Strained Vortex と乱流の統計法則
"Strained Vortex and Statistical Law of Turbulence"

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Abstract

In recent computer simulations it is revealed that homogeneous isotropic turbulence at high Reynolds number is regarded as the field to which the intense vorticity structures, called 'worms', are distributed randomly [1, 2, 3, 4]. It is also reported that 'worm' is approximated as Burgers' vortex under external straining [2, 4] and such intense structure causes intermittency [5]. So the statistical properties of a model velocity field associated with an isolated Burgers' vortex are studied. It is found that, in such a model field, the 2nd-order structure function shows the two-thirds law and the 3rd-order structure function shows the four-fifth law with a negative sign, which are consistent with the Kolmogorov's five-thirds law of the energy spectrum and the negative skewness of the velocity derivative respectively. Furthermore the exponents of higher-order structure functions are found to be consistent with the experimental data showing intermittency.

1 Statistical theory of homogeneous isotropic turbulence

1.1 Kolmogorov's theory (1941)

Let the velocity components $v_i(\boldsymbol{x})$ at each location $\boldsymbol{x}$ be center-valued random variables, while the brackets $\langle (\cdot) \rangle$ denote the ensemble average. We consider the instantaneous statistics of the velocity fluctuations at a fixed time. The center valued property $\langle v_i \rangle = 0$ can be satisfied by means of Galilean transformations which are symmetries of the *Navier-Stokes equation*,

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \nu \triangle \boldsymbol{v}, \tag{1.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \tag{1.2}$$

where ρ is the mass density, p the pressure and ν the *kinematic viscosity*.

We consider now turbulence of *homogeneous* and *isotropic* velocity field. The *longitudinal velocity increment* for the separation $\boldsymbol{\ell}$ and the *pth-order longitudinal structure function* are respectively defined as

$$\delta v_\ell(\boldsymbol{x}, \boldsymbol{\ell}) \equiv \delta \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell}) \cdot \frac{\boldsymbol{\ell}}{\ell}, \tag{1.3}$$

$$S_p(\ell) \equiv \langle (\delta v_\ell(\boldsymbol{x}, \boldsymbol{\ell}))^p \rangle, \tag{1.4}$$

where $\ell \equiv |\boldsymbol{\ell}|$. The dependence of S_p on $\boldsymbol{x}$ and $\boldsymbol{\ell}/\ell$ is dropped because of homogeneity and isotropy. One reason why we consider the longitudinal component is that, in the 2nd- and 3rd-order case, the structure functions made of other components are determined by the longitudinal one.

We introduce the *mean energy dissipation rate* ε defined as, in Cartesian coordinate system,

$$\varepsilon \equiv \left\langle \frac{1}{2} \nu \sum_{ij} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 \right\rangle, \tag{1.5}$$

and the *Kolmogorov dissipation scale* η as, using ν and ε,

$$\eta \equiv \left(\nu^3 / \varepsilon \right)^{1/4}. \tag{1.6}$$

Kolmogorov (1941) made the next two hypotheses in the case of homogeneous isotropic turbulence [6].

Kolmogorov's hypothesis of similarity I *In the range of scales $\ell \ll \eta$ which is called the dissipation range, all the statistical properties are uniquely determined by the scale ℓ, the viscosity ν and the mean energy dissipation rate ε.*

Kolmogorov's hypothesis of similarity II *In the range of scales $\ell \gg \eta$ which is called the inertial range, all the statistical properties are uniquely determined by ℓ and ε only.*

The 2nd-order longitudinal structure function is obtained from the above hypotheses as

$$S_2(\ell) = \frac{\varepsilon}{15\nu}\ell^2 \qquad (\ell \ll \eta) \tag{1.7}$$

$$= C\,\varepsilon^{2/3}\,\ell^{2/3} \quad (\ell \gg \eta), \tag{1.8}$$

where C is a universal dimensionless constant. The behavior of the 2nd-order structure function as the two-thirds power of the distance in the inertial range is called the *two-thirds law*, which holds experimentally for almost any turbulence [7]. This law corresponds to the *five-thirds law* of the energy spectrum.

