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Benard-Marangoni convection with a deformable surface

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1 Introduction

We consider a model of Bénard-Marangoni convection using the Boussinesq equations for the velocity, pressure and temperature :

$$\frac{1}{\mathrm{P_r}}(u_t + u\cdot\nabla u) + \nabla p \;=\; \Delta u - \rho(T)\nabla z\ ,\quad \nabla\cdot u \;=\; 0\ ,\quad T_t + u\cdot\nabla T \;=\; \Delta T$$

in the strip $\{\,-\infty < x < \infty,\ 0 < z < 1+\eta(t,x)\,\}$, where $\rho(T) = \mathrm{G} - \mathcal{R}_a T$ is assumed for the density of the fluid, $\mathrm{P_r}$ is the Prandtl number and $\mathcal{R}_a$ is the Rayleigh number.

We consider the boundary condition $u = 0$ and $T = 1$ on the bottom . The top surface $1+\eta(t,x)$ is deformable and satisfies the kinematic boundary condition

$$\eta_t \;=\; u_3 - u_h\cdot\nabla_h\eta\,\big|_{z=1+\eta(t,x)}\ ,$$

and the stress balance equation is satisfied on it :

$$\Big((p - p_{\mathrm{air}})I - (\nabla u + {}^t\nabla u)\Big)\cdot n \;=\; \sigma H n - (\tau\cdot\nabla)\sigma\tau.$$

Here n and τ are the normal and tangential unit vector of the surface respectively and H is the mean curvature of the surface . The surface stress σ is assumed to be given by

$$\sigma \;\equiv\; \mathrm{W} - \mathcal{M}_a T + \mathrm{V_i}\,(\tau\cdot\nabla)(u\cdot\tau),$$

where $\mathcal{M}_a$ is the Marangoni number and $\mathrm{V_i}$ is the surface viscosity. We also have the boundary condition of temperature $n\cdot\nabla T + \mathrm{B_i}\,T = -1$ on the upper surface .

These equations have a stationary solution

$$\eta = 0\ , \quad u = 0\ , \quad T = \tilde{T}(z) \equiv 1 - z\ , \quad p = \tilde{p}(z) \equiv -\frac{\mathcal{R}_a}{2}(z-1)^2 - \mathrm{G}(z-1) + p_{\mathrm{air}}$$

representing the purely heat conducting state.

We will consider the stability of this stationary state under the assumption that all perturbations are periodic in x . The perturbation (u, p, θ, η) satisfies a nonlinear system, which is transformed to the following quasilinear system on the fixed domain by Beale's transformation provided the positivity of W : (See [1] and [5].)

$$\frac{1}{\mathrm{P_r}} u_t + \nabla p - \Delta u - \mathcal{R}_a \theta \nabla z = F\ , \quad \nabla \cdot u = 0\ , \quad \theta_t - \Delta\theta - u_3 = F_0 \quad \text{in } \Omega\ , \tag{1}$$

$$\eta_t - u_3|_{S_F} = 0\ ,$$

$$pn - (\nabla u + \nabla u^t)\cdot n - (-\mathrm{W}\Delta_h + \mathrm{G})\,\eta\, n - (\mathcal{M}_a \nabla_h(\theta - \eta) - \mathrm{V_i}\,\Delta_h u_h)\tau = f\ ,$$

$$\theta_z + \mathrm{B_i}\,(\theta - \eta) = f_0 \quad \text{on } S_F\ , \tag{2}$$

$$u = 0\ , \theta = 0 \quad \text{on } S_B\ . \tag{3}$$

Here the linear terms are gathered in the left-hand side of the equations, and $\Omega = \{0 < z < 1\}$ is the domain occupied by the fluid at the heat conducting state and $S_F = \{z = 1\}$ and $S_B = \{z = 0\}$ are its boundaries.

We use Sobolev spaces $H^r(\Omega)$ and $H^r(S_F)$ and denote their norm by $\|\cdot\|_r$ and $\|\cdot\|_{r,S_F}$ respectively, and we use the function spaces.

$$\begin{aligned}
K^r(\Omega \times (0,\infty)) &\equiv H^0(0,\infty; H^r(\Omega)) \cap H^{r/2}(0,\infty; H^0(\Omega)) \\
K^r_{-\gamma}(\Omega \times (0,\infty)) &\equiv \{f : e^{\gamma t} f \in K^r(\Omega \times (0,\infty))\}\ , \\
K^{r,\frac{1}{2}}(S_F \times (0,\infty)) &\equiv H^0(0,\infty; H^{r+\frac{1}{2}}(S_F)) \cap H^{r/2}(0,\infty; H^{\frac{1}{2}}(S_F)) \\
K^{r,\frac{1}{2}}_{-\gamma}(S_F \times (0,\infty)) &\equiv \{f : e^{\gamma t} f \in K^{r,\frac{1}{2}}(S_F \times (0,\infty))\}\ .
\end{aligned}$$

2 Existence for nonlinear problems

We have the following for the Laplace transform of the solution of the linearized system .

