfying the properties: For $\forall \epsilon > 0$, $\exists \phi^\epsilon \in C^1$, (x^ϵ, t^ϵ), $\{(x_k^\epsilon, t_k^\epsilon)\}_{k=1}^{N_\epsilon} \subset \mathbf{R}^n \times (0,T)$ (for $\exists$ integer N_ϵ), $\{\theta_k^\epsilon \in (0,1)\}_{k=1}^{N_\epsilon}$ with $\sum_{k=1}^{N_\epsilon} \theta_k^\epsilon = 1$ such that

$$\begin{cases} (i) & W - \phi_\epsilon \text{ attains the minimum at } (x_k^\epsilon, t_k^\epsilon), \\ (ii) & |x_k^\epsilon - x^\epsilon| + |t_k^\epsilon - t^\epsilon| \leq \omega_1(\epsilon)\epsilon, \\ (iii) & |x^\epsilon - \hat{x}| + |t^\epsilon - \hat{t}| \leq \omega_2(\epsilon), \\ (iv) & |D\phi^\epsilon(x_k^\epsilon, t_k^\epsilon)| + |\phi_t^\epsilon(x_k^\epsilon, t_k^\epsilon)| \leq \omega_1(\epsilon)/\epsilon, \\ (v) & \displaystyle\sum_{k=1}^{N_\epsilon} \theta_k^\epsilon (D\phi^\epsilon, \phi_t^\epsilon)(x_k^\epsilon, t_k^\epsilon) = (0,0). \end{cases}$$

Sketch of proof: Set $v_\lambda(x,t) \equiv v(x,t) + \lambda(T-t)$ for $\lambda > 0$. It is sufficient to show that $v_\lambda \le u$ for any $\lambda > 0$. We notice that v_λ satisfies that

$$\begin{cases} (f(x)-\lambda)q + H(p) \ge 0 \text{ for } \forall (p,q) \in D^- v(x,t) \quad ((x,t) \in \mathbf{R}^n \times (0,T)), \\ \qquad\qquad \text{and } v_\lambda(\cdot, T) = v(\cdot, T) \text{ in } \mathbf{R}^n. \end{cases}$$

For the notational simplicity, we shall write v instead of v_λ.

In this note, also for simplicity, we shall prove the assertion assuming that v is continuous. Indeed, the complete proof can be done by approximating v with the "inf & sup"-convolution while we will only adapt "sup"-convolution approximation in the argument below.

Since our argument is local, we may suppose that f is (globally) Lipschitz continuous:

$$|f(x) - f(x')| \le L|x - x'| \quad \forall x,\ x' \in \mathbf{R}^n.$$

Set

$$\gamma_\sigma(r) = \int_0^r h^{-1}(Lt/\sigma)dt.$$

Approximate v by the sup-convolution:

$$v^{\sigma,\tau}(x,t) = \sup_{(y,s)\in \mathbf{R}^n \times (0,T)} \left\{ v(y,s) - \gamma_\sigma(|x-y|) - \frac{(t-s)^2}{2\tau} \right\}.$$

For any fixed $(x,t) \in \mathbf{R}^n \backslash K \times (0,T)$, we choose any $(p,q) \in D^+ v^{\sigma,\tau}(x,t)$. We notice that $v^{\sigma,\tau} \in C^1_-$. See [3] for this fact.

Here, we give the list of properties for C^1_--functions ([3]). For $\phi \in C^1_-$, we have the following:

$$(1) \quad \begin{cases} (i) \ \ D^+\phi \ne \emptyset \ a.e., \\ (ii) \ \ \overline{D^+\phi} \equiv \text{ the graph closure of } D^+\phi \ne \emptyset \text{ in } \mathbf{R}^n \times (0,T), \\ (iii) \ \ \overline{D^+\phi}(x,t) \subset D^-\phi(x,t) \text{ for } \forall (x,t) \in \mathbf{R}^n \times (0,T). \end{cases}$$

Choose $\psi \in C^1$ such that

$$v^{\sigma,\tau} - \psi \text{ attains the maximum at } (x,t) \text{ with } (p,q) = (D\psi(x,t), \psi_t(x,t)).$$

