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# VARIATION OF CURVATURES AND STABILITY OF HYPERSURFACES

*N.H.Abdel-All and Rawya A.Hussien*

*Mathematic Department,Faculty of Science, Assiut University, Assiut, Egypt.*

This article is devoted to some known results on variational problem of hypersurfaces in Euclidean spaces.Our purpose is not only to gather together the known results concerning the stability of compact hypersurfaces, but also to derive them from a uniform point of view,namely,by the method of frame fields and the calculus of differential forms. The normal variation is defined and the variation of the 1st and 2nd quantities are obtained. Thus the variation of the Gauss and mean curvatures are derived. The generalized Gauss and mean curvatures of the parallel hypersurfaces, in terms of the variational problem of a given hypersurface, are given. Finally, compact hyper-surfaces are classified according to the stability with respect to $\int_M H^c * 1$.

## 1. Geometric Prelinimareis

Here, we give a brief review on thegeneral theory of hypersurfaces for later use [1]. Let M be an oriented hypersurface, immersed in an oriented (m+1)-Euclidean space $E^{m+1}$, parametrized by a regular patch

$$X: D \longrightarrow E^{m+1},\ M = X(D) \subset E^{m+1} \tag{1}$$

Here and in the sequel, the Latin and Greek indices run over the ranges (1,2,....,m) and (1,2,....,m+1) respectively. We shall use Einstein's summation convention.

We can choose the field of frames $\mathcal{F}=\{X,e_\alpha\}$ in $E^{m+1}$ such that, restricted to M, the vectors $e_i$ are tangent to M and consequently the vector $e_{m+1}$ is the unit normal field over M in $E^{m+1}$. With respect to the frame field of $E^{m+1}$, choosen above, let $\omega^\alpha$ be the field of dual frames then the structure equations of $E^{m+1}$ are

$$d\omega^\alpha=-\omega^\alpha_\beta\wedge\omega^\beta \quad , \quad d\omega^\beta_\alpha=\omega^\gamma_\alpha\wedge\omega^\beta_\gamma \quad , \quad \omega^\alpha_\beta+\omega^\beta_\alpha=0 \tag{2}$$

where d is the exterior differentiation and $\wedge$ is the exterior product between forms. The fundamental equations of the frame $\mathcal{F}$ are

$$dX=\omega^\alpha e_\alpha \quad , \quad de_\alpha=\omega^\beta_\alpha e_\beta \tag{3}$$

where $\omega^i$ are the dual forms to $e_i$ and $\omega^\alpha_i$ are the connection forms. We restrict these forms to M, then

$$\omega^{m+1}=0, \qquad 0=d\omega^{m+1}=\omega^{m+1}_i\wedge\omega^i \tag{4}$$

Using Cartan's lemma, we write

$$\omega^{m+1}_i=h_{ij}\,\omega^j \quad ,h_{ij}=h_{ji} \tag{5}$$

$$d\omega^i=-\omega^i_j\wedge\omega^j \quad , \quad \omega^j_i+\omega^i_j=0 \tag{6}$$

Using (5), (6) and the 1st equation of (4), one can see that the 1st and 2nd fundamental forms of the immersed manifold M are

$$\begin{aligned} I&=\langle dX,dX\rangle=g_{ij}\,\omega^i\omega^j \quad , \quad g_{ij}=\delta_{ij} \\ II&=-\langle de_{m+1},dX\rangle=\omega^i\omega^{m+1}_i=h_{ij}\,\omega^i\omega^j \end{aligned} \tag{7}$$

respectively, where <,> is the induced metric by the immerstion x, and $\delta_{ij}, \delta^{ij}$ are the well-known Kroneker deltas. Consequently, we have

$$d^2X = (D\omega^i + \omega^j\omega^i_j)\, e_i + II\, e_{m+1} \tag{8}$$

The 3rd fundamental form III is defined as

$$III = \langle de_{m+1}, de_{m+1} \rangle = \delta_{ij}\, \omega^i_{m+1}\, \omega^j_{m+1}$$

and from (5) we obtain

$$III = \gamma_{ij}\, \omega^i\omega^j\ , \ \gamma_{ij} = \sum_k h_{ik}h_{kj} \tag{9}$$

