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Choquet integral models and the multiattribute utility theory

(Choquet 積分モデルと多属性効用理論)

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1 Introduction

Subjective evaluation models using fuzzy integrals with respect to fuzzy measures have been applied in various fields, and their effectiveness has been experimentally proved [2, 7, 8, 9, 10]. Some authors pointed out that the advantage of fuzzy integral models is derived from the non-additivity of fuzzy measures, and wrote such as "in contrast to a linear model, it is not necessary to assume independence in a fuzzy integral model" [7, 8, 9, 10]. In regard to the meaning of "independence" in this intuitive comment, the author [6] has shown from the viewpoint of the multiattribute utility theory that

the fuzzy measure is additive $\Leftrightarrow$ *the attributes are mutually preferentially independent*

in the case that the Choquet integral is adopted as a fuzzy integral. This paper summarizes the preceding results [4, 6] on the Choquet integral model. The proof of the main theorems are shown in Appendix, and the other proofs are omitted; the propositions are immediately derived from definitions and the corollaries are direct consequences of the corresponding theorem and its proof.

2 Fuzzy measures and the Choquet integral

2.1 Basic definitions and properties [5]

Let $(\Theta, \mathcal{F})$ be a measurable space.

Definition 2.1 *A* fuzzy measure *is a set function* $\mu : \mathcal{F} \to [0, \infty]$ *satisfying*

(1) $\mu(\emptyset) = 0$,

(2) $A, B \in \mathcal{F}$ *and* $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

If $\mu(A \cup B) = \mu(A) + \mu(B)$ *whenever* A, $B \in \mathcal{F}$ *and* $A \cap B = \emptyset$, μ *is said to be* additive. *If* $\mu(\Theta) < \infty$, *a fuzzy measure* μ *is said to be* finite.

Throughout the paper we deal only with finite fuzzy measures.

Let μ be a finite fuzzy measure on $(\Theta, \mathcal{F})$.

Definition 2.2 *The* Choquet integral *of a measurable function* $f : \Theta \to \mathbf{R}$ *over* $A \in \mathcal{F}$ *is defined by*

$$(C)\int_A f\,d\mu \triangleq \int_0^\infty \mu(\{f > r\} \cap A)\,dr + \int_{-\infty}^0 [\mu(\{f > r\} \cap A) - \mu(A)]\,dr,$$

where $\{f > r\} \triangleq \{\theta \,|\, f(\theta) > r\}$ *and the two integrals on the right side are both ordinary ones. When the right side is* $\infty + (-\infty)$, *the Choquet integral is not defined. A measurable function* f *is said to be* integrable *iff the Choquet integral of* f *over* Θ *is finite-valued.*

Proposition 2.1 *The Choquet integral of a simple function*

$$f = \sum_{j=1}^m a_j 1_{P_j}$$

is represented by

$$(C)\int_\Theta f\,d\mu = \sum_{j=1}^m a_j \cdot [\mu(A_j) - \mu(A_{j+1})],$$

where $\{P_1, P_2, \ldots, P_m\}$ *is a measurable partition of* Θ, $-\infty < a_1 \leq \cdots \leq a_m < \infty$, $A_j = \bigcup_{k=j}^{m} P_k$ *and* $A_{m+1} = \emptyset$.

The Choquet integral has the following properties.

Proposition 2.2 (1)

$$f \leq g \Rightarrow (C)\int_{\Theta} f\, d\mu \leq (C)\int_{\Theta} g\, d\mu.$$

(2)

$$(C)\int_{\Theta}(af+b)d\mu = a \cdot (C)\int_{\Theta} f\, d\mu + b \cdot \mu(\Theta) \qquad \forall a \geq 0,\ \forall b \in \mathbf{R}.$$

(3) *If* μ *is an ordinary measure*

$$(C)\int_{\Theta} f\, d\mu = \int_{\Theta} f\, d\mu,$$

where the right side is the Lebesgue integral.

