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Clarkson, random Clarkson inequalities and Rademacher type for Banach spaces

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Introduction

In connection with generalized Clarkson's inequalities (Kato [4]; see also [11]), a high-dimensional version of Clarkson-Boas-Koskela-type inequalities (cf. [2], [1], [8]), Tonge [13] presented random Clarkson inequalities for L_p. On the other hand, the authors [6] characterized those Banach spaces satisfying (p, p')-Clarkson's inequality as of (Rademacher) type p with type p constant one, and using this, they [12] proved that if (p, p')-Clarkson's inequality holds in a Banach space X, then the random Clarkson inequality also holds in X, where the unknown absolute constant included in Tonge's original inequalies was replaced by one. We point out here that there are fairly many Banach spaces in which (p, p')-Clarkson's inequality holds (cf. [1], [3], [9]; see also [11]).

In this note, we present a more general high-dimensional version of Clarkson-Boas-koskela-type inequalities for Banach spaces X of type p, and using this, it is shown that if X is of type p, then Tonge-type random Clarkson inequality including an absolute constant holds in X; in this case we can take type p constant as such an absolute constant.

1. Preliminaries

In the following, let p', q', r', ... denote the conjugate numbers of p, q, r, ...

1.1. Clarkson's inequalitites (Clarkson [2]): $\forall x, y \in L_p$

(CI-1) $\quad (\|x + y\|^{p'} + \|x - y\|^{p'})^{1/p'} \leqq 2^{1/p'}(\|x\|^{p} + \|y\|^{p})^{1/p},$

if $1 \leqq p \leqq 2$.

(CI-2) $\quad (\|x + y\|^{p} + \|x - y\|^{p})^{1/p} \leqq 2^{1/p}(\|x\|^{p'} + \|y\|^{p'})^{1/p'},$

if $2 \leqq p \leqq \infty$.

Let $1 \leqq p \leqq 2$. We say that a Banach space X satisfies (p, p')-Clarkson's inequality if (CI-1) holds in X. It is known (cf. [6]) that X satisfies (p, p')-Clarkson's inequality if and only if its dual X' does. This shows that one of the inequalities (CI-1) and (CI-2) implies the others. We note here that Clarkson [2] proved the other several inequalities for L_p, but those follow from (CI-1) or (CI-2).

Boas [1] and Koskela [8] considered some generalizations of Clarkson inequalities for a much wider class of parameters, which we call Clarkson-Boas-Koskela-type inequalities:

1.2. Clarkson-Boas-Koskela's inequalitites (Boas[1], Koskela [8] for L_p; see also [5]): Let $1 \leqq p, r, s \leqq \infty$. Then, $\forall x, y \in L_p$

(CBKI) $\quad (\|x + y\|^{s} + \|x - y\|^{s})^{1/s} \leqq 2^{c(r,s;p)}(\|x\|^{r} + \|y\|^{r})^{1/r},$

where $c(r,s;p) = \max\{1/r', 1/s, 1/r' + 1/s - \min(1/p, 1/p')\}$.

In the inequality (CBKI), let $r = \min\{p, p'\}$ and $s = \max\{p, p'\}$. Then we have the Clarkson inequalities (CI-1) and (CI-2). We note that if $p = 1$ or ∞, then (CBKI) holds in any Banach space X; and if (CI-1) or (CI-2) holds in X, then (CBKI) also holds in X. (In the next section, this facts are claimed in more general forms.)

Now we introduce some generalizations of (CBKI) to higher dimensions.

1. 3. Littlewood matrices $A_n = (\varepsilon_{ij})$ $(2^n \times 2^n)$:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} \quad (n = 1, 2, \dots).$$

1. 4. Generalized Clarkson's inequalitites (Kato [4] for L_p):

Let $1 \leqq p, r, s \leqq \infty$. Then, $\forall x_1, \dots, x_{2^n} \in L_p$

$$\text{(GCI)} \qquad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} x_j \right\|^s \right\}^{1/s} \leqq 2^{nc(r,s;p)} \left\{ \sum_{j=1}^{2^n} \| x_j \|^r \right\}^{1/r}.$$

where $c(r,s;p) = \max\{1/r', 1/s, 1/r' + 1/s - \min(1/p, 1/p')\}$.

For $n = 1$, (GCI) coincides with (CBKI), and so (GCI) may be regarded as a generalization of (CBKI) to higher dimensions. We note that (GCI) can be slightly generalized for a much wider class of parameters r and s. (For the details, see Maligranda and Persson [10].)

In the context of type and cotype for Banach spaces, the authors [6] considered some generalizations of the Clarkson inequality (CI-1) to higher dimensions, and proved that a Banach space X satisfies (p, p')-Clarkson's inequality, $1 \leqq p \leqq 2$, if and only if X is of (Rademacher) type p and its type p constant is one.

1. 5. Type inequalitites: Let $1 \leqq p \leqq 2$. A Banach space X is said to be of (Rademacher) type p if there is a constant $K > 0$ such that

$$\text{(TpI)} \qquad \left(\frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^{p'} \right)^{1/p'} \leqq K \left(\sum_{j=1}^{n} \| x_j \|^{p} \right)^{1/p}.$$

for all $x_1, x_2, \ldots, x_n \in X$.

