数理解析研究所講究録
975 巻 1996 年 65-84

ON A TOY FOCK SPACE GENERATED BY PERMUTATIONS

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ABSTRACT. An example of independence, conditional expectation and martingales in quantum probability theory is constructed on the permutational Fock space which is a kind of discrete "Fock space" generated by all the permutations from the natural numbers. Besides a discrete time analogue of quantum Ito's formula for the permutational Fock space is obtained.

1. INTRODUCTION

The notion of "Fock space" is a nice tool for the purpose of constructing various examples of "independence," "white noise," and "quantum stochastic calculus" in quantum probability theory or noncommutative probability theory.[1,2,3,4] For example, the followings are known.

For a one-particle Hilbert space $\mathcal{H}$, let $\mathcal{H}^{\circ r}$ be the r-th symmetric tensor power, $\mathcal{H}^{\wedge r}$ the r-th antisymmetric tensor power, and $\mathcal{H}^{\otimes r}$ the r-th tensor power. Let $\Phi_{boson} = \oplus_r \mathcal{H}^{\circ r}$ be the boson Fock space, $\Phi_{fermion} = \oplus_r \mathcal{H}^{\wedge r}$ the fermion Fock space, and $\Phi_{free} = \oplus_r \mathcal{H}^{\otimes r}$ the free Fock space. Let us specialize the one-particle Hilbert space $\mathcal{H}$ to be the complex Hilbert space $L^2(\mathbf{R}_+)$ of all L^2-functions of time $t \geq 0$. Then the three Fock spaces are equipped with very rich probabilistic structures. The boson Fock space Φ_{boson} leads to the commuting independence, the noncommuting pair of classical Brownian motions, and the bosonic stochastic calculus of R. L. Hudson and K. R. Parthasarathy[5] with quantum Ito's formula. The fermion Fock space $\Phi_{fermion}$ leads to the anticommuting independence, the noncommuting pair of fermion Brownian motions, and the fermionic stochastic calculus of D. Applebaum and Hudson[6] with fermion Ito's formula. The free Fock space Φ_{free} leads to the free independence of D. Voiculescu,[7] the noncommuting pair of free Brownian motions of R. Speicher,[8] and the free stochastic calculus of B. Kümmerer and Speicher[9] with free Ito's formula. There have been also known several discrete models of Fock space.[1,2] For example, the toy Fock space Φ_{toy} introduced by J. L. Journé and further studied by P. A. Meyer[2,10,11] is an elementary but interesting discrete model for the bosonic stochastic calculus and the fermionic stochastic calculus. Journé's toy Fock space Φ_{toy} is connected with theory of spin systems.[12]

1991 *Mathematics Subject Classification.* Primary 81S25; Secondary 46L89, 46L50, 60G42, 60G48.

Key words and phrases. quantum probability theory, Fock space, Kümmerer independence, conditional expectation, operator martingales, quantum Ito's formula.

This work was supported by Grant-in-Aid for Scientific Research No. 07740157, the Ministry of Education, Science and Culture, Japan.

In this paper, for the purpose of constructing an example of probabilistic notions in discrete time quantum probability theory, we introduce the ***permutational Fock space*** which is a kind of discrete "Fock space" generated by all the permutations from the natural numbers. This "Fock space" can be viewed as a variation of Journé's toy Fock space Φ_{toy}.

The motivation of finding such Fock space is explained as follows. Let us consider the discrete and finite time situation. That is, we take the set $T_n = \{1, 2, \cdots, n\}$ as the set of times, and let the one-particle Hilbert space $\mathcal{H}$ to be the l^2-space $l^2(T_n)$ of all functions of time $k \in T_n$. Let us calculate the dimension of the r-particle space $\mathcal{H}_r$ for each cases of boson ($\mathcal{H}_r = \mathcal{H}^{\circ r}$), fermion ($\mathcal{H}_r = \mathcal{H}^{\wedge r}$), and free ($\mathcal{H}_r = \mathcal{H}^{\otimes r}$). We easily get

$$\dim(\mathcal{H}_r) = \begin{cases} {}_n\mathrm{H}_r = \text{the number of repeated combinations} & \text{(in the boson case)} \\ {}_n\mathrm{C}_r = \text{the number of combinations} & \text{(in the fermion case)} \\ {}_n\Pi_r = \text{the number of repeated permutations} & \text{(in the free case)} \end{cases}$$

This encourages us to guess the existence of "Fock space" $\Phi = \oplus_r \mathcal{H}_r$ such that its r-particle space $\mathcal{H}_r$ satisfies

$$\dim(\mathcal{H}_r) = {}_n\mathrm{P}_r = \text{the number of permutations.}$$

Our aim in this paper is, (i) the construction of such "Fock space" Φ which we call the permutational Fock space, and (ii) the construction, over such "Fock space" Φ, of an example of probabilistic notions in discrete time quantum probability theory, i.e. "independence," "filtration," "conditional expectation," "martingales," "process with independently and identically distributed subalgebras," etc.

The paper is organized as follows. In §2, we construct on the set T of all natural numbers the permutational Fock space Φ with the basic operators, i.e. the creation operators, annihilation operators, conservation operators, and exclusion operators. The pair $(\mathcal{A}, \phi)$ consisting of the C^*-algebra $\mathcal{A}$ generated by all the creation and the annihilation operators with identity and the vacuum state ϕ is a quantum probability space to be studied in this paper. In §3, we obtain the weak Fock expansion theorem for bounded linear operators on the permutational Fock space Φ. In §4, we examine the independence structure arising from the permutational Fock space Φ. We obtain a process $\{\mathcal{A}_k\}_{k\in T}$ with independently and identically distributed subalgebras $\mathcal{A}_k \subset \mathcal{A}$ with respect to the vacuum state ϕ. In §5, we examine the filtration structure $\{\mathcal{A}_{k]}\}_{k\in T}$ of the C^*-algebra $\mathcal{A}$ where $\mathcal{A}_{k]}$ is the C^*-algebra generated by all the creation and annihilation operators up to time k with identity. In §6, we construct the natural conditional expectation $\varepsilon_{k]}$ from the C^*-algebra $\mathcal{A}$ onto the C^*-subalgebra $\mathcal{A}_{k]} \subset \mathcal{A}$ with respect to the vacuum state ϕ. In §7, we obtain the uniform Fock expansion theorem for operators in the C^*-algebra $\mathcal{A}$. We examine the predictable representation theorem for operator martingales on the quantum probability space $(\mathcal{A}, \phi)$. In §8, we examine a discrete time analogue of quantum Ito's formula for discrete time operator processes on the quantum probability space $(\mathcal{A}, \phi)$. The last §9, contains some complementary remarks.

Before closing this introduction, we remark that the independence of the discrete time quantum stochastic process $\{\mathcal{A}_k\}_{k\in T}$ with independently and identically distributed subalgebras $\mathcal{A}_k \subset \mathcal{A}$ obtained in this paper is neither the commuting independence, nor the anticommuting independence, nor the free independence. But it is an example of the Kümmerer independence.[8,14]

2. Definition of Permutational Fock Space

In this section, we give the definition of permutational Fock space.

Let $T = \{1, 2, 3, \cdots\}$ be the set of all positive natural numbers which is interpreted as time. For a natural number $r = 0, 1, 2, \cdots$, we denote by ${}_T\mathrm{P}_r$ the set of all permutations σ of length r from T. We also denote by $\mathrm{Per}(T)$ the disjoint union $\cup_r {}_T\mathrm{P}_r$ under the convention that ${}_T\mathrm{P}_0 = \{\Lambda\}$ the singleton where Λ denotes the null permutaion, i.e. the null string from T. Denote by $\mathcal{H}_r$ the complex Hilbert space $l^2({}_T\mathrm{P}_r)$ of all l^2-functions over the set ${}_T\mathrm{P}_r$. We call $\mathcal{H}_r$ the *r-particle space*, and their direct sum $\Phi = \oplus_r \mathcal{H}_r$ the *permutational Fock space* over T. It contains the natural complete orthonormal system $\{e_\sigma | \sigma \in \mathrm{Per}(T)\}$ labelled by all the permutations σ. A special vector e_Λ is called the *vacuum vector*, and denoted by Ω.

For any permutation $\sigma = (i_1, \cdots, i_r) \in \mathrm{Per}(T)$, we denote the associated set $\{i_1, \cdots, i_r\}$ by $[\sigma]$. We wright $[\sigma] \perp [\tau]$ if and only if $[\sigma] \cap [\tau] = \emptyset$. For any pair of permutations $\sigma = (i_1, \cdots, i_r)$, $\tau = (j_1, \cdots, j_s) \in \mathrm{Per}(T)$ s.t. $\{i_1, \cdots, i_r\} \cap \{j_1, \cdots, j_s\} = \emptyset$, we denote by (σ, τ) a new permutation $(i_1, \cdots, i_r, j_1, \cdots, j_s)$ obtained from composition of σ and τ. Furthermore, when $\tau = (j)$ a permutation of length 1, we wright in the short notation $(\sigma, \tau) = (\sigma, j)$, $(\tau, \sigma) = (j, \sigma)$, and so on. We define the *mutiplication* of basis vectors by

$$e_\sigma e_\tau = \begin{cases} e_{(\sigma,\tau)} & (\text{if } [\sigma] \perp [\tau]), \\ 0 & (\text{otherwise}). \end{cases}$$

This multiplication is extended to the dense subspace $\Phi_{0,0}$ = "the linearspan of $\{e_\sigma | \sigma \in \mathrm{Per}(T)\}$" $\subset \Phi$ through the bilinearity. Furthermore it can be still extended to the dense subspace Φ_0 = "the algebraic direct sum of the r-particle spaces" $\subset \Phi$ because of the boundedness of multiplication of vectors with the fixed particle number, i.e. $\|uv\| \le \|u\|\|v\|$ for $u \in \mathcal{H}_r \cap \Phi_{0,0}$ and $v \in \mathcal{H}_s \cap \Phi_{0,0}$. This multiplication gives to Φ_0 a structure of associative algebra.