The mean, Gauss and scalar curvature functions are

$$H = \frac{1}{m}\, tr\ II = \frac{1}{m}\, h_{ij}\ \delta^{ij} \tag{10}$$

$$K = Det\ (h_{ij}) \tag{11}$$

$$R = m^2H^2 - |II|^2 \tag{12}$$

respectively, where

$$|II| = \left( \sum_{i,j} (h_{ij})^2 \right)^{1/2} \tag{13}$$

is the norm of the 2nd fundamental form II.

The following results are cited from [2]. Let $\Lambda^p$ (M) is the space of p-forms defined on M. The star operator * is define as

$$* : \Lambda^p(M) \longrightarrow \Lambda^{m-p}(M)$$

Using the known properties of the * operator, it is easy to see that the volume element of M with respect to the metric I is

$$*1 = \omega^1 \Lambda\ \omega^2 \Lambda \ldots\ldots \Lambda\omega^m \tag{14}$$

and the codifferential $\delta$: $\Lambda^{P}(M) \longrightarrow \Lambda^{p-1}(M)$, where

$$\delta\omega = (-1)^{mp+m+1} *d*\omega \qquad (15)$$

Thus the Laplace operator $\Delta$ is defind as a map

$\Delta: \Lambda^{P}(M) \longrightarrow \Lambda^{P}(M)$ given by

$$\Delta\omega = (d\delta+\delta d)\omega \qquad (16)$$

For a Zero form $\phi\varepsilon\Lambda^{o}(M)$, $d\phi\varepsilon\Lambda^{1}(M)$is given as

$$d\phi = A_i\omega^i \quad , d\phi(e_j) = A_j \quad , \quad A_i\varepsilon\Lambda^{o}(M) \qquad (17)$$

Exterior differentiation of (17) and using (6), we obtain

$$(dA_i + A_k \omega^k_i)\Lambda\omega^i = o$$

Using Cartan's lemma we have

$$dA_i + A_k\omega^k_i = A_{ij}\omega^j \qquad (18)$$

Combining these relations, it follows

$$A_i(D\omega^i + \omega^j\omega^i_j) + A_{ij}\omega^i\omega^j = 0 \qquad (19)$$

From (15) and (17), the Laplace operator of the function $\phi$ (p=0)is

$$\Delta\phi = *^{-1}d* \, d\phi \qquad (20)$$

Direct computations and using (18), one can prove that [3],[4]

$$\Delta\phi = A_{ij}\delta^{ij} \qquad (21)$$

**2 - Normal variation of the frames**

This section modifies the results on the variational problem which have been introduced in [5], [6]. Let M be a compact hypersurface with piecewise smooth boundary $\partial M$, the function $\phi = 0$ on $\partial M$ and $\int_M \phi * 1 = 0 \ \forall \ \phi \ \varepsilon \ \Lambda^{o}(M)$. We consider a smooth map.

$F: J \times M \longrightarrow E^{m+1}$ such that for $t \in J = [0,1]$ the map $F_t : M \longrightarrow E^{m+1}$, where $F_t(p)=F(t,p)$ for $P \in M$ is an immersion such that $F_o = X$ and $F_t, X$ are coincident on the boundary. The image $F_t(P)$ is represented by the parametrization

$$\bar{X}(t,u^i)=X(u^i) + t\phi(u^i)e_{m+1}(u^i) \tag{22}$$

where $X = X(u^i)$ is a regular patch of M. This representation defines a normal variation of M in $E^{m+1}$ associated with $\phi$. We define the operator $\delta$ as $\delta = \frac{\partial}{\partial t}$ given at $t = 0$. Thus we have the variation vector field $\delta F_t = \phi\, e_{m+1}$ which is denoted by $\delta X$.