The 3rd-order longitudinal structure function becomes

$$S_3(\ell) = -\frac{4}{5}\,\varepsilon\,\ell + 6\nu\frac{dS_2(\ell)}{d\ell}, \tag{1.9}$$

which is derived from the incompressible Navier-Stokes equation (1.1) and (1.2) [8]. In the inertial range $\ell \gg \eta$, the second term of (1.9) is dropped on account of Kolmogorov's second hypothesis, and so-called *four-fifth law* is obtained as

$$S_3(\ell) = -\frac{4}{5}\,\varepsilon\,\ell \quad (\ell \gg \eta). \tag{1.10}$$

For the higher-order structure functions, the following is said from the Kolmogorov's second hypothesis. Let $S_p(\ell)$ be the pth-order longitudinal structure function,

$$S_p(\ell) = C_p\,\varepsilon^{p/3}\,\ell^{p/3} \quad (\ell \gg \eta), \tag{1.11}$$

where C_p is the universal dimensionless constant. In general $S_p(\ell)$ is represented in the form of scaling law with the *scaling exponent* ζ_p as

$$S_p(\ell) \propto \ell^{\zeta_p} \quad (\ell \gg \eta). \tag{1.12}$$

Thus the exponent ζ_p of the Kolmogorov's theory (K41) is, from (1.11),

$$\zeta_p = \frac{p}{3}. \tag{1.13}$$

Experimentally and in the DNS, it is found that ζ_p increase less rapidly with p than the K41 value of (1.13). This fact is called *anomalous scaling* in the inertial range, which means the stronger fluctuations to exist in the smaller scales. Thus the various intermittency models subject to some statistics of the velocity increment or the local dissipation have been suggested after K41, which are summarized in Frisch [7].

1.2 Multifractal model

Parisi and Frisch (1985) presented the *multifractal model* in the following way [9]. Assuming that, in the limit of infinite Reynolds number, there is a set $\mathcal{S}_h \subset \mathbf{R}^3$ of the *fractal dimension* $D(h)$ for each *velocity scaling exponent* h as $\ell \to 0$, that is,

$$\delta v_\ell(\boldsymbol{x}) \propto \ell^h, \quad \boldsymbol{x} \in \mathcal{S}_h. \tag{1.14}$$

From this *multifractal* assumption, the pth-order structure function is expressed as

$$S_p(\ell) \propto \int d\mu(h)\,\ell^{ph+3-D(h)}, \tag{1.15}$$

where $d\mu(h)$ is the measure which gives the weight of the different exponents. In the limit $\ell \to 0$ the power-law with the smallest exponent dominates, thus

$$\zeta_p = \min_h(ph + 3 - D(h)). \tag{1.16}$$

1.3 Log-Poisson model

The *log-Poisson model* with no adjustable parameters was proposed recently by She *et al.* (1994) first based on a phenomenology involving a hierarchy of fluctuation structures associated with vortex filaments, and later the log-Poisson property was noted by Dubrulle (1994) and She *et al.* (1995) independently [10]. The resulting exponent

$$\zeta_p = \frac{p}{9} + 2 - 2\left(\frac{2}{3}\right)^{p/3} \tag{1.17}$$

is in good agreement with the result of the experiment by Anselmet *et al.* [11], for example.

2 A model-vortex field

2.1 Burgers' vortex

We consider a model-field of isolated Burgers' vortex whose circulation and axial stretching rate are the typical values in homogeneous isotropic turbulence, and investigate the structure functions of this field.

In the cylindrical coordinate system (r, θ, z), let the axisymmetric vorticity along z axis $\boldsymbol{\omega}$ and the velocity associated with vorticity $\boldsymbol{v}_\omega$ be respectively

$$\boldsymbol{\omega} = (0, 0, \omega(r)), \tag{2.1}$$

$$\boldsymbol{v}_\omega = (0, v_\theta(r), 0), \tag{2.2}$$

where $\omega = dv_\theta/dr + v_\theta/r$. Imposing the straining of the irrotational and solenoidal velocity field

$$\boldsymbol{v}_e = (-\frac{\sigma}{2}r, 0, \sigma z), \tag{2.3}$$

the total velocity $\boldsymbol{v} = \boldsymbol{v}_\omega + \boldsymbol{v}_e$ is given as

$$\boldsymbol{v} = (-\frac{\sigma}{2}r, v_\theta(r), \sigma z), \tag{2.4}$$

where σ is a positive constant. The *vorticity equation* reduced from the incompressible Navier-Stokes equation (1.1) and (1.2),