Proposition 1 *Assume $r \geq 2$. For small constants $\mathcal{R}_a$ and $\mathcal{M}_a$, there is a positive constant γ such that for non-zero λ in $\{Re\ \lambda > -\gamma\}$ and data F, $F_0 \in H^{r-2}$, f, $f_0 \in H^{r-2+\frac{1}{2}}(S_F)$, there is a unique solution $u, \theta \in H^r, \nabla p \in H^{r-2}, \eta \in H^{r+\frac{1}{2}}(S_F)$ and this solution satisfy*

$$\|u,\theta\|_r + |\lambda|^{\frac{r}{2}}|u,\theta| + \|\nabla p\|_{r-2} + |\lambda|^{\frac{r-2}{2}}|\nabla p| + \|\eta\|_{r+\frac{1}{2},S_F} + |\lambda|^{\frac{r+\frac{1}{2}}{2}}|\eta|_{S_F}$$
$$\leq C\left(\|F,F_0\|_{r-2} + |\lambda|^{\frac{r-2}{2}}|F,F_0|\right) + C\left(\|f,f_0\|_{r-\frac{3}{2},S_F} + |\lambda|^{\frac{r-2}{2}}\|f,f_0\|_{\frac{1}{2},S_F}\right).$$

Here C does not depend on λ. When $\mathrm{V_i}$ is positive, $u_h|_{S_F} \in H^{r+\frac{1}{2}}(S_F)$ and also $\|u_h\|_{r+\frac{1}{2},S_F} + |\lambda|^{\frac{r-2}{2}}\|u_h\|_{2+\frac{1}{2},S_F}$ can be estimated by the right hand side above.

The nonlinear system has F, F_0, f, f_0 in (1)(2) which are quadratic or higher order terms of the unknowns and their derivatives. We have the following for small $\mathcal{R}_a$ and $\mathcal{M}_a$.

Theorem 1 (See [5].) *Assume $\frac{5}{2} < r < 3$.*
(1) When $\mathrm{V_i} > 0$, for small initial conditions $\tilde{u}_0, \tilde{\theta}_0 \in H^{r-1}(\Omega)$, $\eta_0, \tilde{u}_h|_{S_F} \in H^{r-\frac{1}{2}}(S_F)$ which satisfy conditions $\nabla\cdot\tilde{u}_0 = 0$, $\tilde{u}_0, \tilde{\theta}_0\Big|_{S_B} = 0$ and $\int \eta_0 dx = 0$, there exists a global in time solution $u, \theta \in K^r_{-\gamma}, p \in K^{r-2}_{-\gamma}, \eta,\ u_h|_{S_F} \in K^{\frac{r+\frac{1}{2}}{2}}_{-\gamma}(S_F)$.
(2) When $\mathrm{V_i} = 0$, for small initial conditions $\tilde{u}_0, \tilde{\theta}_0 \in H^{r-1}(\Omega)$, $\eta_0 \in H^{r-\frac{1}{2}}(S_F)$ which satisfy conditions $\nabla\cdot\tilde{u}_0 = 0$, $\tilde{u}_0, \tilde{\theta}_0\Big|_{S_B} = 0$ and $\int \eta_0 dx = 0$, there exists a global in time solution $u, \theta \in K^r_{-\gamma}, p \in K^{r-2}_{-\gamma}, \eta \in K^{\frac{r+\frac{1}{2}}{2}}_{-\gamma}(S_F)$.

Remark *The solution constructed in the theorem decays exponentially. Thus, the results say that the purely heat conducting state is stable for small $\mathcal{R}_a$ and $\mathcal{M}_a$.*

3 Eigenvalue problems

Here we want to increase Rayleigh number or Marangoni number in the system (1)-(3) to investigate the instability of the purely heat conducting state. Rewrite the system using the stream function Ψ for the linearized perturbed flow.

$$\Psi = 0\,, \quad \Psi_z = 0\,, \quad \theta = 0 \qquad \text{on} \qquad z = 0\,. \tag{4}$$

$$\Delta\Psi_t + \mathrm{P_r}\mathcal{R}_a\theta_x = \mathrm{P_r}\Delta^2\Psi\,, \quad \theta_t + \Psi_x = \Delta\theta \qquad \text{in} \qquad 0 < z < 1\,. \tag{5}$$

$$\begin{aligned}
\eta_t + \Psi_x &= 0\,, \\
\Psi_{zz} - \Psi_{xx} + \mathcal{M}_a(\theta_x - \eta_x) + \mathrm{V_i}\Psi_{zxx} &= 0\,, \\
-\frac{1}{\mathrm{P_r}}\Psi_{zt} + 3\Psi_{xxz} + \Psi_{zzz} + \mathrm{W}\eta_{xxx} - \mathrm{G}\eta_x &= 0\,, \\
\theta_z + \mathrm{B_i}(\theta - \eta) &= 0 \qquad \text{on} \qquad z = 1\,.
\end{aligned} \tag{6}$$