Here, we try to obtain the fundamental and structural equations of the variation $\bar{X}$. Exterior differentiate (22) and using (3), (4) and (17) we have

$$\begin{aligned} d\bar{X} &= \bar{\omega}^i e_i + \bar{\omega}^{m+1} e_{m+1} \\ \bar{\omega}^i &= (\delta_{ij} - t\phi h_{ij})\omega^j \quad , \quad \bar{\omega}^{m+1} = tA_i \omega^i \end{aligned} \tag{23}$$

The vector valued one-form $d\bar{X}$ can be written as

$$d\bar{x} = \omega^i \bar{e}_i \ , \ \bar{e}_i = \sum_j (\delta_{ij} - t\phi h_{ij}) e_j + tA_i e_{m+1} \tag{24}$$

The vectors $\bar{e}_i$ are tangent vectors to the variation $X = F^{-}(t,p)$ corresponding to the tangent vectors $e_i$ to the surface $X = X(u^i)$, where $\bar{e}_i = F_*(e_i)$ and $F_*$ is the derivative map of F. For simplify the computation, we put $\bar{e}_i = \sum_{\alpha=1}^{m+1} \psi_{i\alpha}(t) e_\alpha$, where $\psi_{i\alpha}(t)$ are linear functions in the parameter t, and are given as

$$\psi_{i\alpha}(t) = \sum_k (\delta_{i\alpha}\, \delta_{\alpha k} - t\phi h_{i\alpha}\, \delta_{\alpha k}) + tA_i\, \delta_{\alpha, m+1} \tag{25}$$

The functions $\psi_{i\alpha}(t)$ having the properties

$$\psi_{i\alpha}(o)= \sum_{k} \delta_{i\alpha}\, \delta_{\alpha k}, \quad \frac{\partial}{\partial t} \psi_{i\alpha}(t)=-\phi \sum_{k} h_{i\alpha}\delta_{\alpha k}+ A_{i}\delta_{\alpha, m+1} \tag{26}$$

The mormal vector $\bar{e}_{m+1}$ to the variation $\bar{X}$ is defined as the vector product of the tangent vectors $\bar{e}_i$ as the following [7]

$$\bar{e}_{m+1}= \sum_{j_i, j_\alpha=1}^{m+1} \sum_{i=1}^{m} \prod_{i=1}^{m} \psi_{ij_\alpha}(t)\, e_{j_1} \Lambda e_{j_2} . \Lambda. \dots . \Lambda e_{j_m} \tag{27}$$

Thus we have

$$\delta e_{m+1}= \left(\frac{\partial}{\partial t} \sum_{j_k=1}^{m} \sum_{j_i=1}^{m+1} \prod_{i=1}^{m} \psi_{ij_k}(t)\, \psi_{ij_{m+1}}(t)\right)(o) e_{j_1} \Lambda e_{j_2} \Lambda \dots \Lambda e_{j_m}$$

After some routin calculations, we get

$$\delta e_{m+1}= \sum_{\substack{j_k=1 \\ j_k \neq j_i}}^{m} \sum_{j_i=1}^{m+1} A_{j_k} e_{j_1} \Lambda e_{j_2} \Lambda. \dots . \Lambda e_{j_m}, (\mathrm{mod}\ e_{m+1})$$

The indices $j_1, j_2, \dots, j_m, j_k$ are taken as an odd permutation of the natural sequence $(1,2,\dots,m+1)$. Therefore $e_{j_1} \Lambda e_{j_2} \Lambda \dots \Lambda e_{j_m} = -e_{j_k}$. Consequently, in a simple form, we have

$$\delta e_{m+1}= - \sum_{i=1}^{m} A_i e_i \quad , \ (\mathrm{mod}\ e_{m+1}) \tag{28}$$