For each time $i \in T$, the *creation operator* d_i^+ is defined by

$$d_i^+ e_{(i_1,i_2,\cdots,i_r)} = \begin{cases} e_{(i,i_1,i_2,\cdots,i_r)} & (\text{if } i \notin \{i_1, i_2, \cdots, i_r\}), \\ 0 & (\text{otherwise}). \end{cases}$$

The creation operator d_i^+ is just the left multiplication operator $e_\sigma \mapsto e_i e_\sigma$ with respect to the above mentioned multiplication in Φ_0.

The *annihilation operator* d_i^- is defined by

$$d_i^- e_{(i_1,i_2,\cdots,i_r)} = \begin{cases} e_{(i_2,\cdots,i_r)} & (\text{if } r \ge 1 \text{ and } i = i_1), \\ 0 & (\text{otherwise}). \end{cases}$$

These operators are bounded linear operators on Φ, and $\|d_i^+\| = \|d_i^-\| = 1$. The annihilation operator is just the adjoint of creation operator: $d_h^- = (d_h^+)^*$. The creation and annihilation operators d_i^+, d_j^- $(i, j \in T)$ satisfy the following relations:

$$\begin{cases} (d_i^+)^2 = (d_i^-)^2 = 0, \\ d_i^- d_j^+ = 0 \ (\text{for } i \ne j), \\ d_i^- d_i^+ = I - \sum_{\sigma \in \mathrm{Per}(T \setminus \{i\})} d_{(\sigma,i)}^+ d_{(\sigma,i)}^-. \end{cases}$$

Here I denotes the identity operator on Φ. The infinite series in the righthand side of the last equality converges in the weak topology of $\mathcal{B}(\Phi)$ which is the space of all bounded linear operators on Φ. These relations may be compared with the CCR (canonical commutation relations)[5] in the boson case, the CAR (canonical anticommutation relations)[6] in the fermion case, and the relations for the Cuntz algebra[9] in the free case. The *conservation operator* d_i° is defined by $d_i^\circ = d_i^+ d_i^-$.

For any permutation $\sigma = (i_1, \cdots, i_r) \in \mathrm{Per}(T)$, put

$$d_\sigma^+ = d_{i_1}^+ \cdots d_{i_r}^+, \quad d_\sigma^- = d_{i_r}^- \cdots d_{i_1}^-, \quad d_\sigma^\circ = d_\sigma^+ d_\sigma^- \quad \text{and} \quad d_\Lambda^+ = d_\Lambda^- = d_\Lambda^\circ = I.$$

Then, the creation and annihilation operators d_σ^+, d_τ^- $(\sigma, \tau \in \mathrm{Per}(T))$ satisfy the following relations:

$$d_\sigma^- d_\tau^+ = \begin{cases} d_\beta^+ - \sum_{[\gamma]\perp[\tau],\ i \in [\alpha]} d_\beta^+ d_{(\gamma,i)}^\circ \\ \qquad (\text{if } \exists \alpha, \beta \in \mathrm{Per}(T) \text{ s.t. } [\alpha]\perp[\beta], \sigma = \alpha,\ \tau = (\alpha, \beta)), \\ d_\beta^- - \sum_{[\gamma]\perp[\sigma],\ i \in [\alpha]} d_{(\gamma,i)}^\circ d_\beta^- \\ \qquad (\text{if } \exists \alpha, \beta \in \mathrm{Per}(T) \text{ s.t. } [\alpha]\perp[\beta], \sigma = (\alpha, \beta),\ \tau = \alpha), \\ 0 \qquad (\text{otherwise}). \end{cases}$$

The serieses in the righthand side converge in the weak topology of $\mathcal{B}(\Phi)$.

For the discussion in the following sections, it is convenient to introduce a new operator. The *exclusion operator* $d_i^\bullet$ is defined by $d_i^\bullet = d_i^- d_i^+$. For any permutation $\sigma \in \mathrm{Per}(T)$, put $d_\sigma^\bullet = d_\sigma^- d_\sigma^+$. Then we have

$$d_\sigma^\bullet e_\tau = \begin{cases} e_\tau & (\text{if } [\sigma]\perp[\tau]), \\ 0 & (\text{otherwise}). \end{cases}$$

Note that $d_\sigma^\bullet = d_\tau^\bullet$ whenever $[\sigma] = [\tau]$. So we can define, for any finite set $U \subset T$, an operator $d_U^\bullet := d_\sigma^\bullet$ using σ s.t. $[\sigma] = U$. The exclusion operator $d_U^\bullet$ is a projection operator and satisfies the following relations:

$$\begin{cases} d_U^\bullet d_\sigma^+ = d_\sigma^+ d_U^\bullet \quad \text{and} \quad d_U^\bullet d_\sigma^- = d_\sigma^- d_U^\bullet \quad (\text{if } U \perp [\sigma]), \\ d_U^\bullet d_V^\bullet = d_{U \cup V}^\bullet. \end{cases}$$

Let $\mathcal{A} = C^*(I, d_i^+, d_i^- | i \in T)$ be the C^*-algebra generated by all the creation and annihilation operators with identity. The C^*-algebra $\mathcal{A}$ has a special state $\phi(\cdot) =< \Omega| \cdot |\Omega >$ called the ***vacuum state***. The pair $(\mathcal{A}, \phi)$ is interpreted as a ***quantum probability space***.

3. Fock expansion

In this section, we prove an expansion theorem for bounded linear operators on the permutational Fock space Φ.

Denote by $\mathcal{B}(\Phi)$ the von Neumann algebra of all bounded linear operators on Φ. Let us show that the space $\mathcal{B}(\Phi)$ has a "basis" $\{d_\sigma^+ d_\tau^-\}$, in the "weak" topology, consisting of normally ordered products of the creation and annihilation operators. The expansion theorem like the following Theorem 3.1 and Theorem 3.2 which assert that every operator in a sufficiently large class can be expanded to the infinite linear combinantion (or integration) of normally ordered products of the basic operators is sometimes called the *Fock expansion theorem.*[13]

At first let us show the following purely "algebraic" Fock expansion. Denote by $L(\Phi_{0,0})$ the set of all linear operators $A : \Phi_{0,0} \to \Phi_{0,0}$ which is possibly unbounded.

Theorem 3.1: *Every operator $A \in L(\Phi_{0,0})$ has the unique expansion of the form*

$$A = \sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^- ,$$

where $a_{\sigma,\tau}$ are scalars, and σ, τ runs over all permutaions from T.

Proof: At first let us show the existence of the expansion. For a given $A \in L(\Phi_{0,0})$, put

$$a_{\sigma,\tau} = \begin{cases} < e_\sigma | A e_\tau > - < e_{\sigma'} | A e_{\tau'} > \\ \qquad\qquad (\text{if } \sigma = (i_1, \cdots, i_r) \neq \Lambda,\ \tau = (j_1, \cdots, j_s) \neq \Lambda, \text{ and } i_r = j_1), \\ < e_\sigma | A e_\tau > \quad (\text{otherwise}), \end{cases}$$

where $\sigma' = (i_1, \cdots, i_{r-1})$ and $\tau' = (j_2, \cdots, j_s)$. Then, the family of scalars $\{a_{\sigma,\tau} | \sigma, \tau \in \mathrm{Per}(T)\}$ satisfy the condition that, for each τ, $a_{\sigma,\tau} = 0$ except for finitely many number of indices σ. By this condition, the operator $\sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^-$ can be well defined as an element in $L(\Phi_{0,0})$. Let us calculate the matrix element $< e_\lambda | \sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^- e_\mu >$, $\lambda, \mu \in \mathrm{Per}(T)$, then we have

$$\begin{aligned} & < e_\lambda | \sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^- e_\mu > \\ &= \sum_{\sigma,\tau} a_{\sigma,\tau} < e_\lambda | d_\sigma^+ d_\tau^- e_\mu > \\ &= \sum_{\substack{\sigma = \Lambda \text{ or } \tau = \Lambda \\ \text{or } \ll \sigma = (i_1, \cdots, i_r) \neq \Lambda, \\ \tau = (j_1, \cdots, j_s) \neq \Lambda, i_r \neq j_1 \gg}} < e_\sigma | A e_\tau >< e_\lambda | d_\sigma^+ d_\tau^- e_\mu > \\ &+ \sum_{\substack{\sigma = (i_1, \cdots, i_r) \neq \Lambda, \\ \tau = (j_1, \cdots, j_s) \neq \Lambda,\ i_r = j_1}} (< e_\sigma | A e_\tau > - < e_{\sigma'} | A e_{\tau'} >) < e_\lambda | d_\sigma^+ d_\tau^- e_\mu > \\ &= \sum_{\sigma,\tau} < e_\sigma | A e_\tau >< e_\lambda | d_\sigma^+ d_\tau^- e_\mu > \\ &- \sum_{\sigma \neq \Lambda, \tau \neq \Lambda, i_r = j_1} < e_{\sigma'} | A e_{\tau'} >< e_\lambda | d_\sigma^+ d_\tau^- e_\mu > \end{aligned}$$

$$= \sum_{\sigma,\tau} < e_\sigma|Ae_\tau >< e_\lambda|d_\sigma^+ d_\tau^- e_\mu >$$
$$- \sum_{\sigma,\tau\in \mathrm{Per}(T), i\in T} < e_\sigma|Ae_\tau >< e_\lambda|d_\sigma^+ d_i^\circ d_\tau^- e_\mu >$$
$$= \sum_{\sigma,\tau} < e_\sigma|Ae_\tau >< e_\lambda|d_\sigma^+(I - \sum_i d_i^\circ) d_\tau^- e_\mu > .$$

Here the operator $E_{\sigma,\tau} = d_\sigma^+(I - \sum_i d_i^\circ)d_\tau^-$ is shown to be an elementary operator in the sense that $E_{\sigma,\tau}e_\rho = e_\sigma$ (if $\rho = \tau$), $= 0$ (if $\rho \neq \tau$). So we get

$$< e_\lambda| \sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^- e_\mu >=< e_\lambda|Ae_\mu > .$$

Hence we get the Fock expansion $A = \sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^-$.