Definition 2.3 [1] $N \in \mathcal{F}$ *is called a* null *set iff*

$$\mu(A \cup N) = \mu(A) \qquad \forall A \in \mathcal{F}.$$

Proposition 2.3 *Let* $N \in \mathcal{F}$. *The following conditions are equivalent to each other.*

(1) N *is a null set.*

(2) *If* f *and* g *are measurable functions such that* $f(\theta) = g(\theta)\ \forall \theta \notin N$, *then*

$$(C)\int_{\Theta} f\, d\mu = (C)\int_{\Theta} g\, d\mu.$$

2.2 Positive sets, semiatoms, and inter-additive partitions [6]

Let μ be a finite fuzzy measure on $(\Theta, \mathcal{F})$.

Definition 2.4 *For $A \subset X$, we define*

$$\mathcal{F} \cap A \triangleq \{F \cap A \,|\, F \in \mathcal{F}\}, \qquad \mathcal{F} \setminus A \triangleq \{F \setminus A \,|\, F \in \mathcal{F}\}.$$

Definition 2.5 *$P \in \mathcal{F}$ is said to be* positive *iff*

$$\mu(A) < \mu(A \cup P) \qquad \forall A \in \mathcal{F} \setminus P.$$

Proposition 2.4 *Let $P \in \mathcal{F}$. The following condtions are equivalent to each other.*

(1) *P is positive.*

(2) *If f and g be measurable functions such that $f(\theta) = a$ $\forall\theta \in P$, $g(\theta) = b$ $\forall\theta \in P$, $a < b$, $f(\theta) = g(\theta)$ $\forall\theta \notin P$, and either f or g is integrable, then*

$$(C)\int_\Theta f \, d\mu < (C)\int_\Theta g \, d\mu.$$

Definition 2.6 [1] *$A \in \mathcal{F}$ is called a* atom *iff A is not a null set and, for any $B \in \mathcal{F} \cap A$, either B or $A \setminus B$ is a null set.*

Definition 2.7 *For $S \in \mathcal{F}$, we define*

$$\mathcal{W}(S) \triangleq \{A \in \mathcal{F} \cap S \,|\, \mu(A \cup B) = \mu(S \cup B) \; \forall B \in \mathcal{F} \setminus S\},$$
$$\mathcal{N}(S) \triangleq \{A \in \mathcal{F} \cap S \,|\, \mu(A \cup B) = \mu(B) \; \forall B \in \mathcal{F} \setminus S\}.$$

$S \in \mathcal{F}$ is called a semiatom *iff S is not a null set and $\mathcal{F} \cap S = \mathcal{W}(S) \cup \mathcal{N}(S)$.*

Definition 2.6 is a natural extension of the definition of atom in the classical measure theory. While an atom is a semiatom, a semiatom is not always an atom. If μ is additive, however, then every semiatom is an atom.

Proposition 2.5 *If S is a semiatom, then, for every measurable function f,*

$$(C)\int_{\Theta} f^S \, d\mu \;=\; (C)\int_{\Theta} f \, d\mu,$$

where

$$f^S(\theta) \;\triangleq\; \begin{cases} \sup_{A \in \mathcal{W}(S)} \inf_{\omega \in A} f(\omega) & \theta \in S \\ f(\theta) & \theta \notin S. \end{cases}$$

Definition 2.8 *$\mathcal{P}$ is called an* inter-additive partition *of Θ iff $\mathcal{P}$ is a finite measurable partition of Θ and*

$$\mu(A) \;=\; \sum_{P \in \mathcal{P}} \mu(A \cap P) \qquad \forall A \in \mathcal{F}.$$

Proposition 2.6 *Let $\mathcal{P}$ be a finite measurable partition of Θ. Then the following two conditions are equivalent to each other.*

(1) *$\mathcal{P}$ is an inter-additive partition of Θ.*

(2) *For every measurable function f,*

$$(C)\int_{\Theta} f \, d\mu \;=\; \sum_{P \in \mathcal{P}} (C)\int_{P} f \, d\mu.$$