## 3. Normal variation of the first fundamental quantities

In this section, we derive the formulas of the variation of the 1st fundamental quantities. The 1st fundamental form of the variation $\bar{X}$ is $\bar{I} = \langle d\bar{X}, d\bar{X}\rangle$ and from (24) we have $\bar{I} = \bar{g}_{ij}\omega^i\omega^j$, where

$$\bar{g}_{ij}(t) =\delta_{ij}-2\, t\phi h_{ij}+t^2(\phi^2\gamma_{ij}+A_iA_j), \quad g_{ij}(o) = \delta_{ij} \tag{29}$$

are the 1st quantities of the variation $\bar{X}$ . It is easy to see that the 1st and 2nd variation of $g_{ij}$ are

$$\delta g_{ij} = -2\phi h_{ij},\quad \delta^2 g_{ij} = \left(\frac{\partial^2}{\partial t^2}\bar{g}_{ij}\right)(o) = 2(\phi^2\gamma_{ij} + A_i A_j) \tag{30}$$

Taking the variation on both sides of the well known relation $\bar{g}_{ij}\,\bar{g}^{ik} = \delta^k_j$ , $\bar{g}^{ik}(o) = g^{ik} = \delta^{ik}$ we obtain

$$\delta\, g_{ij}\,.\,\delta^{ik} + \delta_{ij}\,.\delta g^{ik} = o$$

From (30), one can see that the 1st variation of the contravariant metric tensor $g^{ij}$ is

$$\delta g^{ik} = 2\phi h_{ik} \tag{31}$$

Similarly, using (30), (31), one can obtain the following

$$\delta^2 g^{jk} = 2(3\phi^2\gamma_{jk} - A_k A_j) \tag{32}$$

From the foregoing results, we have

**Lemma 1.** The 1st and 2nd variation of the metric tensors $g_{ij}, g^{ij}$ are given by (30), (31) and (32).

The volume element of the variation $\bar{X}$ is $\bar{*1} = \sqrt{\mathrm{Det}(\bar{g}_{ij})}*1$ and thus $\delta *1 = \left(\frac{\partial}{\partial t}\sqrt{\mathrm{Det}(\bar{g}_{ij})}\right)(o)*1$ .

Using the well known result

$$\frac{\partial}{\partial t}\mathrm{Det}\,(\bar{g}_{ij}) = \mathrm{Det}\,(\bar{g}_{ij}).\ \bar{g}^{ij}\ \frac{\partial \bar{g}_{ij}}{\partial t}$$

and from (30), (10) one can obtain

$$\delta * 1 = -m\,\phi\, H * 1 \tag{33}$$

**4- Parallel hypersurfaces as a variational type.**

Now, we characterize parallel surfaces in terms of the normal variational problem. The two hypersurfaces $M, \tilde{M}$ are called parallel hypersurfaces if one of them is obtained from the other by means of a special normal variation. This variation can be given from (22) where $\phi = 1$ ($A_i = o$, $\forall i$) and $t = c$(const.). Thus, the representation of the hypersurface $\tilde{M}$ which is parallel to the given hypersurface M is

$$\tilde{X}(c, u^i) = X(u^i) + c\, e_{m+1}(u^i) \tag{34}$$

From (29) it follows that the 1st fundamental quantities of the parallel surface $\tilde{M}$ are

$$\tilde{g}_{ij} = (\delta_{ij} - 2ch_{ij}) + c^2\gamma_{ij} \tag{35}$$

The discriminant $\tilde{g}$ of $\tilde{I} = \tilde{g}_{ij}\,\omega^i\omega^j$ is

$$\tilde{g} = \det\left(I - (2ch - c^2\gamma)\right) \tag{36}$$

where h, $\gamma$ and I are square matrices defined as $h = (h_{ij})$, $\gamma = (\gamma_{ij})$, $I = (\delta_{ij})$. But the eigenvalues of $(2ch - c^2\gamma)$ are just $2ck_i - c^2k_i^2$, where $k_i$ are the principal curvatures of the hypersurface M [8]. Thus