Now, let us show the uniqueness of the expansion. For the proof of the unieque-ness, we only have to show the "linear independence" of the family $\{d_\sigma^+ d_\tau^- | \sigma, \tau \in \mathrm{Per}(T)\}$. Assume that $\sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^- = 0$, where the family of scalars $\{a_{\sigma,\tau}|\sigma,\tau \in \mathrm{Per}(T)\}$ satisfy the condition that, for each τ, $a_{\sigma,\tau} = 0$ except for finitely many number of indices σ. Then we have $\sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^- e_\rho = 0$ for all $\rho \in \mathrm{Per}(T)$. By specialization $\rho := \Lambda$, we have $\sum_\sigma a_{\sigma,\Lambda} d_\sigma^+ d_\Lambda^- e_\Lambda = \sum_\sigma a_{\sigma,\Lambda} e_\sigma = 0$ and hence we get $a_{\sigma,\Lambda} = 0$ for all $\sigma \in \mathrm{Per}(T)$. Next, by specialization $\rho := (i)$ from ${}_T\mathrm{P}_1$, we have

$$\sum_\sigma a_{\sigma,\Lambda} d_\sigma^+ d_\Lambda^- e_{(i)} + \sum_\sigma a_{\sigma,(i)} d_\sigma^+ d_{(i)}^- e_{(i)} = 0,$$

and hence we get $a_{\sigma,(i)} = 0$ for all $\sigma \in \mathrm{Per}(T)$. Similarly, by specialization $\rho := (i,j)$ from ${}_T\mathrm{P}_2$, we get $a_{\sigma,(i,j)} = 0$ for all $\sigma \in \mathrm{Per}(T)$, and so on. Finally we get $a_{\sigma,\tau} = 0$ for all $\sigma, \tau \in \mathrm{Per}(T)$. This concludes the "linear independence" of the family $\{d_\sigma^+ d_\tau^-\}$, and hence the uniqueness result. □

Let us investigate the "topological" Fock expansion for operators in the setting of the "rapidly decreasing sequences." We call a family $x = \{x_\sigma\}_{\sigma\in\mathrm{Per}(T)}$ of complex numbers a ***rapidly decreasing family*** if it satisfies

$$\sum_\sigma (\max[\sigma])^k |x_\sigma| < \infty \quad \text{for all } k = 1, 2, 3, \cdots .$$

Denote by $\mathcal{S}$ the set of all rapidly decreasing families $x = \{x_\sigma\}$. Then $\mathcal{S}$ naturally has a structure of countably Hilbert space.[13] Let $\mathcal{S}^*$ be the dual space of $\mathcal{S}$. Denote by $\mathcal{L}(\mathcal{S}, \mathcal{S}^*)$ the space of all continuous linear operators from $\mathcal{S}$ to $\mathcal{S}^*$. The weak convergence operators $A_n \to A$ in $\mathcal{L}(\mathcal{S}, \mathcal{S}^*)$ means $< u|A_n|v > \to < u|A|v >$ for all $u, v \in \mathcal{S}$. In this setting, the following "topological" Fock exapnsion theorem is easily obtained by the similar discussion in the proof of Theorem 3.1 and some simple evaluations.

Theorem 3.2 (Weak Fock Expansion for $\mathcal{B}(\Phi)$): *Every operator $A \in \mathcal{B}(\Phi)$ has the unique expansion of the form*

$$A = \sum_{\sigma,\tau} a_{\sigma,\tau} d_\sigma^+ d_\tau^-,$$

as an element of $\mathcal{L}(\mathcal{S}, \mathcal{S}^*)$, where $a_{\sigma,\tau}$ are scalars, and σ, τ runs over all permutaions from T.

This means that each operator A of the von Neumann algebra $\mathcal{B}(\Phi)$ can be expanded to the Fock expansion in the weak topology of $\mathcal{L}(\mathcal{S}, \mathcal{S}^*)$. Contrary to Theorem 3.2, we will show in §5 (resp. §7) that each operator A of the C^*-algebra $\mathcal{A}_{k]}$ (resp. $\mathcal{A}$) has the another form of "Fock expansion" which is useful in the later discussion.

4. INDEPENDENCE

In this section we examine the independence structure in the quantum probability space $(\mathcal{A}, \phi)$.

The axiomatic theory of "independence" and "white noise" in quantum probability theory was studied by Kümmerer.[14] The "independence" in the sense of Kümmerer was stated as follows.

Definition 4.1: A time indexed family $\{\mathcal{B}_k\}_{k\in T}$ of subalgebras of a unital algebra $\mathcal{B}$ is ***independent in the sense of the Kümmerer*** with respect to a state φ of $\mathcal{B}$, if the factorization for time ordered products holds, i.e.

$$\varphi(B_1 \cdots B_m) = \varphi(B_1) \cdots \varphi(B_m)$$

whenever $B_i \in \mathcal{B}_{k_i}$ and $k_1 < k_2 < \cdots < k_m$.

Such form of independence is referred to as the ***Kümmerer independence.*** The important examples of the Kümmerer independence are the commuting independence,[3] the anticommuting independence, [3] and the free independence.[7,8]

Definition 4.2: A family of subalgebras, $\mathcal{B}_k \subset \mathcal{B}$ is independent in the sense of ***the commuting independence*** if the algebras commute with each other (i.e. $[\mathcal{B}_k, \mathcal{B}_l] = 0$ if $k \neq l$) and $\varphi(B_1 \cdots B_m) = \varphi(B_1) \cdots \varphi(B_m)$ whenever $B_i \in \mathcal{B}_{k_i}$ and $i \neq j$ implies $k_i \neq k_j$.

Definition 4.3: A family of unital subalgebras, $\mathcal{B}_k \subset \mathcal{B}$ is ***freely independent*** if $\varphi(B_1 \cdots B_m) = 0$ whenever $B_i \in \mathcal{B}_{k_i}, k_1 \neq k_2 \neq \cdots \neq k_m$ and $\varphi(B_i) = 0\ \forall i$.

We omit the definition of anticommuting independence.[3] The commuting independence appears in the boson Fock space Φ_{boson}. The anticommuting independence appears in the fermion Fock space $\Phi_{fermion}$. The free independence appears in the free Fock space Φ_{free}.

Let us investigate the independence structure in the permutational Fock space Φ. For each time $k \in T$, let $\mathcal{A}_k = C^*(I, d_k^+, d_k^-)$ be the C^*-subalgebra of $\mathcal{A}$ generated by d_k^+ and d_k^- with identity I. Then the time indexed family of subalgebras $\{\mathcal{A}_k\}_{k\in T}$ is "independent" with respect to the vacuum state ϕ in the following sense.

Theorem 4.4: $\phi(A_1 \cdots A_n) = \phi(A_1) \cdots \phi(A_n)$ *whenever* $A_i \in \mathcal{A}_{k_i}$ *and* $i \neq j$ *implies* $k_i \neq k_j$.

For the proof of this theorem, we prepare a lemma which is easily proved.

Lemma 4.5: The algebra $\mathcal{A}_k$ *has a basis* $\{I, d_k^+, d_k^-, d_k^\circ, d_k^\bullet\}$ *as a vector space.*

Proof of Theorem 4.4: It is sufficient for the proof to restrict ourselves to consider the special case that $k_1 = 1, k_2 = 2, \cdots, k_n = n$. For each $i = 1, \cdots, n$, let

$$A_i = \sum_{\sigma_i, \tau_i \in \mathrm{Per}(\{i\})} a^{(i)}_{\sigma_i, \tau_i} d^+_{\sigma_i} d^-_{\tau_i} + a^{(i)} d^\bullet_i$$

be the expansion of A_i by the basis $\{1, d_i^+, d_i^-, d_i^+d_i^-, d_i^\bullet\}$ of $\mathcal{A}_i$, where $\{i\}$ is the singleton consisting of only one element i. Since $A_i \in \mathcal{B}(\Phi)$, A_i has the another expansion

$$A_i = \sum_{\sigma,\tau\in \mathrm{Per}(T)} b_{\sigma,\tau}^{(i)} d_\sigma^+ d_\tau^-$$

by the "basis" $\{d_\sigma^+ d_\tau^- | \sigma, \tau \in \mathrm{Per}(T)\}$ of $\mathcal{B}(\Phi)$. The coefficients b's can be represented by a's :

$$b_{\sigma,\tau}^{(i)} = \begin{cases} a_{\Lambda,\Lambda}^{(i)} + a^{(i)} & (\text{for } \sigma = \tau = \Lambda), \\ a_{\sigma,\tau}^{(i)} & (\text{for } (\sigma,\tau) = (\Lambda,(i)) \text{ or } (\sigma,\tau) = ((i),\Lambda)), \\ a_{(i),(i)}^{(i)} - a^{(i)} & (\text{for } \sigma = \tau = (i)), \\ -a^{(i)} & (\text{for } \sigma = \tau = (\rho, i) \text{ where } \rho \in \mathrm{Per}(T \setminus \{i\}) \text{ and } \rho \neq \Lambda), \\ 0 & (\text{otherwise}). \end{cases}$$

Note that $b_{\sigma,\tau}^{(i)} = 0$ if

$\ll \tau \neq \Lambda$, and the right terminal of τ is not equals $i \gg$

or $\ll \tau \neq \Lambda$, the right terminal of τ is i, and $\sigma \neq \tau \gg$.