We can consider Ψ , θ and η of the following form because of the periodicity condition in x

$$\Psi = \varphi(z)\exp(\,\mathrm{i}nx + \lambda t\,)\,,$$

$$\theta = \theta(z)\exp(\,\mathrm{i}nx + \lambda t\,)\,, \qquad \eta = \eta\exp(\,\mathrm{i}nx + \lambda t\,)\,.$$

Thus the instability problem (4)-(6) is reduced to the eigenvalue problem of the ODE for φ , θ and η :

$$\varphi(0) = 0\,, \quad \varphi'(0) = 0\,, \quad \theta(0) = 0 \qquad \text{on} \qquad z = 0\,. \tag{7}$$

$$\begin{aligned}
\mathrm{P_r}(\varphi'''' - 2n^2\varphi'' + n^4\varphi) &= \mathrm{P_r}\mathcal{R}_a\mathrm{i}n\theta + \lambda(\varphi'' - n^2\varphi)\,, \\
\theta'' - n^2\theta &= \mathrm{i}n\varphi + \lambda\theta \qquad \text{in} \qquad 0 < z < 1\,.
\end{aligned} \tag{8}$$

$$\begin{aligned}
\lambda\eta + \mathrm{i}n\varphi(1) &= 0\,, \\
\varphi''(1) - \mathrm{V_i}n^2\varphi'(1) + n^2\varphi(1) + \mathcal{M}_a\mathrm{i}n(\theta(1) - \eta) &= 0\,, \\
\varphi'''(1) - \frac{1}{\mathrm{P_r}}\lambda\varphi'(1) - 3n^2\varphi'(1) - (\mathrm{W}n^2 + \mathrm{G})\mathrm{i}n\eta &= 0\,, \\
\theta'(1) + \mathrm{B_i}(\theta(1) - \eta) &= 0 \qquad \text{on} \qquad z = 1\,.
\end{aligned} \tag{9}$$

By this formulation, the original problem of stability is reduced to investigate the behavior of the real part of the eigenvalue λ when the parameters $\mathcal{R}_a$, $\mathcal{M}_a$ and n vary. The key problem is to find the critical Rayleigh number

$$\mathcal{R}_a = \mathcal{R}_c \qquad \text{at which} \qquad \lambda = \pm\mathrm{i}\omega \quad (\,\omega \in \mathbf{R}\,) \tag{10}$$

for certain periodicity in x , namely n fixed , and further to show

$$\left.\frac{\partial \operatorname{Re}\lambda}{\partial \mathcal{R}_a}\right|_{\mathcal{R}_a=\mathcal{R}_c} > 0 . \tag{11}$$

By this motion of eigenvalue and by the fact that the original evolution problem for the linearized system forms a sectorial operator, we see that a sufficient condition given in the papers [2], [3], [6] and [8] for the occurence of the stationary bifurcation or the Hopf bifurcation for the infinite dimensional system holds . Hence, we see that
the heat conducting state becomes unstable for $\mathcal{R}_a > \mathcal{R}_c$ *and the stationary bifurcation or the Hopf bifurcation occurs at* $\mathcal{R}_a = \mathcal{R}_c$ *according as* $\omega = 0$ *or* $\omega \neq 0$ *respectively.*

Criterion for existence of critical eigenvalue

In order to justify the above argument about the instability and the bifurcation we use the method given in [9] to prove the existence of the purely imaginary eigenvalue and the critical Rayleigh number in a small neighbourhood of the computed purely imaginary eigenvalue and critical Rayleigh number based on the Newton method.
To obtain the eigenvalue and the eigenfunction for (7) – (9), we use the shooting method, i.e., we consider the fundamental solutions of the initial value problem for (8) in $z \geq 0$ and express the eigenfunction by the solutions as

$$\varphi = a\varphi_1(z) + b\varphi_2(z) + c\varphi_3(z) , \quad \theta = a\theta_1(z) + b\theta_2(z) + c\theta_3(z) , \quad z > 0, \tag{12}$$

where $\varphi_j(z)$, $\theta_j(z)$, $j = 1,2,3$ satisfy (8) in $z > 0$ and the initial conditions at $z = 0$

$$\begin{cases} \varphi_j(0) = 0 , & \varphi_j'(0) = 0 , & \theta_j(0) = 0 , & j = 1,2,3, \\ \varphi_1''(0) = 1 , & \varphi_1'''(0) = 0 , & \theta_1'(0) = 0 , \\ \varphi_2''(0) = 0 , & \varphi_2'''(0) = 1 , & \theta_2'(0) = 0 , \\ \varphi_3''(0) = 0 , & \varphi_3'''(0) = 0 , & \theta_3'(0) = 1 , \end{cases} \tag{13}$$

a , b and c are constants to be determined. In order that the function (12) is the eigenfunction, it must satisfy the condition (9) . This condition is written as follows