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla} \boldsymbol{v} + \nu \triangle \boldsymbol{\omega}, \tag{2.5}$$

has the exact steady solution in the form

$$\omega(r) = \omega_0 \exp\left(-\frac{\sigma r^2}{4\nu}\right) = \frac{\Gamma}{\pi r_{\rm B}^2} \exp\left(-\frac{r^2}{r_{\rm B}^2}\right), \tag{2.6}$$

$$v_\theta(r) = \frac{2\nu\,\omega_0}{\sigma r}\left\{1 - \exp\left(-\frac{\sigma r^2}{4\nu}\right)\right\} = \frac{\Gamma}{2\pi r}\left\{1 - \exp\left(-\frac{r^2}{r_{\rm B}^2}\right)\right\}, \tag{2.7}$$

where the *Burgers' radius* $r_{\rm B}$ defined as $1/e$ radius of $\omega(r)$ is

$$r_{\rm B} = 2\left(\frac{\nu}{\sigma}\right)^{1/2} \tag{2.8}$$

and the *circulation* Γ is

$$\Gamma \equiv \int_0^\infty \omega(r)2\pi r dr = \pi r_B^2 \omega_0. \tag{2.9}$$

This is called the *Burgers' vortex* [12]. The Burgers' vortex is also the asymptotic solution for the arbitrary initial axisymmetric vorticity distribution in the case of uniform strain as is above[13].

The *rate of strain tensors* defined as, in the Cartesian coordinate system,

$$e_{ij}(\boldsymbol{x}) \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}(\boldsymbol{x}) + \frac{\partial v_j}{\partial x_i}(\boldsymbol{x})\right), \tag{2.10}$$

is calculated as

$$\boldsymbol{e} = \begin{pmatrix} -\frac{\sigma}{2} & \frac{1}{2}\left(\frac{dv_\theta}{dr} - \frac{v_\theta}{r}\right) & 0 \\ \frac{1}{2}\left(\frac{dv_\theta}{dr} - \frac{v_\theta}{r}\right) & -\frac{\sigma}{2} & 0 \\ 0 & 0 & \sigma \end{pmatrix}. \tag{2.11}$$

The axial *stretching rate* of vorticity has a positive constant value as

$$\frac{\sum_{ij}\omega_i(r)\,e_{ij}(r)\,\omega_j(r)}{\sum_k \omega_k^2(r)} = \sigma. \tag{2.12}$$

The squared strength of $\boldsymbol{e}$ is evaluated as a function of r,

$$e^2(r) \equiv \sum_{ij} e_{ij}^2(r) = \frac{3}{2}\sigma^2 + \frac{1}{2}\left(\frac{dv_\theta}{dr}(r) - \frac{v_\theta(r)}{r}\right)^2, \tag{2.13}$$

thus the *local energy dissipation rate* is given as

$$\varepsilon_{\text{loc}}(r) \equiv 2\nu e^2(r) = \nu\left\{3\sigma^2 + \left(\frac{dv_\theta}{dr}(r) - \frac{v_\theta(r)}{r}\right)^2\right\}. \tag{2.14}$$

Because the azimuzal velocity profile is obtained by (2.7),

$$\frac{dv_\theta}{dr}(r) - \frac{v_\theta(r)}{r} = \frac{\Gamma}{\pi r_B^2}\left\{\exp\left(-r^2/r_B^2\right) - \frac{1 - \exp(-r^2/r_B^2)}{r^2/r_B^2}\right\}. \tag{2.15}$$

If Γ is large enough in comparison with σ, the energy of the Burgers' vortex is strongly dissipated around the Burgers' radius while at the center of vortex scarcely dissipated as shown in figure 1.
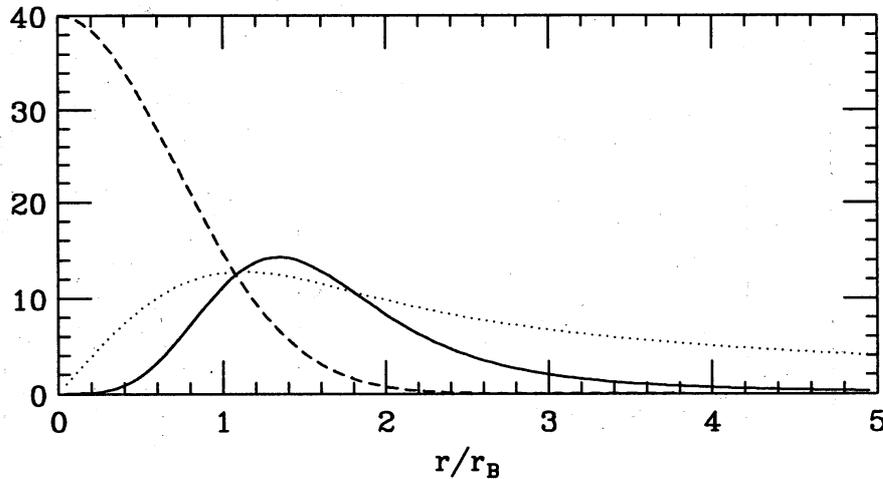
Figure 1: Dashed line, the axial vorticity ω; dotted line, the azimuthal velocity v_θ; solid line, the local energy dissipation rate $\varepsilon_{\mathrm{loc}}$ of the Burgers' vortex. $r_B = 1$, $\nu = 0.1$ and $R_\Gamma \equiv \Gamma/\nu = 1257$.