Let us consider the vacuum expectation

$$< \Omega | A_1 \cdots A_n | \Omega > = \sum_{\sigma_1,\tau_1,\cdots,\sigma_n,\tau_n} b_{\sigma_1,\tau_1}^{(1)} \cdots b_{\sigma_n,\tau_n}^{(n)} < \Omega | d_{\sigma_1}^+ d_{\tau_1}^- \cdots d_{\sigma_n}^+ d_{\tau_n}^- | \Omega > .$$

For the vector $d_{\tau_n}^- \Omega$ not to vanish, τ_n must be equal to the null permutation Λ. For the term

$$b_{\sigma_1,\tau_1}^{(1)} b_{\sigma_2,\tau_2}^{(2)} \cdots b_{\sigma_n,\Lambda}^{(n)} < \Omega | \cdots | \Omega >$$

not to vanish, it is necessary that $b_{\sigma_n,\Lambda}^{(n)} \neq 0$, and hence it is necessary that $\sigma_n = \Lambda$ or $\sigma_n = (n)$. Next, for the vector $d_{\tau_{n-1}}^- d_{\sigma_n}^+ d_\Lambda^- \Omega$ not to vanish, it is necessary that $\tau_{n-1} = \Lambda$ or $\tau_{n-1} = (n)$. By the way, for the term $b_{\sigma_1,\tau_1}^{(1)} b_{\sigma_2,\tau_2}^{(2)} \cdots b_{\sigma_n,\Lambda}^{(n)} < \Omega | \cdots | \Omega >$ not to vanish, it is necessary that $b_{\sigma_{n-1},\tau_{n-1}}^{(n-1)} \neq 0$, and hence it is necessary that

$\ll \tau_{n-1} = \Lambda \gg$,

or $\ll \tau_{n-1} \neq \Lambda$, and the right terminal of τ_{n-1} equals $n-1 \gg$.

Therefore, for the term $b_{\sigma_1,\tau_1}^{(1)} b_{\sigma_2,\tau_2}^{(2)} \cdots b_{\sigma_n,\Lambda}^{(n)} < \Omega | \cdots | \Omega >$ not to vanish, it is necessary that $\tau_{n-1} = \Lambda$. Repeating this discussion, we can see that, for the term $b_{\sigma_1,\tau_1}^{(1)} b_{\sigma_2,\tau_2}^{(2)} \cdots b_{\sigma_n,\Lambda}^{(n)} < \Omega | \cdots | \Omega >$ not to vanish, it must be hold that

$$\tau_1 = \tau_2 = \cdots = \tau_n = \Lambda.$$

This implies that $\sigma_i = \Lambda$ or $\sigma_i = (i)$ for each $i = 1, \cdots, n$. In such case we have $< \Omega | d_{\sigma_1}^+ d_{\tau_1}^- \cdots d_{\sigma_n}^+ d_{\tau_n}^- | \Omega > = 0$ if there exists i such that $\sigma_i = (i)$. Therefore the only term which survives is $b_{\Lambda,\Lambda}^{(1)} \cdots b_{\Lambda,\Lambda}^{(n)} < \Omega | d_\Lambda^+ d_\Lambda^- \cdots d_\Lambda^+ d_\Lambda^- | \Omega >$. So we have

$$< \Omega | A_1 \cdots A_n | \Omega > = b_{\Lambda,\Lambda}^{(1)} \cdots b_{\Lambda,\Lambda}^{(n)} = < \Omega | A_1 | \Omega > \cdots < \Omega | A_n | \Omega > .$$

This implies the factorization: $\phi(A_1 \cdots A_n) = \phi(A_1) \cdots \phi(A_n)$. □

Corollary 4.6: The family of C^*-subalgebras $\{\mathcal{A}_k\}_{k\in T}$ in the quantum probability space $(\mathcal{A},\phi)$ is independent in the sense of Kümmerer.

The time indexed family of subalgebras $\{\mathcal{A}_k\}_{k\in T}$ can be viewd as a discrete time quantum i.i.d. process, i.e. a process with independently and identically distributed subalgebras, in the Kümmerer independence.

Remark 4.7: The following simple example shows that the independence arising from the permutational Fock space is not the free independence. Put $A_1 = d_1^-$, $A_2 = d_2^-$, $A_3 = I - d_1^\bullet$, $A_4 = d_2^+$ and $A_5 = d_1^+$, then we have $A_i \in \mathcal{A}_{k_i}$ $(i = 1,\cdots,5)$ with $k_1 \neq k_2 \neq k_3 \neq k_4 \neq k_5$ and $\phi(A_i) = 0$ $(i = 1,2,\cdots,5)$ but $\phi(A_1A_2A_3A_4A_5) = 1$. Besides it is easy to see that the independence in the permutational Fock space is neither the commuting independence nor the anticommuting independence.

5. Filtration

In this section, we examine the filtration structure of the quantum probability space $(\mathcal{A},\phi)$.

Put $T_k = \{1,2,\cdots,k\}$, and let $\mathcal{A}_{k]} = C^*(I, d_i^+, d_i^- | i \in T_k)$ be the C^*-subalgebra of $\mathcal{A}$ generated by all the creation and annihilation operators up to time k, with identity. Then the increasing family $\{\mathcal{A}_{k]}\}_{k\in T}$ of C^*-subalgebras of $\mathcal{A}$ can be interpreted as the *filtration* of $\mathcal{A}$. Let us find the canonical basis of the algebra $\mathcal{A}_{k]}$. Put $L_k = \{(\sigma,U,\tau) \mid \sigma,\tau \in \mathrm{Per}(T_k),\ U \subset T_k,\ U\perp[\sigma]\cup[\tau]\}$. Then we have

Theorem 5.1 (Fock Expansion for $\mathcal{A}_{k]}$): *Every operator $A \in \mathcal{A}_{k]}$ has the unique expansion of the form*

$$A = \sum_{(\sigma,U,\tau)\in L_k} a_{\sigma,U,\tau} d_\sigma^+ d_U^\bullet d_\tau^-,$$

where $a_{\sigma,U,\tau}$ are scalars.

Proof: At first let us show that the set $\{d_\sigma^+ d_U^\bullet d_\tau^-\}_{(\sigma,U,\tau)\in L_k} \cup \{0\}$ is stable under the multiplication. Let us compute the product $(d_\sigma^+ d_U^\bullet d_\tau^-)(d_{\sigma'}^+ d_{U'}^\bullet d_{\tau'}^-)$, then

$$(d_\sigma^+ d_U^\bullet d_\tau^-)(d_{\sigma'}^+ d_{U'}^\bullet d_{\tau'}^-) = \begin{cases} d_\sigma^+ d_U^\bullet (d_\beta^+ d_\alpha^\bullet) d_{U'}^\bullet d_{\tau'}^- \\ \quad (\text{if } \exists \alpha,\beta \in \mathrm{Per}(T_k) \text{ s.t. } [\alpha]\perp[\beta], \tau = \alpha,\ \sigma' = (\alpha,\beta)), \\ d_\sigma^+ d_U^\bullet (d_\alpha^\bullet d_\beta^-) d_{U'}^\bullet d_{\tau'}^- \\ \quad (\text{if } \exists \alpha,\beta \in \mathrm{Per}(T_k) \text{ s.t. } [\alpha]\perp[\beta], \tau = (\alpha,\beta),\ \sigma' = \alpha), \\ 0 \quad (\text{otherwise}). \end{cases}$$

Here we used the multiplication formula given in §2. Furthermore the factors in the above expression can be rewritten as

$$d_U^\bullet d_\beta^+ = \begin{cases} d_\beta^+ d_U^\bullet & (\text{for } [\beta]\perp U), \\ 0 & (\text{otherwise}), \end{cases} \qquad d_\beta^- d_{U'}^\bullet = \begin{cases} d_{U'}^\bullet d_\beta^- & (\text{for } [\beta]\perp U'), \\ 0 & (\text{otherwise}). \end{cases}$$

This implies that $(d^+_\sigma d^\bullet_U d^-_\tau)(d^+_{\sigma'} d^\bullet_{U'} d^-_{\tau'})$ is equal to either $d^+_\sigma d^+_\beta d^\bullet_U d^\bullet_\alpha d^\bullet_{U'} d^-_{\tau'}$ or $d^+_\sigma d^\bullet_U d^\bullet_\alpha d^\bullet_{U'} d^-_\beta d^-_{\tau'}$ or 0. Hence we get the stability of $\{d^+_\sigma d^\bullet_U d^-_\tau\}_{(\sigma,U,\tau)\in L_k} \cup \{0\}$ under the multiplication. This implies that $\{d^+_\sigma d^\bullet_U d^-_\tau\}_{(\sigma,U,\tau)\in L_k}$ is a generating system for the vector space $\mathcal{A}_{k]}$.

Let us show the linear independence of the family $\{d^+_\sigma d^\bullet_U d^-_\tau\}_{(\sigma,U,\tau)\in L_k}$. Assume that $\sum_{(\sigma,U,\tau)\in L_k} a_{\sigma,U,\tau} d^+_\sigma d^\bullet_U d^-_\tau = 0$. Using

$$d^\bullet_U = I - \sum_{\substack{(\gamma,i)\ s.t.\\ [\gamma]\perp U,\ \gamma\in \mathrm{Per}(T),\\ i\in U}} d^\circ_{(\gamma,i)},$$

we have

$$\begin{aligned}
&\sum_{(\sigma,U,\tau)\in L_k} a_{\sigma,U,\tau}\ d^+_\sigma d^\bullet_U d^-_\tau \\
&= \sum_{\sigma,\tau\in \mathrm{Per}(T_k)} a_{\sigma,\emptyset,\tau} d^+_\sigma d^-_\tau + \sum_{\substack{U\neq\emptyset\\ U\subset T_k}} \sum_{\substack{[\sigma]\perp U\\ [\tau]\perp U\\ \sigma,\tau\in \mathrm{Per}(T_k)}} a_{\sigma,U,\tau} d^+_\sigma d^\bullet_U d^-_\tau \\
&= \sum_{(\sigma,U,\tau)\in L_k} a_{\sigma,U,\tau} d^+_\sigma d^-_\tau - \sum_{U\neq\emptyset} \sum_{\substack{[\sigma]\perp U\\ [\tau]\perp U}} \sum_{\substack{(\gamma,i)\ s.t.\\ [\gamma]\perp U,\ i\in U\\ \gamma\in \mathrm{Per}(T)}} a_{\sigma,U,\tau} d^+_\sigma d^\circ_{(\gamma,i)} d^-_\tau \\
&= \sum_{(\sigma,U,\tau)\in L_k} a_{\sigma,U,\tau} d^+_\sigma d^-_\tau - \sum_{U\neq\emptyset} \sum_{\substack{[\sigma]\perp U\\ [\tau]\perp U}} \sum_{\substack{(\gamma,i)\ s.t.\\ [\gamma]\perp U,\ i\in U,\\ \gamma\in \mathrm{Per}(T_k)}} a_{\sigma,U,\tau} d^+_\sigma d^\circ_{(\gamma,i)} d^-_\tau \\
&\quad - \sum_{U\neq\emptyset} \sum_{\substack{[\sigma]\perp U\\ [\tau]\perp U}} \sum_{\substack{(\gamma,i)\ s.t.\\ [\gamma]\perp U,\ i\in U,\\ \gamma\in \mathrm{Per}(T)\setminus \mathrm{Per}(T_k)}} a_{\sigma,U,\tau} d^+_\sigma d^\circ_{(\gamma,i)} d^-_\tau .
\end{aligned}$$