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} \eta \\ a \\ b \\ c \end{pmatrix} = 0 \,, \tag{14}$$

where the coefficients a_{ij} are explicitly given by $\varphi_k(1)$, $\varphi_k'(1)$, $\varphi_k''(1)$, $\varphi_k'''(1), \theta_k(1)$, $\theta_k'(1)$ $k = 1, 2, 3$. In order that (12) is nontrivial, it is necessary that

$$\det\, A \;\equiv\; \det(a_{ij}) \;=\; 0 \,, \tag{15}$$

and this is sufficient for (12) to be the eigenfunction. Thus, we now come to search the values of $\mathcal{R}_a = \mathcal{R}_c$, $\lambda = i\omega_c$ satisfying (15) , for the fixed parameters $\mathcal{M}_a, \mathrm{P_r}$, G and n . We define

$$\det\, A \;=\; \mathcal{F}(\mathcal{R}_a, \lambda\,;\; \mathcal{M}_a, \mathrm{P_r}, \mathrm{G}, n)\,.$$

Noting that (15) can be rewritten as

$$\mathcal{F}(\mathcal{R}_a, \lambda) \;=\; \mathcal{F}(\mathcal{R}_0, \lambda_0) \;+\; \frac{\partial \mathcal{F}}{\partial \mathcal{R}_a}(\mathcal{R}_a - \mathcal{R}_0) \;+\; \frac{\partial \mathcal{F}}{\partial \lambda}(\lambda - \lambda_0) \;=\; 0 \,,$$

we can state our criterion for existence of the critical eigenvalue based on the simplified Newton method as follows :

Theorem *Suppose, for a small $\varepsilon > 0$, there exist $\mathcal{R}_0$ and λ_0 such that*

$$\|\mathcal{F}(\mathcal{R}_0,\ \lambda_0)\| \;<\; \varepsilon\,. \tag{16}$$

Put

$$L_0 \;\equiv\; \left(\overline{\frac{\partial \mathcal{F}}{\partial \mathcal{R}_a}}(\mathcal{R}_0, \lambda_0) \;,\; \overline{\frac{\partial \mathcal{F}}{\partial \lambda}}(\mathcal{R}_0, \lambda_0) \right) \,, \tag{17}$$

where the bar means an appropriate approximation of the quantity. Suppose further that, for a small δ, there is a ρ_1 such that the estimate

$$\|\, D\mathcal{F}(\mathcal{R}_a\,,\ \lambda) \;-\; L_0 \,\| \;<\; \delta \tag{18}$$

holds for any $(\mathcal{R}_a\ ,\ \lambda)$ such that

$$(\mathcal{R}_a \;-\; \mathcal{R}_0)^2 \;+\; |\lambda \;-\; \lambda_0|^2 \;<\; \rho_1^2\,.$$

For ε, ρ_1, δ and L_0 as above, if it holds that

$$\| L_0^{-1} \| \left(\frac{\varepsilon}{\rho_1} + \delta \right) \le 1 , \tag{19}$$

then there exist some $\mathcal{R}_c$ and λ_c in the ρ_1-neighborhood of $\mathcal{R}_0$ and λ_0 satisfying

$$\mathcal{F}(\mathcal{R}_c , \lambda_c) = 0 . \tag{20}$$

To utilize this criterion to our problem, we need to justify the following steps:
(i) To find appropriate values $\mathcal{R}_0$ and λ_0, we use the shooting method , i.e., an approximate eigenvalue, eigenfunction and critical Rayleigh number of the problem (7)-(9) are obtained by numerical computation using the fourth order Taylor finite difference scheme for the fundamental solutions and the Newton method.
(ii) To estimate ε we need the interval analysis by a computer software for the bound of round-off errors in the computation of the fundamental solutions and the theory of pseudo trajectory to estimate the difference between the genuine fundamental solutions and the numerically computed ones.
(iii) At this pair of $\mathcal{R}_0$, λ_0, find an approximate derivative L_0 and estimate the norm $\| L_0^{-1} \|$;
(iv) Estimate δ for which the estimate (18) holds in the ρ_1-neighborhood of $\mathcal{R}_0$ and λ_0;
(v) For these values in (i, ii, iii, iv), prove that the criterion (19) holds.