Let $(\hat{x},\hat{t}) \in \mathbf{R}^n \times (0,T)$ satisfy $v^{\sigma,\tau}(x,t) = v(\hat{x},\hat{t}) - \gamma_\sigma(|x-\hat{x}|) - (t-\hat{t})^2/(2\tau)$. We claim that

$$(p,q) \in D^+ v(\hat{x},\hat{t}).$$

Indeed,

$$v^{\sigma,\tau}(y,s)-\psi(y,s)\le v^{\sigma,\tau}(x,t)-\psi(x,t)\quad (\text{for } \forall (y,s)\in \mathbf{R}^n\times(0,T))$$

implies

$$\begin{aligned}(2)\qquad & v(y',s')-\gamma_\sigma(y'-y)-\frac{(s'-s)^2}{2\tau}-\psi(y,s)\\ &\le v(\hat{x},\hat{t})-\gamma_\sigma(\hat{x}-x)-\frac{(\hat{t}-t)^2}{2\tau}-\psi(x,t)\end{aligned}$$

for any (y',s') and $(y,s)\in \mathbf{R}^n\times(0,T)$. We plug $(y',s')=(\hat{x},\hat{t})$ in (2) to derive

$$\left(D\gamma_\sigma(\hat{x}-x),\frac{\hat{t}-t}{\tau}\right)=(D\psi(x,t),\psi_t(x,t)).$$

On the other hand, by plugging $(y,s)=(x,t)$ in (2), the left hand side of the above belongs to $D^+v(\hat{x},\hat{t})$. (We note that this claim does not hold if we adapt the standard "mollifier" approximation instead of the sup-convolution.)

We moreover note that, since the definition of $v^{\sigma,\tau}$ yields $(\hat{t}-t)^2\le 4\tau\|v\|_\infty$, the above observation gives the estimate

$$(3)\qquad |q|\le 2\sqrt{\frac{\|v\|_\infty}{\tau}}.$$

We also note that, by $(A)-(ii)$,

$$(4)\qquad H(p)\le -h(|D\gamma_\sigma(\hat{x}-x)|)=-\frac{L|x-\hat{x}|}{\sigma}\le\frac{f(x)-f(\hat{x})}{\sigma}.$$

Setting $W(y,s)\equiv v(y,s)-\gamma_\sigma(y-x)-(s-t)^2/(2\tau)$, we apply the Barron-Jensen Lemma to this W at the maximum point $(\hat{x},\hat{t})$. Then, for each $\epsilon>0$, we can choose $\phi^\epsilon\in C^1$ etc. satisfying $(i)-(v)$ with the same notation as in the lemma;

$$(D\phi^\epsilon(x_k^\epsilon,t_k^\epsilon),\phi_t(x_k^\epsilon,t_k^\epsilon))\in D^-W(x_k^\epsilon,t_k^\epsilon).$$

Thus,

$$(p_k^\epsilon,q_k^\epsilon)\equiv\left(D\gamma_\sigma(x_k^\epsilon-x)+D\phi^\epsilon(x_k^\epsilon,t_k^\epsilon),\frac{t_k^\epsilon-t}{\tau}+\phi_t^\epsilon(x_k^\epsilon,t_k^\epsilon)\right)\in D^-v(x_k^\epsilon,t_k^\epsilon).$$

Hence, from the definition, we have

$$f(x_k^\epsilon)(q_k^\epsilon - \lambda) + H(p_k^\epsilon) \geq 0.$$

Multiplying θ_k^ϵ in the above, taking the summation from 1 to N_ϵ, from the concavity of H with (v) of the lemma, we have

$$\sum_{k=1}^{N_\epsilon} \theta_k^\epsilon f(x_k^\epsilon)(q_k^\epsilon - \lambda) + H\left(\sum_{k=1}^{N_\epsilon} \theta_k^\epsilon D\gamma_\sigma(x_k^\epsilon - x)\right) \geq 0.$$