The third term of the last equality can be rewritten as

$$- \sum_{\substack{(\lambda,\mu,i,\nu)\ s.t.\\ \lambda,\nu\in \mathrm{Per}(T_k),\\ i\in T_k,\\ \mu\in \mathrm{Per}(T)\setminus \mathrm{Per}(T_k),\\ \text{“the left terminal of }\mu\text{”}\in T\setminus T_k,\\ [\lambda]\cup[\nu],\ [\mu],\ \{i\}\ :\ \text{mutually disjoint}}} \Bigg(\sum_{\substack{(\sigma,U,\tau)\ s.t.\\ i\in U\subset T_k,\\ U\perp [\lambda]\cup[\mu]\cup[\nu],\\ \sigma,\tau\in \mathrm{Per}(T_k),\\ \exists\rho\in T_k\ s.t.\ \lambda=(\sigma,\rho),\nu=(\tau,\rho)}} a_{\sigma,U,\tau} \Bigg)\ d^+_\lambda d^\circ_{(\mu,i)} d^-_\nu .$$

By the “linear independence” of $\{d^+_\alpha d^-_\beta | \alpha,\beta\in \mathrm{Per}(T)\}$, we get

$$\sum_{\substack{(\sigma,U,\tau)\ s.t.\\ i\in U\subset T_k,\\ U\perp [\lambda]\cup[\mu]\cup[\nu],\\ \sigma,\tau\in \mathrm{Per}(T_k),\\ \exists\rho\in T_k\ s.t.\ \lambda=(\sigma,\rho),\nu=(\tau,\rho)}} a_{\sigma,U,\tau} = 0$$

for all quadrupple (λ,μ,i,ν) s. t. $\lambda,\nu\in \mathrm{Per}(T_k)$, $i\in T_k$, $\mu\in \mathrm{Per}(T)\setminus \mathrm{Per}(T_k)$, “the left terminal of μ” $\in T\setminus T_k$ and the triple $[\lambda]\cup[\nu]$, $[\mu]$, $\{i\}$ is mutually disjoint.

Let us show that $a_{\sigma,U,\tau} = 0$ for all $(\sigma, U, \tau) \in L_k$. For each triple (λ, i, ν) s.t. $\lambda, \nu \in T_k$, $i \in T_k$ and $i \notin [\lambda] \cup [\nu]$, put

$$l(\lambda,\nu) = \max\{\text{length}(\rho) | \exists \lambda', \nu', \rho \in T_k \ \ s.t. \ \lambda = (\lambda', \rho), \nu = (\nu', \rho)\},$$

where length(ρ)$= r$ for $\rho \in {}_T\mathrm{P}_r$. Let us first cinsider the case of $l(\lambda,\nu) = 0$. Put $\mu := \mu_1$, where μ_1 is any permutation s. t.

$$[\mu_1] = T_{k+1} \setminus ([\lambda] \cup [\nu] \cup \{i\}),$$

then we get $a_{\lambda,\{i\},\nu} = 0$. Put $\mu := \mu_2$, where μ_2 is any permutation s. t.

$$[\mu_2] = T_{k+1} \setminus ([\lambda] \cup [\nu] \cup \{i,j\})$$

with $i \neq j$ and $\{i,j\} \subset T_k \setminus ([\lambda] \cup [\nu])$, then we get

$$a_{\lambda,\{i\},\nu} + a_{\lambda,\{i,j\},\nu} = 0,$$

and hence $a_{\lambda,\{i,j\},\nu} = 0$. Repeating this discussion, we have

$$\begin{cases} a_{\lambda,\{i\},\nu} = 0, \\ a_{\lambda,\{i,j\},\nu} = 0, \\ a_{\lambda,\{i,k\},\nu} = 0, \\ \quad \cdots\cdots \\ a_{\lambda,\{i,j,k\},\nu} = 0, \\ \quad \cdots\cdots \\ a_{\lambda,T_k\setminus([\lambda]\cup[\nu]),\nu} = 0. \end{cases}$$

Hence we get $a_{\lambda,U,\nu} = 0$ for all triple $(\lambda, U, \nu) \in L_k$ s.t. $l(\lambda,\nu) = 0$ and $U \ni i$. Next let us consider the case of $l(\lambda,\nu) = 1$. For $\lambda = (i_1, \cdots, i_r)$ and $\nu = (j_1, \cdots, j_s)$, put $\lambda' = (i_1, \cdots, i_{r-1})$ and $\nu' = (j_1, \cdots, j_{s-1})$. By the discussion similar to the preceeding one, we get

$$\begin{cases} a_{\lambda,\{i\},\nu} + a_{\lambda',\{i\},\nu'} = 0, \\ a_{\lambda,\{i\},\nu} + a_{\lambda',\{i\},\nu'} + a_{\lambda,\{i,j\},\nu} + a_{\lambda',\{i,j\},\nu'} = 0, \\ a_{\lambda,\{i\},\nu} + a_{\lambda',\{i\},\nu'} + a_{\lambda,\{i,k\},\nu} + a_{\lambda',\{i,k\},\nu'} = 0, \\ \quad \cdots\cdots \\ a_{\lambda,\{i\},\nu} + a_{\lambda',\{i\},\nu'} + a_{\lambda,\{i,j\},\nu} + a_{\lambda',\{i,j\},\nu'} + a_{\lambda,\{i,k\},\nu} + a_{\lambda',\{i,k\},\nu'} \\ +a_{\lambda,\{i,j,k\},\nu} + a_{\lambda',\{i,j,k\},\nu'} = 0, \\ \quad \cdots\cdots \\ \displaystyle\sum_{\substack{U \ s.t. \\ i \in U \subset T_k, \\ U \perp [\lambda] \cup [\nu],}} a_{\lambda,U,\nu} + \sum_{\substack{U \ s.t. \\ i \in U \subset T_k, \\ U \perp [\lambda] \cup [\nu],}} a_{\lambda',U,\nu'} = 0. \end{cases}$$

Hence we get

$$a_{\lambda,\{i\},\nu} = a_{\lambda,\{i,j\},\nu} = a_{\lambda,\{i,k\},\nu} = \cdots = a_{\lambda,\{i,j,k\},\nu} = \cdots = a_{\lambda,T_k\setminus([\lambda]\cup[\nu]),\nu} = 0.$$

Here we used the assumption $l(\lambda',\nu') = 0$. So we have $a_{\lambda,U,\nu} = 0$ for all triple $(\lambda, U, \nu) \in L_k$ s.t. $l(\lambda,\nu) = 1$ and $U \ni i$. By the induction on the number $l = l(\lambda,\nu)$, we finally get

$$a_{\lambda,U,\nu} = 0 \quad (\text{for } (\lambda,U,\nu) \in L_k,\ [\lambda] \cup [\nu] \perp \{i\},\ U \ni i).$$

So we have

$$a_{\sigma,U,\tau} = 0 \quad (\text{for } (\sigma,U,\tau) \in L_k,\ U \neq \emptyset).$$

This necessarily implies $\sum_{\sigma,\tau\in\mathrm{Per}(T_k)} a_{\sigma,\emptyset,\tau} d_\sigma^+ d_\tau^- = 0$, and hence we also get $a_{\sigma,\emptyset,\tau} = 0$. So we get the linear independence of the family $\{d_\sigma^+ d_U^\bullet d_\tau^-\}_{(\sigma,U,\tau)\in L_k}$. So we conclude that this family is a basis of $\mathcal{A}_{k]}$ □

6. Conditional Expectation

In this section, we construct a kind of "conditional expectation" $\varepsilon_{k]} : A \mapsto \varepsilon_{k]}(A)$ from the algebra $\mathcal{A}$ onto the subalgebra $\mathcal{A}_{k]}$ with respect to the vacuum state ϕ. The conditional expectation $\varepsilon_{k]}$ will be used in the next section to define the notion of operator martingales.

Before constructing the conditional expectation, let us first examine the decomposition of the algebra $\mathcal{A}_{k]}$. Put $\mathcal{B}_{k]} = C^*(d_\sigma^+, d_\sigma^- | \sigma \in \mathrm{Per}(T_k), [\sigma] \ni k)$, then the algebra $\mathcal{B}_{k]}$ can be decomposed to the following form as a vector space.