3 Preference relations and value functions [3]

The *preference* relation is one of the most important concepts in the utility theory. A preference relation $\succeq$ is a binary relation on the set X of objects to be evaluated and $x \succeq y$ means that x is prefered or indifferent to y for a decision maker. A preference relation is assumed to be a weak order:

Definition 3.1 *A binary relation $\succeq$ on a set X is called a* weak order *iff it has the following two properties.*

comparability: *either* $x \succeq y$ *or* $y \succeq x$ $\forall x, y \in X$.

transitivity: $x \succeq y$ & $y \succeq z \;\Rightarrow\; x \succeq z$ $\forall x, y, z \in X$.

The *strong preference* relation $\succ$ and the *indefference* relation $\sim$ are defined respectively by

$$x \succ y \overset{\triangle}{\iff} \text{not } y \succeq x, \qquad x \sim y \overset{\triangle}{\iff} x \succeq y \;\&\; y \succeq x.$$

When the objects are characterized by n attributes, the set X is assumed to be given by $X = \prod_{i=1}^{n} X_i$. Each index i (or each factor X_i) is called an *attribute.* We write $I \triangleq \{1, 2, \cdots, n\}$, and $X_J \triangleq \prod_{j \in J} X_j$ for any non-empty subset J of I. Since $X_{\{i\}} = X_i$, we sometimes denote $\{i\}$ by i for convenience, and $I \setminus i$ means $I \setminus \{i\}$. For any non-empty proper subset J of I, we denote by x_J the projection of $x = (x_1, x_2, \cdots, x_n) \in X$ to X_J, and write $x = (x_J, x_{I\setminus J})$.

Definition 3.2 *An attribute i is said to be* essential *iff there exist $x_i, y_i \in X_i$ and $x_{I\setminus i} \in X_{I\setminus i}$ such that $(x_i, x_{I\setminus i}) \succ (y_i, x_{I\setminus i})$. An attribute which is not essential is said to be* inessential.

Definition 3.3 *Let $\emptyset \neq J \subsetneqq I$. We say J is* preferentially independent *of $I \setminus J$ (or X_J is* preferentially independent *of $X_{I\setminus J}$) iff, for every pair x_J and y_J of elements of X_J,*

$$(x_J, x_{I\setminus J}) \succeq (y_J, x_{I\setminus J}) \text{ for some } x_{I\setminus J} \in X_{I\setminus J} \;\Rightarrow\; (x_J, x_{I\setminus J}) \succeq (y_J, x_{I\setminus J}) \text{ for all } x_{I\setminus J} \in X_{I\setminus J}.$$

The attributes in I (or $X_1, X_2, \cdots, X_n$) are said to be mutually preferentially independent *iff, for every non-empty proper subset J of I, J is preferentially independent of $I \setminus J$.*

Definition 3.4 *A function $u : X \to \mathbf{R}$ is called a* value function *(or an* ordinal utility function*) if*

$$x \succeq y \;\Leftrightarrow\; v(x) \geq v(y) \qquad \forall x, y \in X.$$

Definition 3.5 *A value function v is said to be* additive *iff for each $i \in I$ there exist a real-valued function v_i on X_i and a nonnegative real number k_i such that*

$$v(x) = \sum_{i \in I} k_i \cdot v_i(x_i) \qquad \forall x \in X.$$

4 Choquet-integral value functions

Definition 4.1 *A* Choquet-integral value function *is a value function v which can be represented by*

$$v(x) = (C)\int_I v_i(x_i)\, d\mu \qquad \forall x \in X, \tag{1}$$

where v_i is a real-valued function on X_i, $i \in I$, and μ is a finite fuzzy measure on the power set 2^I of I. Note that the integrand is the function $v_{(\cdot)}(x_{(\cdot)}) : i \mapsto v_i(x_i)$.

By Proposition 2.2(3), if μ is an ordinary measure, a Choquet-integral value function coincides with an additive one (Definition 3.5); $k_i = \mu(\{i\})\ \forall i \in I$.