2.2 Calculation

Calculation of average of the velocity increment between two separated points is as follows, suggested first by Kambe *et al.* [14]. First choose a reference point $\boldsymbol{x} = (x, 0, z)$ in the Cartesian coordinate system, where the dependence on the y component can be dropped from the axisymmetry of the velocity field, and choose a running point $\boldsymbol{x} + \boldsymbol{\ell}$ at a distance ℓ from $\boldsymbol{x}$. Next the velocity increment between the two points is calculated. Last, the average of the pth-order longitudinal velocity increment over the sphere of radius ℓ centered at $\boldsymbol{x}$ is taken, and then volume averaging is carried out by shifting the reference point.

Let the location on the spherical surface centered at the reference point $\boldsymbol{x}$ be $\boldsymbol{\ell} = (\ell, \vartheta, \phi)$ in the spherical coordinate system as shown in figure 2 (a). The components of $\boldsymbol{\ell}$ in the Cartesian coordinate system are written as

$$\begin{cases} \ell_x = \ell \sin \vartheta \cos \phi \\ \ell_y = \ell \sin \vartheta \sin \phi \\ \ell_z = \ell \cos \vartheta. \end{cases} \tag{2.16}$$
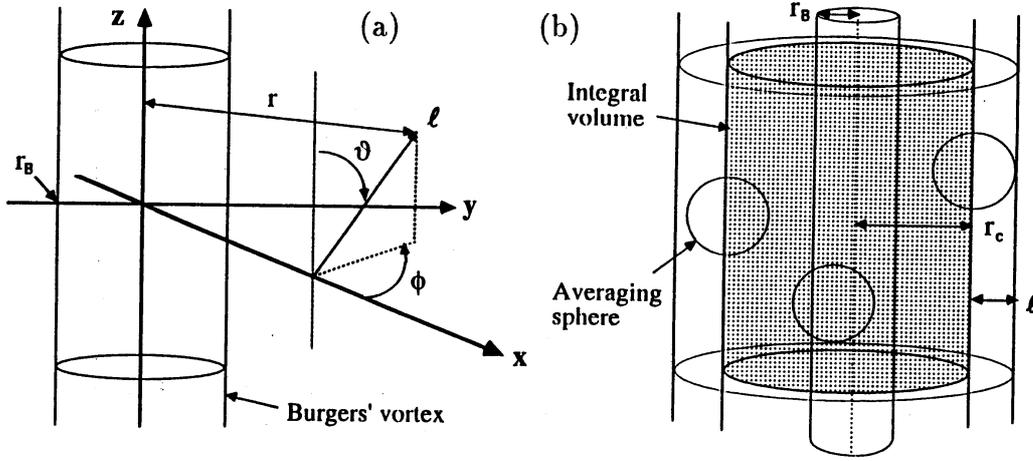
Figure 2: (a) The coordinate system at spherical average with respect to the running point $x + \ell$ where the reference point x and the separation length ℓ are fixed. (b) The integral space with respect to the reference point x.