Proposition 6.1:

$$\begin{aligned}\mathcal{B}_{k]} \;=\; & \bigoplus_{\sigma,\tau\in\mathrm{Per}(T_{k-1})} d^+_{(\sigma,k)}\mathcal{A}_{k-1]}d^-_{(\tau,k)} \;\oplus\; \bigoplus_{\sigma\in\mathrm{Per}(T_{k-1})} d^+_{(\sigma,k)}\mathcal{A}_{k-1]} \\ & \oplus \bigoplus_{\tau\in\mathrm{Per}(T_{k-1})} \mathcal{A}_{k-1]}d^-_{(\tau,k)} \;\oplus\; \mathcal{A}_{k-1]}d^\bullet_k.\end{aligned}$$

Proof: At first let us show that the set

$$\{d^+_{(\sigma,k)}Ad^-_{(\tau,k)}, d^+_{(\sigma,k)}B, Cd^-_{(\tau,k)}, Dd^\bullet_k | A,B,C,D \in \mathcal{A}_{k-1]}, \sigma,\tau \in \mathrm{Per}(T_{k-1})\} \cup \{0\}$$

is stable under the multiplication. Using the fact that $d_k^\bullet$ is a projection and formulas

$$\begin{aligned} d_\sigma^+ d_k^\bullet &= d_k^\bullet d_\sigma^+ \quad (\text{for } k \notin [\sigma]), \quad d_\sigma^- d_k^\bullet = d_k^\bullet d_\sigma^- \quad (\text{for } k \notin [\sigma]), \\ d_k^+ d_k^\bullet &= d_k^+, \quad d_k^\bullet d_k^- = d_k^-, \end{aligned}$$

we get the multiplication rule as follows:

$$(d^+_{(\sigma,k)}Ad^-_{(\tau,k)})(d^+_{(\sigma',k)}A'd^-_{(\tau',k)}) = \begin{cases} d^+_{(\sigma,k)}(Ad^\bullet_\tau A')d^-_{(\tau',k)} & (\tau = \sigma'), \\ 0 & (\text{otherwise}), \end{cases}$$

$$(d^+_{(\sigma,k)}Ad^-_{(\tau,k)})(d^+_{(\sigma',k)}B') = \begin{cases} d^+_{(\sigma,k)}(Ad^\bullet_\tau B') & (\tau = \sigma'), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\begin{aligned} &(d^+_{(\sigma,k)}Ad^-_{(\tau,k)})(C'd^-_{(\tau',k)}) = 0, \qquad (d^+_{(\sigma,k)}Ad^-_{(\tau,k)})(D'd^\bullet_k) = 0, \\ &(d^+_{(\sigma,k)}B)(d^+_{(\sigma',k)}A'd^-_{(\tau',k)}) = 0, \qquad (d^+_{(\sigma,k)}B)(d^+_{(\sigma',k)}B') = 0, \\ &(d^+_{(\sigma,k)}B)(C'd^-_{(\tau',k)}) = d^+_{(\sigma,k)}(BC')d^-_{(\tau',k)}, \end{aligned}$$

$$(d^+_{(\sigma,k)}B)(D'd^\bullet_k) = d^+_{(\sigma,k)}(BD'),$$

$$(Cd^-_{(\tau,k)})(d^+_{(\sigma',k)}A'd^-_{(\tau',k)}) = \begin{cases} (CA')d^-_{(\tau',k)} & (\tau = \sigma'), \\ 0 & (\text{otherwise}), \end{cases}$$

$$(Cd^-_{(\tau,k)})(d^+_{(\sigma',k)}B') = \begin{cases} (Cd^\bullet_\tau B')d^\bullet_k & (\tau = \sigma'), \\ 0 & (\text{otherwise}), \end{cases}$$

$$(Cd^-_{(\tau,k)})(C'd^-_{(\tau',k)}) = 0, \qquad (Cd^-_{(\tau,k)})(D'd^\bullet_k) = 0,$$

$$(Dd^\bullet_k)(d^+_{(\sigma',k)}A'd^-_{(\tau',k)}) = 0, \qquad (Dd^\bullet_k)(d^+_{(\sigma',k)}B') = 0,$$

$$(Dd^\bullet_k)(C'd^-_{(\tau',k)}) = (DC')d^-_{(\tau',k)}, \quad (Dd^\bullet_k)(D'd^\bullet_k) = (DD')d^\bullet_k,$$

where $A, B, C, D, A', B', C', D' \in \mathcal{A}_{k-1]}$. This implies that vector space $\mathcal{C} = \sum_{\sigma,\tau} d^+_{(\sigma,k)}\mathcal{A}_{k-1]}d^-_{(\tau,k)} + \sum_\sigma d^+_{(\sigma,k)}\mathcal{A}_{k-1]} + \sum_\tau \mathcal{A}_{k-1]}d^-_{(\tau,k)} + \mathcal{A}_{k-1]}d^\bullet_k$ forms an algebra. The equality $\mathcal{B}_{k]} = \mathcal{C}$ is obvious. We also get the direct sum property from the linear independence of the finite family of operators $\{d^+_\sigma d^\bullet_U d^-_\tau\}_{(\sigma,U,\tau)\in L_k}$. □

Besides the algebra $\mathcal{A}_{k]}$ can be decomposed to the direct sum of $\mathcal{B}_{j]}$'s ($j \leq k$) with $\mathbf{C}I$.

Proposition 6.2: $\mathcal{A}_{k]} = \mathbf{C}I \oplus \mathcal{B}_{1]} \oplus \cdots \oplus \mathcal{B}_{k]}$.

Corollary 6.3:

$$\begin{aligned}\mathcal{A}_{k]} &= \mathcal{A}_{k-1]} \oplus \bigoplus_{\sigma,\tau\in\mathrm{Per}(T_{k-1})} d^+_{(\sigma,k)}\mathcal{A}_{k-1]}d^-_{(\tau,k)} \oplus \bigoplus_{\sigma\in\mathrm{Per}(T_{k-1})} d^+_{(\sigma,k)}\mathcal{A}_{k-1]} \\ &\quad \oplus \bigoplus_{\tau\in\mathrm{Per}(T_{k-1})} \mathcal{A}_{k-1]}d^-_{(\tau,k)} \oplus \mathcal{A}_{k-1]}d^\bullet_k.\end{aligned}$$

Using the decomposition of the algebra $\mathcal{A}_{k]}$, let us construct a conditional expectation. Let $\varepsilon_{k-1,k} : \mathcal{A}_{k]} \to \mathcal{A}_{k-1]}$ be a linear map defined by

$$\varepsilon_{k-1,k}(A) = \begin{cases} A & (\text{if } A \in \mathcal{A}_{k-1]}), \\ 0 & (\text{if } A \in \oplus_{\sigma,\tau\in\mathrm{Per}(T_{k-1})} d^+_{(\sigma,k)}\mathcal{A}_{k-1]}d^-_{(\tau,k)}), \\ 0 & (\text{if } A \in \oplus_{\sigma\in\mathrm{Per}(T_{k-1})} d^+_{(\sigma,k)}\mathcal{A}_{k-1]}), \\ 0 & (\text{if } A \in \oplus_{\tau\in\mathrm{Per}(T_{k-1})} \mathcal{A}_{k-1]}d^-_{(\tau,k)}), \\ B & (\text{if } A = Bd^\bullet_k \in \mathcal{A}_{k-1]}d^\bullet_k). \end{cases}$$

For $j < k$, put $\varepsilon_{j,k} = \varepsilon_{j,j+1} \circ \cdots \circ \varepsilon_{k-1,k}$, then the following holds.

Proposition 6.4: The map $\varepsilon_{j,k} : \mathcal{A}_{k]} \to \mathcal{A}_{j]}$, $j < k$, satisfies the followings for all $A \in \mathcal{A}_{k]}$ and all $B, B_1, B_2 \in \mathcal{A}_{j]}$:
(1) $\varepsilon_{j,k}(B) = B$; (2) $\varepsilon_{j,k}(A^*) = \varepsilon_{j,k}(A)^*$; (3) $\varepsilon_{j,k}(A^*A) \geq 0$;
(4) $\| \varepsilon_{j,k}(A) \| \leq \| A \|$; (5) $\varepsilon_{j,k}(A^*A) \geq \varepsilon_{j,k}(A)^*\varepsilon_{j,k}(A)$;
(6) $\phi(\varepsilon_{j,k}(A)) = \phi(A)$; (7) $\varepsilon_{j,k}(B_1AB_2) = B_1\varepsilon_{j,k}(A)B_2$.

Proof: We only have to show that the map $\varepsilon_{k-1,k}$ satisfies (1), $\cdots$, (7).
(1) Obvious from the definition.
(2) Let

$$A = A_0 + \sum_{\sigma,\tau} d^+_{(\sigma,k)}A_{\sigma,\tau}d^-_{(\tau,k)} + \sum_\sigma d^+_{(\sigma,k)}B_\sigma + \sum_\tau C_\tau d^-_{(\tau,k)} + Dd^\bullet_k,$$

where $A_0, A_{\sigma,\tau}, B_\sigma, C_\tau, D \in \mathcal{A}_{k-1]}$, then

$$A^* = A_0^* + \sum_{\sigma,\tau} d^+_{(\tau,k)} A^*_{\sigma,\tau} d^-_{(\sigma,k)} + \sum_\sigma B^*_\sigma d^-_{(\sigma,k)} + \sum_\tau d^+_{(\tau,k)} C^*_\tau + D^* d^\bullet_k,$$

and hence $\varepsilon_{k-1,k}(A^*) = A_0^* + D^* = \varepsilon_{k-1,k}(A)^*$.
(3) By the multiplication rule discussed in Proof of Theorem 6.1, we get

$$\begin{aligned}\varepsilon_{k-1,k}(A^*A) &= A_0^*A_0 + A_0^*D + D^*A_0 + D^*D + \sum_{\sigma\in\mathrm{Per}(T_{k-1})} B^*_\sigma B_\sigma \\ &= (A_0+D)^*(A_0+D) + \sum_\sigma B^*_\sigma B_\sigma \geq 0.\end{aligned}$$

(4) From $\varepsilon_{k-1,k}(I) = I$ and (3), we get $\| \varepsilon_{k-1,k} \| = 1$. Hence $\| \varepsilon_{k-1,k}(A) \| \leq \| A \|$.
(5) Since $\varepsilon_{k-1,k}(A)^*\varepsilon_{k-1,k}(A) = (A_0+D)^*(A_0+D)$, we get

$$\varepsilon_{k-1,k}(A^*A) - \varepsilon_{k-1,k}(A)^*\varepsilon_{k-1,k}(A) = \sum_\sigma B^*_\sigma B_\sigma \geq 0.$$

(6) We have

$$\begin{aligned}&< \Omega|A|\Omega > \\ &=< \Omega|A_0 + \sum_{\sigma,\tau} d^+_{(\sigma,k)} A_{\sigma,\tau} d^-_{(\tau,k)} + \sum_\sigma d^+_{(\sigma,k)} B_\sigma + \sum_\tau C_\tau d^-_{(\tau,k)} + Dd^\bullet_k|\Omega > \\ &=< \Omega|(A_0 + Dd^\bullet_k)|\Omega >=< \Omega|(A_0+D)|\Omega >=< \Omega|\varepsilon_{k-1,k}(A)|\Omega > .\end{aligned}$$