In this section, we assume that the preference relation $\succeq$ has a Choquet-integral value function (Eq. (1)), and use the following conditions.

(C1) For any $J \subset I \setminus \{i\}$, there exist $x, y \in X$ and $r, s \in \mathbf{R}$ such that $J = \{j \in I \mid v_j(x_j) > r\}$ and $J \cup \{i\} = \{j \in I \mid v_j(y_j) > s\}$.

(C2) The intersection $\bigcap_{i \in I} v_i(X_i)$ of the ranges of v_i's contains at least two distinct points.

(C3) The intersection $\bigcap_{i \in I} v_i(X_i)$ of the ranges of v_i's is not nowhere dense.

Note that the relationship between Conditions (C1–3) is geven as follows:

$$(C3) \Rightarrow (C2) \Rightarrow (C1).$$

Theorem 4.1 [4] *If either* $v_i(x_i) = \text{const.}\ \forall x_i \in X_i$ *or* $\{i\}$ *is a null set, then the attribute* i *is inessential. Moreover, if Condition* (C1) *is satisfied, the converse holds: if the attribute* i *is inessential, then either* $v_i(x_i) = \text{const.}\ \forall x_i \in X_i$ *or* $\{i\}$ *is a null set.*

Theorem 4.2 [6] *Let* $\emptyset \neq J \subsetneq I$. *If either* J *is a positive semiatom or* $\{J, I \setminus J\}$ *is an inter-additive partition of* I, *then* J *is preferentially independent of* $I \setminus J$. *Moreover, if Condition* (C3) *is satisfied, then the converse holds: if* J *is preferentially independent of* $I \setminus J$, *then either* J *is a positive semiatom or* $\{J, I \setminus J\}$ *is an inter-additive partition of* I.

Corollary 4.1 *Let* i *be an essential attribute in* I. *If* $\{i\}$ *is positive, then* i *is preferentially independent of* $I \setminus i$. *Moreover, if Condition* (C2) *is satisfied, then the converse holds.*

Corollary 4.2 (1) *Assume that the set* I *of the attributes has exactly two essential attributes* i *and* j. *If* $\{i\}$ *and* $\{j\}$ *are both positive, then the attributes are mutually preferentially independent. Moreover, if Condition* (C2) *is satisfied, then the converse holds.*

(2) *If* μ *is additive, then the attributes are mutually preferentially independent. Moreover, if the set* I *has at least three essential attributes, and if Condition* (C3) *is satisfied, then the converse holds.*

5 Conclusion

In this paper, we have investigated preference relations which have Choquet-integral value functions. The main result is that, under a natural condition, the attributes are mutually preferentially independent iff the fuzzy measure is additive. Therefore, since a fuzzy measure is not assumed to be additive, we can say "it is not necessary to assume *preferential independence* in a Choquet integral model."

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Appendix

Proof of Theorem 4.1. If v_i is constant, obviously the attribute i is inessential. If $\{i\}$ is a null set, the result follows directly from Proposition 2.3.

We now prove the converse. Assume that v_i is not constant, and let $J \subset I \setminus \{i\}$. It is sufficent to prove that $\mu(J \cup \{i\}) = \mu(J)$. By Condition (C1), there exist x, $y \in X$ and r, $s \in \mathbf{R}$ such that $J = \{j \in I \mid v_j(x_j) > r\}$ and $J \cup \{i\} = \{j \in I \mid v_j(y_j) > s\}$. For $j \in I \setminus \{i\}$, we define

$$z_j \triangleq \begin{cases} x_j \vee y_j & j \in J \\ x_j \wedge y_j & j \notin J \cup \{i\}, \end{cases}$$

where $\vee$ and $\wedge$ are binary operations on X_k, $k \in I$, defined respectively by

$$x_k \vee y_k \triangleq \begin{cases} x_k & v_k(x_k) \geq v_k(y_k) \\ y_k & v_k(x_k) < v_k(y_k), \end{cases} \qquad x_k \wedge y_k \triangleq \begin{cases} x_k & v_k(x_k) \leq v_k(y_k) \\ y_k & v_k(x_k) > v_k(y_k). \end{cases}$$