The longitudinal velocity increment $\delta v_\ell(x, \ell)$ is calculated from the equations (1.3) and (2.7) as

$$
\begin{aligned}
\delta v_\ell(x, \ell, \vartheta, \phi) &= \sigma \ell \frac{3\cos^2 \vartheta - 1}{2} + \left(\frac{v_\theta(r)}{r} - \frac{v_\theta(x)}{x} \right) x \sin \vartheta \sin \phi \\
&= \frac{\nu}{r_B} \left[\frac{4\ell}{r_B} P_2(\cos \vartheta) + \frac{R_\Gamma}{2\pi} \left\{ \frac{1 - \exp(-r^2/r_B^2)}{r^2/r_B^2} \right.\right. \\
&\qquad \left.\left. - \frac{1 - \exp(-x^2/r_B^2)}{x^2/r_B^2} \right\} \frac{x}{r_B} \sin \vartheta \sin \phi \right], \quad (2.17)
\end{aligned}
$$

$$
r^2 = (x + \ell \sin \vartheta \cos \phi)^2 + (\ell \sin \vartheta \sin \phi)^2, \qquad (2.18)
$$

where P_2 is the second-order Legendre function and $R_\Gamma = \Gamma/\nu$ is the *Vortex Reynolds number* based on its total circulation. Note that a dependence on z is dropped in the expression at this moment. The spherical average is thus calculated as

$$
\langle (\delta v_\ell)^p \rangle_{\mathrm{sp}} (x, \ell) \equiv \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{0}^{\pi} (\delta v_\ell)^p \sin \vartheta \, d\vartheta \, d\phi. \qquad (2.19)
$$

Choosing some region of space as a sample space, we take an average $\langle(\cdot)\rangle$ over the space. One sample is the cylindrical space of radius r_c centered at the vortex axis shown in figure 2 (b). In the cylindrical coordinate system (r, θ, z), the average is

$$\langle(\cdot)\rangle \equiv \lim_{z_c \to \infty} \frac{1}{2\pi r_c^2 z_c} \int_{-z_c}^{z_c} \int_{-\pi}^{\pi} \int_0^{r_c} (\cdot) \, r \, dr \, d\theta \, dz. \qquad (2.20)$$

The pth-order longitudinal structure function is given as follows,

$$S_p(\ell, r_c) \equiv \frac{2}{r_c^2} \int_0^{r_c} \langle(\delta v_\ell)^p\rangle_{sp} \, x \, dx. \qquad (2.21)$$
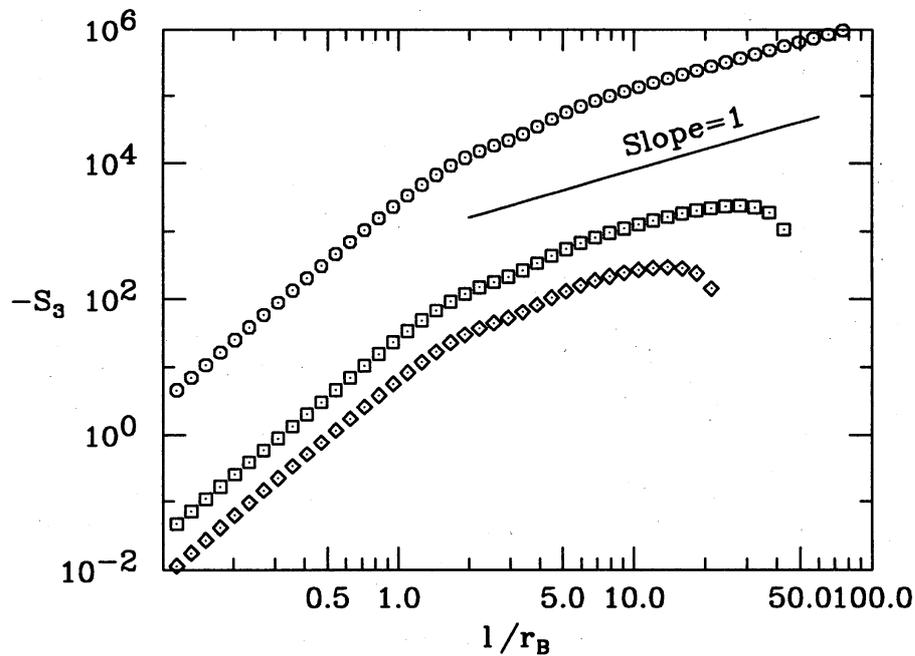

Figure 3: The 3rd-order structure function times -1. $\diamondsuit$, $R_\Gamma = 628$; $\square$, $R_\Gamma = 1257$; and $\odot$, $R_\Gamma = 12566$.