(7) We have

$$\begin{aligned}B_1AB_2 &= B_1(A_0 + \sum_{\sigma,\tau} d^+_{(\sigma,k)} A_{\sigma,\tau} d^-_{(\tau,k)} + \sum_\sigma d^+_{(\sigma,k)} B_\sigma + \sum_\tau C_\tau d^-_{(\tau,k)} + Dd^\bullet_k)B_2 \\ &= B_1A_0B_2 + (B_1DB_2)d^\bullet_k \\ &\quad +\{\sum_{\sigma,\tau} B_1 d^+_{(\sigma,k)} A_{\sigma,\tau} d^-_{(\tau,k)} B_2 + \sum_\sigma B_1 d^+_{(\sigma,k)} B_\sigma B_2 + \sum_\tau B_1 C_\tau d^-_{(\tau,k)} B_2\}.\end{aligned}$$

Since the last term $\{\cdots\}$ vanishes under $\varepsilon_{k-1,k}$, we get

$$\varepsilon_{k-1,k}(B_1AB_2) = B_1A_0B_2 + B_1DB_2 = B_1(A_0+D)B_2 = B_1\varepsilon_{k-1,k}(A)B_2. \quad \square$$

The system of maps $\{\varepsilon_{j,k} | j < k\}$ is consistent, i.e. $\varepsilon_{j,k} \circ \varepsilon_{k,l} = \varepsilon_{j,l}$ for $j < k < l$. So we can define in the natural way a map $\tilde{\varepsilon}_{j]} : \cup_k \mathcal{A}_{k]} \to \mathcal{A}_{j]}$ as the extension of each $\varepsilon_{j,k}$. Furthermore a map $\varepsilon_{j]} : \mathcal{A} \to \mathcal{A}_{j]}$ can be defined in the natural way as the continuous extension of $\tilde{\varepsilon}_{j]}$ because of the norm continuity of $\tilde{\varepsilon}_{j]}$. Then the following is easily obtained.

Theorem 6.5: *The map $\varepsilon_{j]} : \mathcal{A} \to \mathcal{A}_{j]}$ satisfies the followings for all $A \in \mathcal{A}$ and all $B, B_1, B_2 \in \mathcal{A}_{j]}$:*
(1) $\varepsilon_{j]}(B) = B$; (2) $\varepsilon_{j]}(A^*) = \varepsilon_{j]}(A)^*$; (3) $\varepsilon_{j]}(A^*A) \geq 0$;
(4) $\| \varepsilon_{j]}(A) \| \leq \| A \|$; (5) $\varepsilon_{j]}(A^*A) \geq \varepsilon_{j]}(A)^*\varepsilon_{j]}(A)$;
(6) $\phi(\varepsilon_{j]}(A)) = \phi(A)$; (7) $\varepsilon_{j]}(B_1AB_2) = B_1\varepsilon_{j]}(A)B_2$.

Hence it is natural to interpret the map $\varepsilon_{k]}$ as a *conditional expectation* from the algebra $\mathcal{A}$ onto the subalgebra $\mathcal{A}_{k]}$ with respect to the vacuum state ϕ.

Remark 6.6: Conditional expectation for an operator algebra was originally introduced by H. Umegaki[15] in the theory of von Neumann algebras. Note that the map $\varepsilon_{k]}$ satisfies the properties of conditional expectation in the sense of Umegaki, up to the faithfulness of the state ϕ and the weak continuity of the map. The map $\varepsilon_{k]}$ is rather a conditional expectation in the C^*-algebraic setting.

7. Operator Matingales

In this section, we discuss about operator martingales on the quantum probability space $(\mathcal{A}, \phi)$.

Let us introduce the notion of adapted operators, previsible operators and operator martingales. These notions are defined in the parallel way to those in the Journés toy Fock space.[2] An operator $A \in \mathcal{A}$ is called a ***k*-adapted operator** if $A \in \mathcal{A}_{k]}$. An operator $A \in \mathcal{A}$ is called a ***k*-previsible operator** if $A \in \mathcal{A}_{k-1]}$. Let $\{M_k\}_{k\in T}$ be a time indexed family of adapted operators $M_k \in \mathcal{A}_{k]}$. We call it an *operator martingale* if $\varepsilon_{k]}(M_{k'}) = M_k$ for $k \leq k'$. A time indexed family $\{u_k\}_{k\in T}$ of operators is called a *previsible process* if each u_k is a k-previsible operator.

The *creation process* $D_k^+ = \sum_{j=1}^{k} d_j^+$, the *conservation process* $D_k^\circ = \sum_{j=1}^{k} d_j^\circ$ and the *annihilation process* $D_k^- = \sum_{j=1}^{k} d_j^-$ are important examples of operator martingale. Note that the process $D_k^\bullet = \sum_{j=1}^{k} d_j^\bullet$ is not an operator martingale (it is rather a *submartingale*) but the process $\prod_{j=1}^{k} d_j^\bullet$ is an operator martingale. Let us define the *inclusion operator* $e_k^\bullet$ by $e_k^\bullet = I - d_k^\bullet$, then the process $E_k^\bullet = \sum_{j=1}^{k} e_j^\bullet$ is an operator martingale. For a finite subset $U \subset T$, put $e_U^\bullet = \prod_{k\in U} e_k^\bullet$. The explicit action of the inclusion operator $e_U^\bullet$ is

$$e_U^\bullet e_\sigma = \begin{cases} e_\sigma & (\text{if } U \subset [\sigma]), \\ 0 & (\text{otherwise}). \end{cases}$$

We get two expansions:

$$d_U^\bullet = \sum_{V\subset U} (-1)^{\#(V)} e_V^\bullet \qquad \text{and} \qquad e_U^\bullet = \sum_{V\subset U} (-1)^{\#(V)} d_V^\bullet.$$

Using this expansion, we can see that the family $\{d_\sigma^+ e_U^\bullet d_\tau^-\}_{(\sigma,U,\tau)\in L_k}$ becomes another basis of the algebra $\mathcal{A}_{k]}$. We also get the following uniform Fock expansion for operators in the C^*-algebra $\mathcal{A}$ because of the the norm continuity of the conditional expectation $\varepsilon_{j]} : \mathcal{A} \to \mathcal{A}_{j]}$.

Theorem 7.1 (Uniform Fock Expansion): *Each operator $M \in \mathcal{A}$ has the unique expansion of the form*

$$M = \lim_{j\to\infty} \sum_{(\sigma,U,\tau)\in L_j} m_{\sigma,U,\tau} d_\sigma^+ e_U^\bullet d_\tau^-$$

in the uniform topology.

In Journé's toy Fock space Φ_{toy}, the following predictable representation theorem for operator martingales holds.[2] Let a_j^+ (resp. a_j^-, a_j°) be the creation operator (resp. annihilation operator, conservation operator) in Journé's toy Fock space Φ_{toy}.[2]

Theorem 7.2 (Predictable Representation[2]): *Each operator martingale $\{M_k\}_{k\in T}$ on Φ_{toy} can be uniquely expanded to the form*

$$M_k = mI + \sum_{j=1}^{k} u_j^+ a_j^+ + \sum_{j=1}^{k} u_j^- a_j^- + \sum_{j=1}^{k} u_j^\circ a_j^\circ \quad (k = 1, 2, \cdots),$$

where m is a scalar, $\{u_k^+\}_{k\in T}$, $\{u_k^-\}_{k\in T}$ and $\{u_k^\circ\}_{k\in T}$ are previsible processes on Φ_{toy}.

This means that each operator martingale on Φ_{toy} has the canonical expansion as a sum of triple of discrete time "stochastic integrals" of previsible processes where the "integrators" are the three basic martingales, i.e. the creation process $A_k^+ = \sum_{j=1}^{k} a_j^+$, the annihilation process $A_k^- = \sum_{j=1}^{k} a_j^-$ and the conservation process $A_k^\circ = \sum_{j=1}^{k} a_j^\circ$.

In the case of permutational Fock space Φ, the similar representation holds for only a special class of martingales. Denote by $\mathcal{D}(T_k)$ the set of all triples $(\sigma, U, \tau) \in L_k$ s.t. $\sigma = (i_1, \cdots, i_r)$ with $i_1 > \cdots > i_r$ and $\tau = (j_1, \cdots, j_s)$ with $j_1 > \cdots > j_s$. Then the following is easily obtained.

Proposition 7.3: Let $\{M_k\}_{k\in T}$ be an operator martingale such that each M_k has the Fock expansion of the form

$$M_k = \sum_{(\sigma,U,\tau)\in\mathcal{D}(T_k)} m_{\sigma,U,\tau}^{(k)} d_\sigma^+ e_U^\bullet d_\tau^-,$$

then $\{M_k\}_{k\in T}$ can be expanded to the form

$$M_k = mI + \sum_{j=1}^{k} d_j^+ u_j^+ + \sum_{j=1}^{k} u_j^- d_j^- + \sum_{j=1}^{k} d_j^+ u_j^\circ d_j^- + \sum_{j=1}^{k} u_j^\bullet e_j^\bullet \quad (k = 1, 2, \cdots),$$

where m is a scalar, $\{u_j^+\}_{j\in T}$, $\{u_j^-\}_{j\in T}$, $\{u_j^\circ\}_{j\in T}$ and $\{u_j^\bullet\}_{j\in T}$ are previsible processes in the algebra $\mathcal{A}$.

For a general martingale, another form of predictable representation holds.

Proposition 7.4: Each operator martingale $\{M_k\}_{k\in T}$ can be expanded to the form

$$\begin{aligned} M_k = mI \; &+ \sum_{j=1}^{k} \sum_{\sigma\in Per(T_{j-1})} d_{(\sigma,j)}^+ u_{(\sigma,j)}^+ + \sum_{j=1}^{k} \sum_{\tau\in Per(T_{j-1})} u_{(\tau,j)}^- d_{(\tau,j)}^- \\ &+ \sum_{j=1}^{k} \sum_{\sigma,\tau\in Per(T_{j-1})} d_{(\sigma,j)}^+ u_{(\sigma,j),(\tau,j)}^\circ d_{(\tau,j)}^- + \sum_{j=1}^{k} u_j^\bullet e_j^\bullet. \end{aligned}$$

where m is a scalar, $u_{(\sigma,j)}^+$, $u_{(\tau,j)}^-$, $u_{(\sigma,j),(\tau,j)}^\circ$ and $u_j^\bullet$ are j-previsible operators.