Following these steps we see that there exist the exact eigenvalue $\lambda = i\omega_c$ and the critical Rayleigh number $\mathcal{R}_a = \mathcal{R}_c$ for (7) - (9) in the ρ_1 -neighborhood of numerically computed values ($\mathcal{R}_0$, λ_0) in (i).

In order to verify the condition (11) we have to use such arguments as in [9] which uses the adojoint system of the equations to (7) - (9), which is given by the following :

$$\psi(0) = 0 , \quad \psi'(0) = 0 , \quad \zeta(0) = 0 \qquad \text{on} \quad z = 0 . \tag{21}$$

$$\mathrm{P_r}\,(\psi'''' - 2n^2\psi'' + n^4\psi) = \mathrm{P_r}\, i\, n \zeta + \overline{\lambda}(\psi'' - n^2\psi) , \tag{22}$$

$$\zeta'' - n^2 \zeta = \mathcal{R}_a i\, n \psi + \overline{\lambda} \zeta \qquad \text{in} \quad 0 < z < 1 .$$

$$\begin{aligned}
\overline{\lambda}\xi \;-\; i\,n\,\psi(1) \;+\; \frac{1}{\mathrm{W}\,n^2+\mathrm{G}}\{\mathcal{M}_a i\,n\,\psi'(1) \;+\; \mathrm{B_i}\,\zeta(1)\} &= 0\ , \qquad (23)\\
\psi''(1) + \mathrm{V_i}\,n^2\,\psi'(1) + n^2\psi(1) &= 0\ ,\\
\psi'''(1) - \frac{\overline{\lambda}}{\mathrm{P_r}}\psi'(1) - 3n^2\psi'(1) + (\mathrm{W}n^2+\mathrm{G})\,i\,n\,\xi &= 0\ ,\\
\zeta'(1) \;+\; \mathrm{B_i}\,\zeta(1) + \mathcal{M}_a i\,n\,\psi'(1) &= 0 \qquad \text{on} \qquad z = 1\ .
\end{aligned}$$

For notational convenience we write the eigenvalue λ_c and the eigenfunction $\Phi = (\eta, \varphi, \theta)$ with the critical Rayleigh number $\mathcal{R}_c$ for the system of equations (8) and the boundary conditions (7), (9) as

$$L\,\Phi \;=\; 0 \quad \text{and} \quad B\,\Phi \;=\; 0\ . \qquad (24)$$

Let us denote the eigenvalue λ_c and the eigenfunction $\Psi = (\xi, \psi, \zeta)$ which satisfy the the adjoint problem (21) - (23)

$$L^*\Psi \;=\; 0 \quad \text{and} \quad B^*\Psi \;=\; 0\ .$$

Taking the derivative of (24) with respect to the Rayleigh number and the $L^2(0,1)$-inner product with Ψ, we obtain

$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_c} \;=\; -\,\frac{\left(\frac{\partial L}{\partial\mathcal{R}}\,\Phi\,,\,\Psi\right)_{L^2}}{\left(\frac{\partial L}{\partial\lambda}\,\Phi\,,\,\Psi\right)_{L^2}}\ .$$

Example 1. We take $\mathrm{G} = 400$, $\mathrm{W} = 0$, $\mathrm{P_r} = 1$, $\mathrm{V_i} = 0$, $\mathrm{B_i} = 0$ and $\mathcal{M}_a = 0$.

n	λ	$\mathcal{R}_0$	$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right\|_{\mathcal{R}=\mathcal{R}_0}$
1.0	0.0	1108.1082	0.00601 956
2.0	0.0	670.28924	0.01176 430
2.08558	0.0	668.99825	0.01227 492
3.0	0.0	782.78265	0.01623 625
4.0	0.0	1131.0427	0.01723 006

In Figure 1 the four curves correspond to the neutral curves for $n = 2, 3, 1, 4$. Figure 2 shows the neutral curve for the smallest Rayleigh number by the proper choice of $n \in \mathbf{R}$. For this gravity G we see that the stationary bifurcation occurs when $\mathcal{R}_a$ or $\mathcal{M}_a$ increases across this curve.