The structure functions are calculated at $R_\Gamma = 628, 1257$ and 12566. Upper limit of integral $r_c = 3r_B$. In figure 3 the 3rd-order structure functions

are shown. The inertial range begin around $\ell \sim r_B$ and is wider as R_Γ is larger. It is found that the 3rd-order exponent ζ_3 is about unity independent of R_Γ. This scaling is consistent with Kolmogorov's four-fifth law. In the figure 4 other exponents ζ_p up to $p = 25$ are shown in comparison with K41, log-Poisson model, DNS and experiment. The lager R_Γ is, the more ζ_p deviates from K41. The tendency of the even exponents to extend the odd ones in this model is the same in the experiment by Anselmet *et al.*.
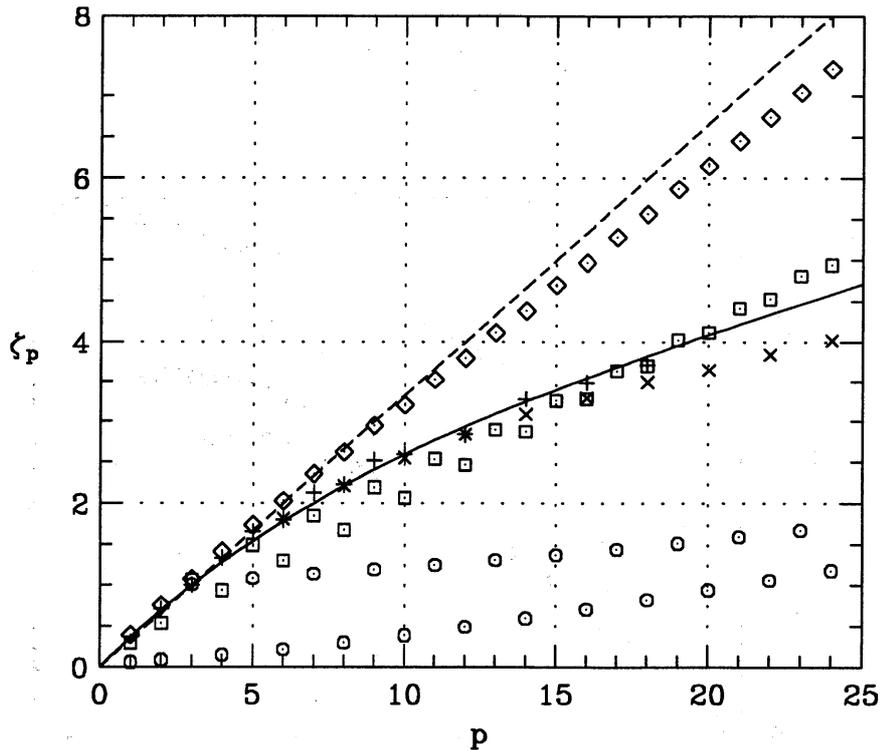

Figure 4: The exponent ζ_p of the structure function S_p shown by $\Diamond$, $\square$ and $\odot$ which are as in figure 3. dashed line, K41 (1.13); solid line, log-Poisson model (1.17); $\times$, DNS at $R_\lambda = 150$, Vincent & Meneguzzi (1991) [1]; $+$, jet at $R_\lambda = 852$, Anselmet *et al.* (1984) [11].

If the vortex is absent, therefore $v_\theta = 0$, we have $S_p(\ell) = C_p\,\sigma^p\ell^p \propto \ell^p$ from (2.18) and (2.19), where C_p is a exact constant. On the other hand,

if the external strain is absent so that $\sigma = 0$, we find that the structure functions of odd order are identically zero by use of (2.7), (2.18) and (2.19). Hence the scaling consistent with the homogeneous isotropic turbulence as obtained above is considered to result from the combined field of the vortex under external straining.

The probability distribution functions of the vortex Reynolds number R_Γ and the Burgers' radius r_B of worms are obtained by Jimenéz *et al.* [2] in DNS and by Belin *et al.* [5] experimentally. Especially $R_\Gamma / R_\lambda^{1/2}$ distributes independent of R_λ. Taking that p.d.f.s into account we expect to obtain the scaling nearer that of experiments which is K41 like scaling of low-order at any R_λ and the higher-order scaling with the larger deviation from K41.

3 Summary

The conclusions of this study are summarized as follows.

1. This model field of the isolated Burgers' vortex with moderate circulation shows that the 2nd-order structure function has about two-thirds scaling exponent.

2. The 3rd-order structure function of this model have a negative sign and unity exponent independent of the vortex Reynolds number R_Γ.

3. The scaling exponents of the high-order structure function obtained from the model-strained vortex field deviates increasingly from K41 as R_Γ is larger, i.e. a Burgers' vortex causes more and more intermittency of turbulence as it's circulation is larger.

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