Since v_i is not constant, there exists a $w_i \in X_i$ such that $v_i(x_i) \neq v_i(w_i)$, and we define

$$z_i \triangleq x_i \wedge w_i, \qquad z_i' \triangleq x_i \vee w_i.$$

Moreover, we define $v_k \triangleq v_k(z_k)$, $k \in I$, and

$$
\begin{aligned}
M = \{j_1, j_2, \ldots, j_m\} &\triangleq \{j \in I \,|\, v_i < v_j < v_i(z_i')\}, \quad \text{where } v_{j_1} \le v_{j_2} \le \cdots \le v_{j_m}, \\
v_{j_0} &\triangleq v_i, \\
v_{j_{m+1}} &\triangleq v_i(z_i'), \\
M_{j_{m+1}} &\triangleq \{j \,|\, v_{j_{m+1}} \le v_j\}, \\
M_k &\triangleq \{j_k, \ldots, j_m\} \cup M_{j_{m+1}}, \quad k = 1, 2, \ldots, m.
\end{aligned}
$$

The inessentiality of i implies that $(z_i, z_{I\setminus i}) \sim (z_i', z_{I\setminus i})$, and hence that

$$v(z_i', z_{I\setminus i}) - v(z_i, z_{I\setminus i}) = 0.$$

Since v is a Choquet-integral value function (Eq. (1)), it follows from Proposition 2.1 that

$$\sum_{k=1}^{m+1} (v_{j_k} - v_{j_{k-1}})[\mu(M_k \cup \{i\}) - \mu(M_k)] = 0.$$

By the definition of z, there exists an integer k such that $v_{j_k} > v_{j_{k-1}}$ and $M_k = J$, and therefore $\mu(J \cup \{i\}) - \mu(J) = 0$. ∎

Proof of Theorem 4.2. If J is a positive semiatom, J is preferentially independent of $I \setminus J$ by Propositions 2.4 and 2.5. If $\{J, I \setminus J\}$ is an inter-additive partition of Θ, the desired result follows from Proposition 2.6.

We prove the converse. Suppose that $\bigcap_{i\in I} v_i(X_i)$ is not nowhere dense and that J is preferentially independent of $I \setminus J$. We first prove that $\{J, I \setminus J\}$ is an inter-additive partition of Θ when J is not a semiatom. If J is a null set, then $\{J, I\setminus J\}$ is an inter-additive partition of Θ, so we assume that J is not null. By Condition (C3) and Proposition 2.2 (2) we can assume that $[0,1] \subset \overline{\bigcap_{i\in I} v_i(X_i)}$ and $0, 1 \in \bigcap_{i\in I} v_i(X_i)$. Let $K \subset J$, $L \subset I \setminus J$,

$a, b \in \bigcap_{i \in I} v_i(X_i)$, and $0 < a < b < 1$. Then there exist x, y, $z \in X$ such that

$$v_i(x_i) = \begin{cases} 0 & i \in J \setminus K \\ b & i \in K \\ 1 & i \in L \\ 0 & \text{otherwise,} \end{cases} \qquad v_i(y_i) = \begin{cases} a & i \in J \\ a & i \in L \\ 0 & \text{otherwise,} \end{cases} \qquad v_i(z_i) = 0 \quad \forall i \in I.$$

Therefore we obtain

$$\begin{aligned} & (x_J, x_{I \setminus J}) \succsim (y_J, x_{I \setminus J}) \\ \Leftrightarrow\ & v(x_J, x_{I \setminus J}) \gtreqless v(y_J, x_{I \setminus J}) \\ \Leftrightarrow\ & b\mu(K \cup L) + (1-b)\mu(L) \\ & \gtreqless a\mu(J \cup L) + (1-a)\mu(L) \\ \Leftrightarrow\ & (b-a)[\mu(K \cup L) - \mu(L)] \gtreqless a[\mu(J \cup L) - \mu(K \cup L)]. \end{aligned}$$