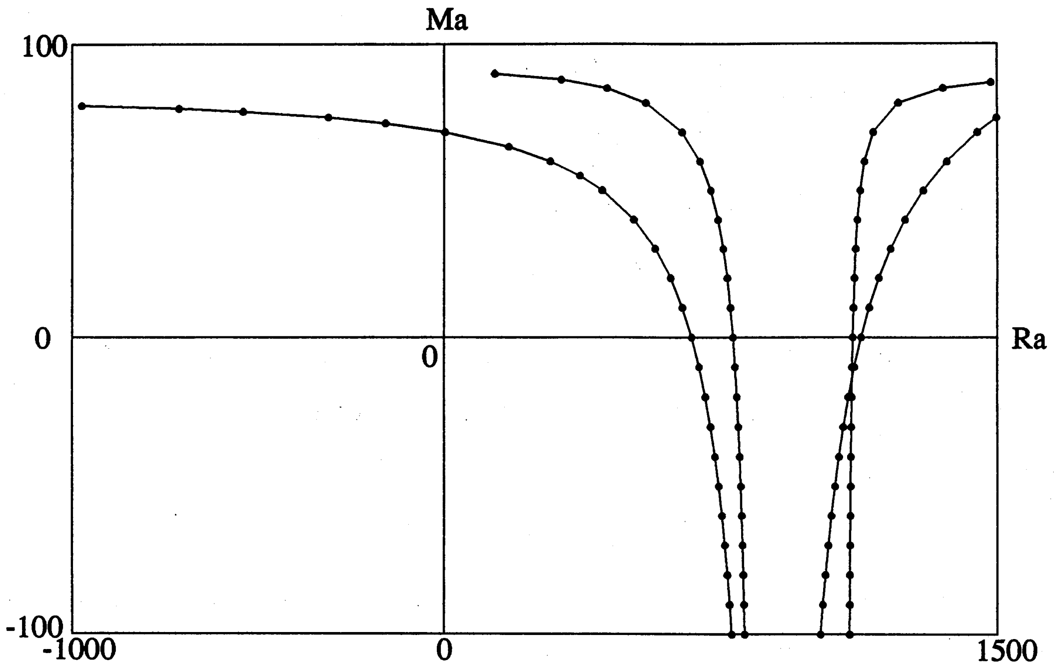

Figure 1

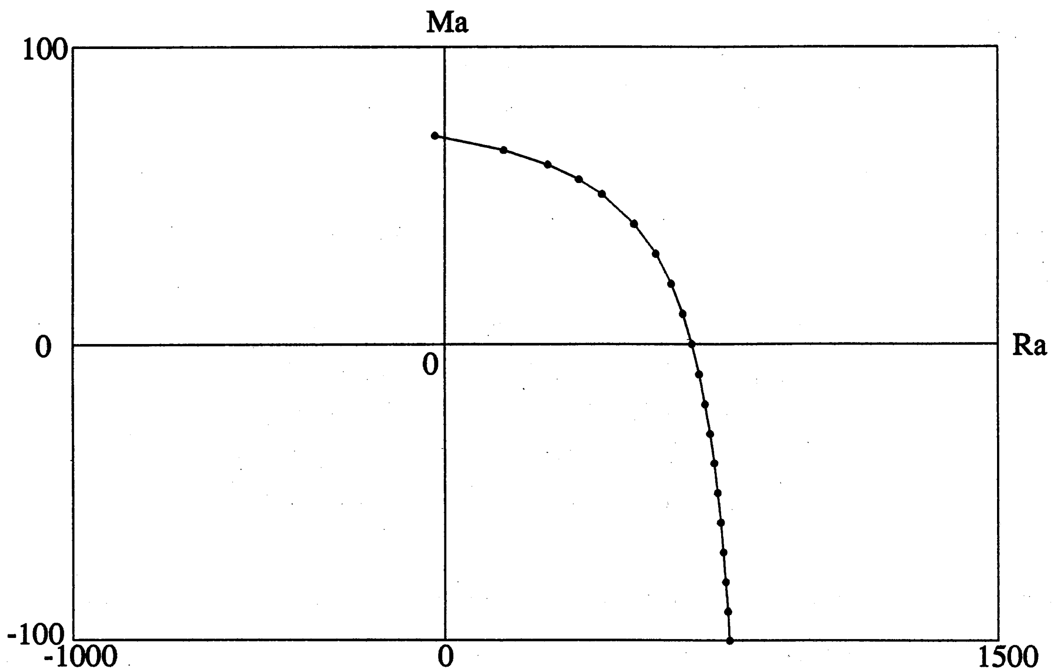

Figure 2

Example 2. We take $G = 100$, $W = 0$, $P_r = 1$, $V_i = 0$, $B_i = 0$ and $\mathcal{M}_a = 0$.

n	λ_0	$\mathcal{R}_0$	$\frac{\partial\lambda}{\partial\mathcal{R}}\Big\vert_{\mathcal{R}=\mathcal{R}_0}$
0.5	i × 2.91543 59	447.81500	0.00323 041 − i × 0.00182 891
0.93201	i × 4.41412	390.84911	0.00739 433 − i × 0.00639 798
1.0	i × 4.55206 09	391.30728	0.00739 433 − i × 0.00639 798
2.0	i × 5.15597 17	424.67690	0.01092 757 − i × 0.01304 263
3.0	i × 5.83570 00	514.01005	0.01003 548 − i × 0.01216 266
4.0	i × 6.52511 06	749.27424	0.00818 050 − i × 0.00902 531

Figure 3 and 4 show the neutral curves for $n = 1$ and for $n = 2$ respectively. The white circle corresponds to the purely imaginary eigenvalue $\lambda = i\omega$, and the black ones do to $\lambda = 0$. Figure 5 shows the neutral curves for the smallest Rayleigh number by the proper choice of $n \in \mathbf{R}$. For this gravity G we see that the Hopf bifurcation occurs for $\mathcal{M}_a \geq -35$ and that the stationary bifurcation does for $\mathcal{M}_a \leq -45$ when $\mathcal{R}_a$ increases across the corresponding curve.