$$\tilde{g} = \prod_i (1 - ck_i)^2 \tag{37}$$

The element of area $*\tilde{1}$ of the surface $\tilde{M}$ is $*\tilde{1} = \sqrt{\tilde{g}}\, * 1$, and thus we have $*\tilde{1} = P_m(c) * 1$ where

$$P_m(c) = \sum_{i=1}^{m+1} (-1)^{i-1} H_{i-1} C^{i-1} \tag{38}$$

and $H_i = \sum_{j_1 < j_2 < \dots < j_i} k_{j_1} k_{j_2} \dots k_{j_i}$, $H_o = 1$, $1 \le i \le m$

are the higher order Gaussian curvatures.

From (37) it follows that $\tilde{g} \neq 0$. Thus the vector valued function $\tilde{X} = \tilde{X}(c, u^i)$ define a one- parameter family of parallel hypersurfaces. From the construction of the parallel hypersurfaces, it is easily to see that, the unit normal vector field to the hypersurface $\tilde{M}$ is coincident with $e_{m+1}$ (the normal vector field of M). The shape operator $S_{\tilde{P}}(\tilde{v})$ of the hypersurface $\tilde{M}$ at the point $\tilde{P}$ in the direction of the tangent vector $\tilde{V}$ is defind as $S_{\tilde{P}}(\tilde{v}) = -de_{m+1}(\tilde{v})$. From (24) one can prove the well known theorem [9]

$$S_P(e_i) = S_{\tilde{P}}(\tilde{e}_i) \quad , \quad \tilde{e}_i = \sum_j (\delta_{ij} - ch_{ij}) e_j \tag{39}$$

for every $e_i$ and $\tilde{e}_i$ in the tangent spaces $T_p(M)$ and $T_{\tilde{p}}(\tilde{M})$ respectively. Therefore

$$S_{\tilde{P}}(\tilde{e}_i) = -de_{m+1}(e_i) = -\omega^j_{m+1}(e_i) e_j \tag{40}$$

Taking account of (5) we have

$$S_{\tilde{P}}(\tilde{e}_i) = \sum_j \left( h_{jk} \omega^k(e_i) \right) e_j = \sum_j h_{ji} e_j$$

The inverse transformation of (39) is

$$e_j = \sum_k \frac{\hat{h}_{kj}}{p_m(c)} \bar{e}_k \tag{41}$$

where $\hat{h}_{kj}$ is the cofactor of the element $\delta_{kj} - ch_{kj}$ in the matrix $(\delta_{kj} - ch_{kj})$. Henceforth $S_{\tilde{P}}(\tilde{e}_i) = \sum_k \tilde{h}_{ik} \tilde{e}_k$

where $\tilde{h}_{ik} = \sum_j \frac{h_{ji} \hat{h}_{kj}}{p_m(c)}$ are the elements of the matrix $\tilde{h} = (\tilde{h}_{ik})$

of the shape operator S. Because the mapping $\tilde{x}$ is not preserve the lines of curvatures ($h_{kj} \neq o$, $j \neq k$) these results are an extension to the results which have been obtained in [10]. Thus, we reach to the following

**Theorem 1.** The Gauss and mean curvature functions $\tilde{K}$ and $\tilde{H}$ of the parallel surface $\tilde{M}$ are

$$\tilde{K} = \frac{\mathrm{Det}\left(\sum_j h_{ji}\hat{h}_{kj}\right)}{\left(p_m(c)\right)^m}, \quad \tilde{H} = \frac{\mathrm{tr}\left(\sum_j h_{ji}\hat{h}_{kj}\right)}{m\, p_m(c)} \text{ respectively}$$