Similarily we have

$$\begin{aligned} (x_J, y_{I \setminus J}) \succsim (y_J, y_{I \setminus J}) &\Leftrightarrow (b-a)\mu(K) \gtreqless a[\mu(J \cup L) - \mu(K \cup L)], \\ (x_J, z_{I \setminus J}) \succsim (y_J, z_{I \setminus J}) &\Leftrightarrow (b-a)\mu(K) \gtreqless a[\mu(J) - \mu(K)]. \end{aligned}$$

The preferential independence implies that the three inequalities above are equivalent to one another. From the assumption that $[0,1] \subset \overline{\bigcap_{i \in I} v_i(X_i)}$ it follows that, for any $K \subset J$ and $L \subset I \setminus J$,

$$\begin{aligned} & [\mu(K \cup L) - \mu(L)] : [\mu(J \cup L) - \mu(K \cup L)] \\ = \ & \mu(K) : [\mu(J \cup L) - \mu(K \cup L)] \qquad \text{(A.1)} \\ = \ & \mu(K) : [\mu(J) - \mu(K)]. \end{aligned}$$

Since J is neither a semiatom nor a null set, there exist $K_0 \subset J$ and L_1, $L_2 \subset I \setminus J$ such that $\mu(L_1) < \mu(K_0 \cup L_1)$ and $\mu(K_0 \cup L_2) < \mu(J \cup L_2)$. From Eq. (A.1) and the inequality $\mu(L_1) < \mu(K_0 \cup L_1)$ it follows that $\mu(K_0) > 0$. Similarily it follows from the inequality

$\mu(K_0 \cup L_2) < \mu(J \cup L_2)$ that $\mu(J) - \mu(K_0) > 0$. Let L be an arbitrary subset of $I \setminus J$. Since $\mu(K_0) > 0$ and $\mu(J) - \mu(K_0) > 0$, by Eq. (A.1) we obtain that

$$\begin{aligned} \mu(K_0 \cup L) - \mu(L) &= \mu(K_0), \\ \mu(J \cup L) - \mu(K_0 \cup L) &= \mu(J) - \mu(K_0), \end{aligned}$$

and hence that

$$\mu(J \cup L) = \mu(J) + \mu(L). \tag{A.2}$$

Now consider an arbitrary $K \subset J$. If $\mu(K) = 0$, it follows from Eq. (A.1) that $\mu(K \cup L) - \mu(L) = 0$, and hence that $\mu(K \cup L) = \mu(K) + \mu(L)$. If $\mu(K) > 0$, it follows from Eq. (A.1) that

$$\mu(J \cup L) - \mu(K \cup L) = \mu(J) - \mu(K),$$

and therefore from Eq. (A.2) that $\mu(K \cup L) = \mu(K) + \mu(L)$. This proves that $\{J, I \setminus J\}$ is an inter-additive partition.

We now prove that J is positive when it is a semiatom. Since J is not a null set, there exists an $L \subset I \setminus J$ such that $\mu(L) < \mu(J \cup L)$. Let $M \subset I \setminus J$. Then we can choose $x, y \in X$ such that

$$v_i(x_i) = \begin{cases} 1 & i \in J \\ 1 & i \in L \\ 0 & \text{otherwise,} \end{cases} \qquad v_i(y_i) = \begin{cases} 0 & i \in J \\ 1 & i \in M \\ 0 & \text{otherwise.} \end{cases}$$

Since $v(x_J, x_{I\setminus J}) = \mu(J \cup L) > \mu(L) = v(y_J, x_{I\setminus J})$, it follows that $(x_J, x_{I\setminus J}) \succ (y_J, x_{I\setminus J})$. Hence the preferential independence implies that $(x_J, y_{I\setminus J}) \succ (y_J, y_{I\setminus J})$, and therefore that $\mu(J \cup M) = v(x_J, y_{I\setminus J}) > v(y_J, y_{I\setminus J}) = \mu(M)$. ∎