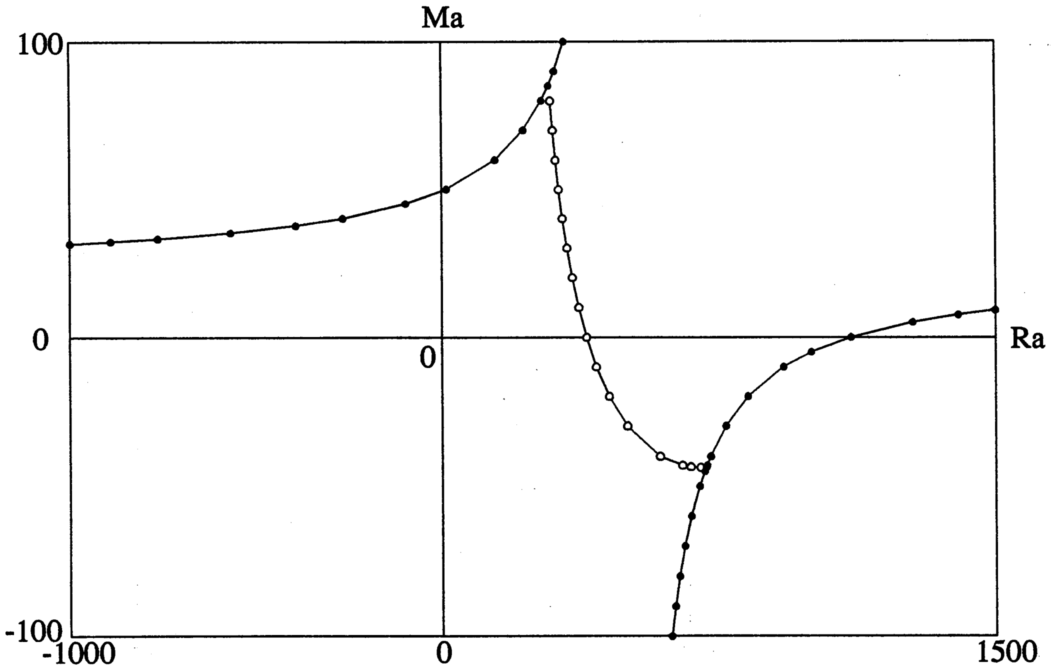

Figure 3

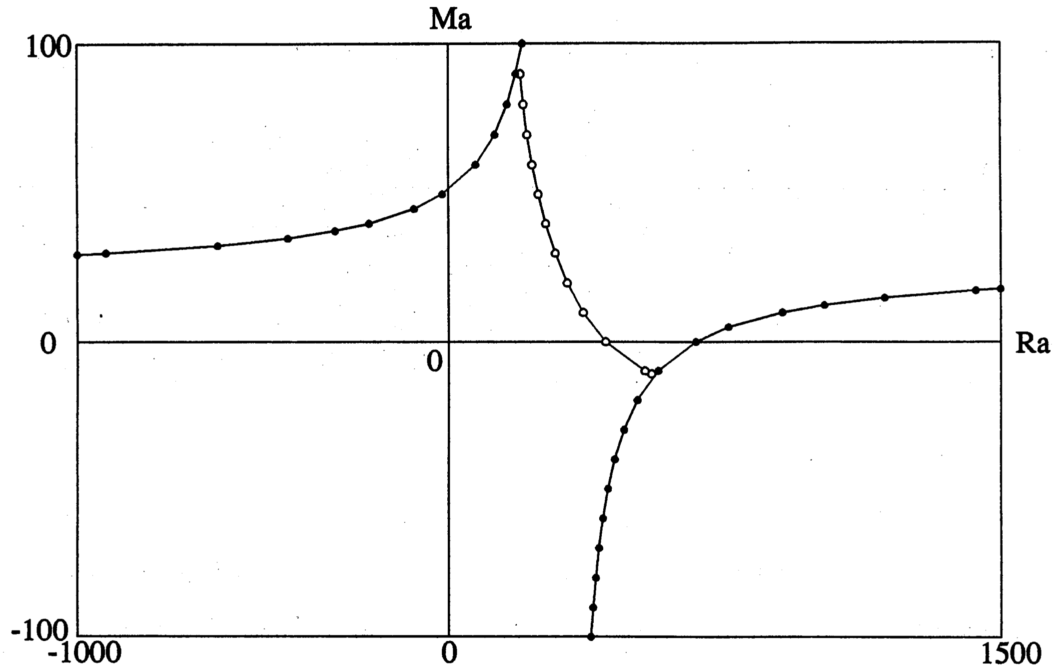

Figure 4

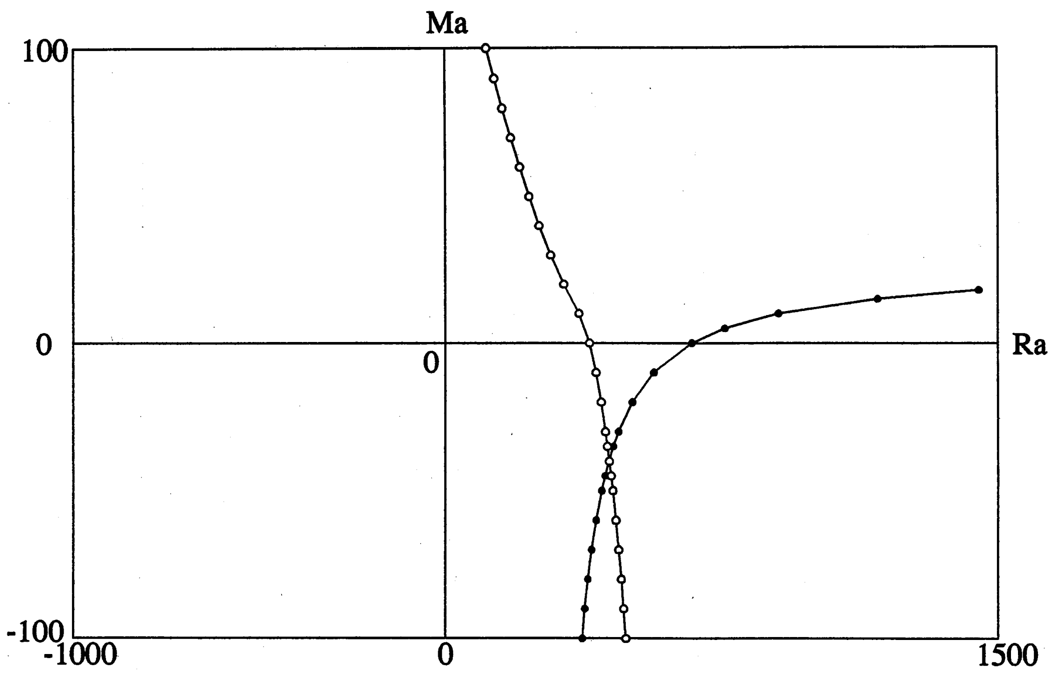

Figure 5

Example 3. We give another interesting example taking $G = 100$, $W = 0$, $P_r = 1$, $V_i = 0$, $B_i = 0$ and $\mathcal{M}_a \approx -43.73$, and $n = \pm 1$.

λ_0	$\mathcal{R}_0$	M_0	$\frac{\partial \lambda}{\partial \mathcal{R}}\Big\vert_{\mathcal{R}=\mathcal{R}_0}$
$\pm i \times 0.29905\,335$	712.52096	−43.735	$-0.00047\,645 \mp i \times 0.20119\,028$
0.0	713.26319	−43.735	162.83867
0.0	713.27868	−43.73	−352.57294

Thus it suggests an existence of the double zero eigenvalue of the determinant at $\mathcal{R}_a \approx 713$, $\mathcal{M}_a \approx -43.73$.

Example 4. We give another example taking $G = 100$, $W = 0$, $P_r = 1$, $V_i = 0$, $B_i = 0$ and $\mathcal{M}_a \approx 8$, and $n =$ 1 or 2 .

n	$\lambda_{0'}$	$\mathcal{R}_0$	M_0	$\frac{\partial \lambda}{\partial \mathcal{R}}\Big\vert_{\mathcal{R}=\mathcal{R}_0}$
1.0	$i \times 4.43652\,66$	374.05568	8.0	$0.00814\,675 - i \times 0.00575\,477$
1.0	$i \times 4.43483\,51$	373.85774	8.1	$0.00815\,640 - i \times 0.00574\,645$
2.0	$i \times 5.75871\,35$	374.21235	8.0	$0.01251\,047 - i \times 0.00987\,391$
2.0	$i \times 5.76349\,04$	373.67749	8.1	$0.01252\,959 - i \times 0.00984\,381$

It suggests the neutral curves $\lambda = i\omega_1$ for $n = 1$ and $\lambda = i\omega_2$ for $n = 2$ intersect at $\mathcal{R}_a \approx 374$ and $\mathcal{M}_a \approx 8.0$.

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