The smallest constant K satisfying (TpI) is said to be type p constant and denoted by $T_p(X)$. (The notion of type p is also defined by usual Rademacher functions, but it is the same as the above definition.)

2. Clarkson-Boas-Koskela-type inequalitites and Rademacher type for Banach spaces

In this section, we present a high-dimensional version of Clarkson-Boas-Koskela-type inequalites for a Banach space X.

2. 1. Clarkson-Boas-Koskela-type inequalitites *with n elements*:
Let $1 \leqq p, r, s \leqq \infty$. Then, $\forall\, x_1, x_2, \ldots, x_n \in X$

$$\text{(nCBKI)} \quad \left(\sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^{s} \right)^{1/s} \leqq 2^{n/s} n^{c(r,s;p)-1/s} \left(\sum_{j=1}^{n} \| x_j \|^{r} \right)^{1/r},$$

where $c(r, s;p) = \max\{1/r', \ 1/s, \ 1/r' + 1/s - \min(1/p, 1/p')\}$.

2. 2. Theorem. Let $1 \leqq p \leqq 2$. Then for a Banach space X, the following statements are equivalent.

(i) (p, p')-Clarkson's inequality holds in X.

(ii) X is of type p and its type p constant is one.

(iii) For each r and s with $1 \leqq r, s \leqq \infty$, (nCBKI) holds in X.

2. 3. Corollary (Maligranda and Persson [10]). Let $1 \leqq p \leqq \infty$. Then for each r and s with $1 \leqq r, s \leqq \infty$, (nCBKI) holds in L_p.

2. 4. Remark. Corollary 2. 3 is also true in the case $r < 1$ $(r \neq 0)$ or $0 < s < 1$. In this case, $c(r,s;p)$ is replaced by $c(u,v;p)$, where $u = \max\{r,1\}$ and $v = \max\{s,1\}$, see [10]. We can show that Theorem 2. 2 is also true for such cases.

Now we introduce (nCBKI) including an absolute constant, and give a generalization of Theorem 2. 2.

2. 5. Theorem. Let $1 \leqq p \leqq 2$. If a Banach space X is of type p, and its type p constant $T_p(X) = K$, then (nCBKI) with the constant K holds in X, that is, for each r and s with $1 \leqq r, s \leqq \infty$, and for all $x_1, x_2, \ldots, x_n \in X$,

$$\text{(nCBKI)} \quad \left(\sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^s \right)^{1/s} \leqq K\, 2^{n/s} n^{c(r,s;p)-1/s} \left(\sum_{j=1}^{n} \| x_j \|^r \right)^{1/r}$$

holds, where $c(r,s;p) = \max\{1/r', 1/s, 1/r' + 1/s - 1/p'\}$.

2. 6. Remark. In Theorem 2. 5, it is easy to see that if (nCBKI) with an absolute constant K holds in X, then $T_p(X) \leqq K$ (put $r = p$ and $s = p'$). This means that $T_p(X)$ coincides with the smallest constant K satisfying (nCBKI); the constant K depends on X and p, but it is independent of r, s and n. We note that Theorem 2. 5 is also true in the case $r < 1$ $(r \neq 0)$ or $0 < s < 1$. In this case, $c(r,s;p)$ is replaced by $c(u,v;p)$, where $u = \max\{r,1\}$ and $v = \max\{s,1\}$.

3. Random Clarkson inequalitites and Rademacher type for Banach spaces

In [13], Tonge presented random Clarkson inequalities for L_p. For a Banach space X, these inequalities are stated as follows:

3. 1. Random Clarkson inequalitites: Let $1 \leqq p \leqq 2$ and $1 \leqq r, s \leqq \infty$. Let $A = (a_{ij})$ be an $n \times n$ matrix, where a_{ij} are independent identically distributed random variables taking the values ± 1 with equal probability 1/2. Then, **E** denoting mathematical expectation, for any $x_1, x_2, \ldots, x_n$ in X.

$$\text{(RCI)} \qquad E\left(\sum_{i=1}^{n} \left\| \sum_{j=1}^{n} a_{ij}x_j \right\|^s\right)^{1/s} \leqq Kn^{c(r,s;p)}\left(\sum_{j=1}^{n} \| x_j \|^r\right)^{1/r},$$

where $c(r,s;p) = \max\{1/r', 1/s, 1/r' + 1/s - 1/p'\}$, and K is a positive absolute constant depending only on p and X.

3. 2. Remark. In [13], Tonge proved that (RCI) holds in L_p. The authors [12] recently generalized this result for more general Banach spaces X, and proved that if X satisfies (p, p')-Clarkson inequality, then (RCI) holds in X, where the unknown absolute constant K was replaced by one.

Using Theorem 2. 5, we can easily prove the following Tonge-type random Clarkson inequality for Banach spaces of type p.

3. 3. Theorem. Let $1 \leqq p \leqq 2$. If a Banach space X is of type p, and its type p constant $T_p(X) = K$, then (RCI) with the constant K holds in X; and conversely, if (RCI) holds in X, then X is of type p.

3.4. **Remark.** If X satisfies (p, p')-Clarkson inequality, then it is of type p with type p constant one (see Theorem 2.2). Hence, the authors' result [12] mentioned above follows from Theorem 3.3. (For the details, see Kato and Takahashi [7]).

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