Though the process with independent increments $\{D_k^\bullet\}_{k\in T}$ is not an operator martingale, the process $\{D_k^\bullet\}_{k\in T}$ is useful to describe the discrete time analogue of quantum Ito's formmula in the next section because it allow us to make quantum Ito table simple.

8. Analogue of Quantum Ito's Formula

In this section, we examine in the quantum probability space $(\mathcal{A}, \phi)$ a discrete time analogue of quantum Ito's foumula.

Generally speaking, quantum Ito's formula which is given in the continuous time quantum stochastic calculus[5,6,9] is the rule to compute the product M_tN_t of two stochastic integrals M_t and N_t. In a differential form, quantum Ito's formula can be viewed as the rule to compute the product dM_tdN_t of two stochastic differentials dM_t and dN_t, appearing in the following equation:

$$d(M_tN_t) = (dM_t)N_t + M_t(dN_t) + dM_tdN_t.$$

The corresponding multiplication table described by the stochastic differentials of the basic operator processes is often called the quantum Ito table.

Well, in our case of discrete time situation, the increment of the product M_kN_k of two discrete time "stochastic integrals" M_k and N_k is equal to

$$M_kN_k - M_{k-1}N_{k-1} = \xi_kN_{k-1} + M_{k-1}\eta_k + \xi_k\eta_k.$$

Here $\xi_k = M_k - M_{k-1}$ and $\eta_k = N_k - N_{k-1}$. Hence we interpret the rule for computing the product $\xi_k\eta_k$ of increments of two discrete time "stochastic integrals" M_k and N_k as a ***discrete time quantum Ito's formula.***

For simplicity, we restrict ourselves to consider a special class of operator processes (= discrete time analogue of stochastic integrals) which is stable under the pointwise multiplication and which contains the basic operator processes, i.e. the creation process $\{D_k^+\}_{k\in T}$, the conservation process $\{D_k^\circ\}_{k\in T}$, the annihilation process $\{D_k^-\}_{k\in T}$, and the exclusion process $\{D_k^\bullet\}_{k\in T}$.

Let $\{M_k\}_{k\in T}$ and $\{N_k\}_{k\in T}$ be two operator processes of the following form:

$$\begin{aligned} M_k &= mI + \sum_{j=1}^{k} A_jd_j^+B_jd_j^-C_j + \sum_{J=1}^{k} D_jd_j^+E_j + \sum_{j=1}^{k} F_jd_j^-G_j + \sum_{j=1}^{k} H_jd_j^\bullet, \\ N_k &= nI + \sum_{j=1}^{k} A'_jd_j^+B'_jd_j^-C'_j + \sum_{J=1}^{k} D'_jd_j^+E'_j + \sum_{j=1}^{k} F'_jd_j^-G'_j + \sum_{j=1}^{k} H'_jd_j^\bullet, \end{aligned}$$

where m, n are scalars, and A_j, B_j, C_j, D_j, E_j, F_j, G_j, H_j, A'_j, B'_j, C'_j, D'_j, E'_j, F'_j, G'_j, H'_j are j-previsible operators from $\mathcal{A}_{j-1]}$. Then the pointwise product $\{M_kN_k\}_{k\in T}$ of two operator processes $\{M_k\}_{k\in T}$ and $\{N_k\}_{k\in T}$ becomes to a finite sum of operator processes where each summand is an operator process of the above form. So the set of all such operator processes is stable under the multiplication.

Now let us define a continuous linear map $\delta : \mathcal{A} \to \mathcal{A}$ by

$$\delta(d_\sigma^+e_U^\bullet d_\tau^-) = \begin{cases} e_U^\bullet & (\text{if } \sigma = \tau = \Lambda), \\ 0 & (\text{otherwise}), \end{cases}$$

then we get

$$d_k^- A d_k^+ = \delta(A) d_k^\bullet \quad (\text{for } A \in \mathcal{A}_{k-1]}).$$

Using the map δ, we have the following.

Theorem 8.1 (Analogue of Quantum Ito's Formula): *Let $\{M_k\}_{k\in T}$ and $\{N_k\}_{k\in T}$ be two operator processes given above, then the pointwise product $\{M_kN_k\}_{k\in T}$ is calculated based on the followoning rule.*

$$\begin{aligned}
&(Ad_j^+Bd_j^-C)(A'd_j^+B'd_j^-C') = Ad_j^+(B\delta(CA')B')d_j^-C',\\
&(Ad_j^+Bd_j^-C)(D'd_j^+E') = Ad_j^+(B\delta(CD')E'),\\
&(Ad_j^+Bd_j^-C)(F'd_j^-G') = 0, \qquad (Ad_j^+Bd_j^-C)(H'd_j^\bullet) = 0,\\
&(Dd_j^+E)(A'd_j^+B'd_j^-C') = 0, \quad (Dd_j^+E)(D'd_j^+E) = 0,\\
&(Dd_j^+E)(F'd_j^-G') = Dd_j^+(EF')d_j^-G',\\
&(Dd_j^+E)(H'd_j^\bullet) = Dd_j^+(EH'),\\
&(Fd_j^-G)(A'd_j^+B'd_j^-C) = (F\delta(GA')B')d_j^-C',\\
&(Fd_j^-G)(D'd_j^+E') = (F\delta(GD')E')d_j^\bullet\\
&(Fd_j^-G)(F'd_j^-G') = 0, \qquad (Fd_j^-G)(D'd_j^\bullet) = 0,\\
&(Hd_j^\bullet)(A'd_j^+B'd_j^-C') = 0, \qquad (Hd_j^\bullet)(D'd_j^+E') = 0,\\
&(Hd_j^\bullet)(F'd_j^-G') = (HF')d_j^-G',\\
&(Hd_j^\bullet)(H'd_j^\bullet) = (HH')d_j^\bullet,
\end{aligned}$$

where $A, B, C, D, E, F, G, H, A', B', C', D', E', F', G', H'$ are j-previsible operators from $\mathcal{A}_{j-1]}$. This rule is summarized in a quantum Ito table:

	$A'd_j^+B'd_j^-C'$	$D'd_j^+E'$	$F'd_j^-G'$	$H'd_j^\bullet$
$Ad_j^+Bd_j^-C$	$Ad_j^+(B\delta(CA')B')d_j^-C'$	$Ad_j^+(B\delta(CD')E')$	0	0
Dd_j^+E	0	0	$Dd_j^+(EF')d_j^-G'$	$Dd_j^+(EH')$
Fd_j^-G	$(F\delta(GA')B')d_j^-C'$	$(F\delta(GD')E')d_j^\bullet$	0	0
$Hd_j^\bullet$	0	0	$(HF')d_j^-G'$	$(HH')d_j^\bullet$.

This is a discrete time analogue of quantum Ito's formula.

9. Additional Remarks

Finally we close this paper with some complementary remarks in the quantum probability space $(\mathcal{A}, \phi)$.

Remark 9.1 : We can naturally define the *canonical pair* p_k, q_k $(k \in T)$ by $q_k = d_k^+ + d_k^-$ and $p_k = i(d_k^+ - d_k^-)$. Here i is the imaginary unit. The spectrum of the self-adjoint operator q_k (resp. p_k) is $\mathrm{Sp}(q_k) = \{-1, 0, +1\}$ (resp. $\mathrm{Sp}(p_k) = \{-1, 0, +1\}$) and the probability distribution on the spectrum of operator q_k (resp. p_k) under the vacuum expectation ϕ is $\mathrm{P}(\{-1\}) = \mathrm{P}(\{+1\}) = 1/2$ and $\mathrm{P}(\{0\}) = 0$. So the operator q_k (resp. p_k) takes values ±1 almost surely in the vacuum state ϕ. Hence the operator q_k (resp. p_k) can be interpreted as a *quantum Bernoulli random variable.* Besides the operator process $\{q_1 + \cdots + q_k\}_{k\in T}$ (resp. $\{p_1 + \cdots + p_k\}_{k\in T}$) can be interpreted as a *quantum random walk* because it is a process with stationary independent increments in the Kümmerer independence. It is easily proved that the limit distribution of central limit type for this random walk is the Wigner semicircle law with mean 0 and variance 1.[16]

Remark 9.2 : In the permutational Fock space Φ, there exists naturally a unitary operator $\mathcal{F}$ defined by $\mathcal{F}e_\sigma = i^{\#([\sigma])}e_\sigma$. The operator $\mathcal{F}$ can be interpreted as a kind of *Fourier transform* because of the relations:

$$\mathcal{F}^{-1}d_k^+\mathcal{F} = -id_k^+, \quad \mathcal{F}^{-1}d_k^-\mathcal{F} = id_k^-, \quad \mathcal{F}^{-1}d_k^\circ\mathcal{F} = d_k^\circ, \quad \mathcal{F}^{-1}d_k^\bullet\mathcal{F} = d_k^\bullet,$$
$$\mathcal{F}^{-1}q_k\mathcal{F} = -p_k, \quad \mathcal{F}^{-1}p_k\mathcal{F} = q_k.$$

Remark 9.3 : Put $X_k = q_k$, $Y_k = p_k$ and $Z_k = d_k^\bullet - d_k^\circ$, $(k \in T)$. Then, for each $k \in T$, the triple (X_k, Y_k, Z_k) is a realization of the ***angular momentum commutation relations*** up to the factor 2. That is

$$[X_k, Y_k] = 2iZ_k, \quad [Y_k, Z_k] = 2iX_k, \quad [Z_k, X_k] = 2iY_k.$$

Since the family of subalgebras $\{\mathcal{A}_k\}_{k\in T}$ is the discrete time quantum i.i.d. process in the Kümmerer independence, this representation can be interpreted as a noncommuting, non anticommuting, non freely independent, but "independent" system of spins. Compare this with the case of Journé's toy Fock space Φ_{toy} where the corresponding two representations of angular momentum commutation relations form the commuting spins and the anticommuting spins.[2]

Further discussions and generalizations of the permutational Fock space Φ will be presented elsewhere.[17,18]

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