Thus, by (ii) and (v) (again) of the lemma, we see

$$\begin{aligned} 0 \leq\ & f(x^\epsilon)\left(\sum_{k=1}^{N_\epsilon} \theta_k^\epsilon \frac{t_k^\epsilon - t}{\tau} - \lambda\right) + L\omega_1(\epsilon)\epsilon\left(\lambda + \sum_{k=1}^{N_\epsilon} \theta_k^\epsilon |q_k^\epsilon|\right) \\ & + H\left(\sum_{k=1}^{N_\epsilon} \theta_k^\epsilon D\gamma_\sigma(x_k^\epsilon - x)\right). \end{aligned}$$

Note that

$$|q_k^\epsilon| \leq \frac{C}{\tau} + \frac{\omega_1(\epsilon)}{\epsilon} \quad \text{for some } C > 0.$$

Hence, with this estimate, sending $\epsilon \to 0$ in the above, we have

$$f(\hat{x})\,(q - \lambda) + H(p) \geq 0. \tag{5}$$

Case 1: $x \neq \hat{x}$

We first claim that $q \geq \lambda$. Indeed, otherwise, (5) yields $p = 0$, which implies $x = \hat{x}$ since $p = D\gamma_\sigma(\hat{x} - x)$.

Hence, by (4), we have

$$f(\hat{x})(q - \lambda) \leq\leq (f(x) - \sigma H(p))(q + \lambda).$$

Combining these inequalities with (3), we have

$$f(x)(q - \lambda) + \left(1 - 2\sigma\sqrt{\frac{\|v\|_\infty}{\tau}}\right) H(p) \geq 0. \tag{6}$$

Case 2: $x = \hat{x}$

In this case, (5) immediately yields also (6).

(In order to derive (6), in [5], we used a more careful observation. However, with that argument, we have to care about the dependence of parameters. But, using the above argument (= considering 2 cases), we can obtain (3.14) in [5] (for each k).)

If we take σ and $\tau > 0$ so that $(1 - 2\sigma\sqrt{\|v\|_\infty/\tau}) \geq 1/2$, by (6), we then see

$$-H(p) \leq 2f(x)q \leq 4f(x)\sqrt{\frac{\|v\|_\infty}{\tau}}.$$

Hence, we have

$$f(x)\left(q - \frac{\lambda}{2}\right) + \left(1 - 2\sigma\sqrt{\frac{\|v\|_\infty}{\tau}} + \frac{\lambda}{8}\sqrt{\frac{\tau}{\|v\|_\infty}}\right) H(p) \geq 0.$$

Therefore, for appropriate choice of σ and $\tau > 0$, we have

$$q + \frac{H(p)}{f(x)} \geq \frac{\lambda}{2}.$$

By (ii) and (iii) of (1), we see

$$F(x, t; v^{\sigma,\tau}) \geq \frac{\lambda}{2},$$

which is a contradiction if we suppose that $u - v^{\sigma,\tau}$ takes its negative minimum at $(x, t) \in \mathbf{R}^n \times (0, T)$.

To conclude our proof, we have to avoid the case when the above minimum is taken at $x = \infty$ or $t = 0$. (Once we can do it, we get $u \geq v^{\sigma,\tau}$, which yileds our assertion by sending σ and $\tau \to 0$.)

For this purpose, we need more "test" functions. However, since those techniques are rather standard, we shall stop our proof here. See [5] for the details.

References

[1] G. Barles, Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations, *Nonlinear Anal.*, **20** (1993), 1123-1134.

[2] E. N. BARRON & R. JENSEN, Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians, *Comm. Partial Differential Equation*, **15** (1990), 1713-1742.

[3] H. ISHII & M. RAMASWAMY, Uniqueness results for a class of Hamilton-Jacobi equations with singular coefficients, *Comm. Partial Differential Equation*, **20** (1996), 2187-2213.

[4] A. SICONOLFI, A first order Hamilton-Jacobi equation with singularity and the evolution of level sets, *Comm. P. D. E.*, **20** (1995), 277-307.

[5] S. KOIKE, Semicontinuous viscosity solutions for Hamilton-Jacobi equations with a degenerate coefficient, to appear in *Differential Integral Equation*.