**5- The variation of the second fundamental quantities.**

Here, new approach to the variation of the 2nd fundamental quantities $h_{ij}$ is adapted. For this purpose, the 1st equation of (23) can be written as

$$d\bar{X} = dX + tA_i\omega^i e_{m+1} + t\phi\omega^i_{m+1} e_i$$

Exterior differentiation gives

$$d^2\bar{X} = d^2X - t\phi \sum_i (\omega_i^{m+1})^2 e_{m+1} + \theta^i e_i$$

where $\theta^i = t(2A_i\omega^i\omega^i_{m+1} + \phi\, \omega^j_{m+1}\omega^i_j + \phi\, d\omega^i_{m+1})$. From (5) it follows

$$d^2\bar{X} = d^2X - t\phi \sum_i h_{ij}h_{ik}\omega^j\omega^k e_{m+1} + \theta^i e_i$$

Thus, the 1st variation of the vector valued 2-form $d^2X$ is

$$\delta d^2X = -\phi \sum_i h_{ij}h_{ik}\omega^j\omega^k e_{m+1} \quad , \ (\mathrm{mod}\ e_i) \tag{42}$$

From the 2nd equality of (7) we get

$$\delta\ II = \delta h_{ij}\omega^i\omega^j = \delta\langle d^2X,\ e_{m+1}\rangle \tag{43}$$

Using (28) and (42) we obtain

$$\delta h_{ij}\omega^i\omega^j = -\phi \sum_i h_{ij}h_{ik}\omega^j\omega^k - A_i(D\omega^i + \omega^j\omega^i_j)$$

Substitution in (19) gives

$$\delta h_{ij}\omega^i\omega^j = A_{ij}\omega^i\omega^j - \phi \sum_i h_{ij}h_{ik}\omega^j\omega^k$$

and from which we have

$$\delta h_{ij} = \Lambda_{ij} - \phi \sum_k h_{ik}h_{kj} \tag{44}$$

The Gaussian curvature $\bar{K}$ of the variation $\bar{X}$ is

$\bar{K}$ = Det $(\bar{h}_{ij})$/ Det $(\bar{g}_{ij})$ and the variation $\delta K$ is defined as

$$\delta K = \frac{\partial}{\partial t}\left(\text{Det } (\bar{h}_{ij}) / \text{ Det } (\bar{g}_{ij})\right)(o)$$

Using the ruls of differentiation to the determinant function, one can see that

$$\delta K = \text{Det } (h_{ij}) \left(\sum_{i,j} h_{ij} \ \delta h_{ij} + 2\phi \ \delta^{ij} \ h_{ij}\right)$$

From (9), (10), (11) and (44) we have

$$\delta K = K \sum_{i,j} h_{ij}(\Lambda_{ij} - \phi \ \gamma_{ij}) + 2 \ \phi \ m \ H \ K \tag{45}$$

Thus, we have

**Theorem 2.** The variation of the Gauss curvature is given from (45). There is no variation for surface with null Gaussian curvature.

The contraction on both sides of (44) yields

$$\delta h_{ii} = \Lambda_{ii} - \phi \sum_k (h_{ik})^2 \tag{46}$$

Summing over i and using (13) we get

$$\delta \sum_i h_{ii} = \Delta \ \phi - \phi \ |II|^2 \tag{47}$$

The mean curvature function $\bar{H}$ of the hypersurface $\bar{M}$ is defined from $m\bar{H} = \bar{g}^{ij}(t)\bar{h}_{ij}(t)$, where $\bar{h}_{ij}(t) = \langle d^2\bar{X}, \frac{\bar{e}_{m+1}}{\sqrt{\bar{g}}} \rangle (e_i, e_j)$ are the 2nd fundamental quantities of the variation $\bar{X}$ and $\bar{h}_{ij}(o)=h_{ij}$. Thus, the variation $\delta H$ of the mean curvature function H is given from

$$m\,\delta H = \frac{\partial}{\partial t}\left(\bar{g}^{ij}(t)\,\bar{h}^{ij}(t)\right)(o)$$

Explicitly

$$m\,\delta H = \delta h_{ij}.\ \delta^{ij} + h_{ij}.\ \delta g^{ij}$$

Taking account (13) and (31), we have

$$m\,\delta H = \sum_i \delta h_{ii} + 2\phi\,|II|^2$$

and from (45), (12) it follows that

$$m\delta H = \Delta\phi + \phi\,(m^2H^2 - R) \tag{48}$$

Thus, we have

**Theorem 3**. The variation of the mean curvature is given from (48).

Now we shall give a necessary and sufficient condition for the immersion X to have stability with respect to

$$\int_M H^c * 1,\ c \geq o\ .$$

From (33) and (48) one can see that

$$\delta \int_M H^c{*}1 = \frac{1}{m}\int_M \left\{cH^{c-1}\Delta\,\phi + m^2\phi\,(c-1)H^{c+1} - c\,\phi\,H^{c-1}\,R\right\}*1$$

Since the immersion X is compact, we use the Green theorem and thus we have

$$\delta\int_M H^c*1 = \frac{1}{m}\int_M \phi \left\{c\Delta H^{c-1}+m^2(c-1)H^{c+1}-cH^{c-1}R\right\}*1 \tag{49}$$

The immersion X is stable with respect to $\int_M H^c *1$ if and only if the right-hand side of(49)is identically zero $\forall\ \phi\ \varepsilon\ \Lambda^o(M)$, that is

$$c\Delta H^{c-1}+m^2(c-1)H^{c+1}-cH^{c-1}R=o \tag{50}$$

The integral condition (50) has been obtaind in [6] using the methods of tensor analysis. Thus, we reach to the proof of the main theorem

**Theorem 4.** The oriented closed immersion $X : M \longrightarrow E^{m+1}$ is stable with respect to the integral $\int_M H^c*1$ if and only if the condition (50) is valid.

Putting c=m=2 in (50), it is easy to see that the closed surfaces in $E^3$are the solutions of the differential equation [11]

$$\Delta H +2H\ (H^2-K)=o \tag{51}$$

We take the tours in $E^3$ as an application of this result, where

$$\omega^1= bdu^1, \omega^2=f(u^1)du^2,\ f(u^1) =a + b \cos u^1$$

$$\omega_1^2 =-\sin u^1du^1, \omega_1^3 = b\ \omega^1, \omega_2^3 = \cos u^1\ f(u^1)\ \omega^2$$

$$*\ \omega^1 = \omega^2, *\ \omega^2= \omega^1$$

$$H = \frac{a+2b \cos u^1}{2bf(u^1)}\ ,\quad K = \frac{\text{Cos} u^1}{bf(u^1)}$$

From (20) one can obtain

$$\Delta H = \frac{-ab-a^2\cos u^1}{2b^2\left(f(u^1)\right)^3}$$

Thus, the condition (51) is valid for the tours such that

$$a = b\sqrt{2}$$

**6-Stability of hypersurfaces with constant mean curvature.**

Let $X: M \longrightarrow E^{m+1}$ has a constant mean currature, the immersion X is stable with respect to the integral $\int_M H^c * 1$ if

$$m^2(c-1)H^C - cR = o \qquad (52)$$

which characterizes the Wiengarten surfaces. Thus, we have

**Lemma 2.** The hypersurfaces with constant mean curvatures which are stable with respect to the integral $\int_M H^c * 1$ are of type Wiengarten.

Here, some main results on hypersurfaces of constant mean curvature are summerized. In the case where C=o, we have

**Lemma 3.** The hypersurfaces which are stable with respect to $\int_M *1$ are the minimal hypersurfaces.

In the case where C=1, the condition (52) is degenerate to R=2K=o, which characterizes the developable surfaces. Thus, we have

**Lemma 4** The developable surfaces are the only hypersurfaces with constant mean curvature and stable with respect to the integral $\int_M H*1$ .

These results have been proved in[12] using the methods of the associated characteristic Euler Lagrange vector. From the above lemmas, we have a classificiation given from

**Theorem 5.** The immersion with constant mean curvature display many similarities with minimal immersions in $E^{m+1}$. They are both solution to the variational problem of minimizing the area function.

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