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Consistency Proof via Pointwise Induction

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Abstract

We show that the consistency of the first order arithmetic PA follows from the pointwise induction up to the Howard ordinal. Our proof differs from U. Schmerl [S]: We do not need Girard's Hierarchy Comparison Theorem. A modification on ordinal assignment to proofs by Gentzen and Takeuti [T] is made so that one step reduction on proofs exactly corresponds to the stepping down $\alpha\mapsto\alpha[1]$ in ordinals. Also a generalization to theories ID_q of finitely iterated inductive definitions is proved.

We show that the consistency of the first order arithmetic PA follows from the pointwise induction up to the Howard ordinal. Our proof differs from U. Schmerl [S]: We do not need Girard's Hierarchy Comparison Theorem.

Let P be a proof of the empty sequent in PA or the second order arithmetic $\Pi_1^1-CA_0$. For such a proof P let $o(P)$ denote the ordinal assigned to P and $r(P)$ a reduct of P defined by Gentzen and Takeuti [T]. $r(P)$ is again a proof of the empty sequent and $o(r(P))<o(P)$. Then the reduction $r:P\mapsto r(P)$ is close to but does not fit perfectly the stepping down $\alpha\mapsto\alpha[n]$ defined by Buchholz [B2].[1] We need to tune these functions o and r to stepping down in order to have $o(r(P))=o(P)[n]$ for an n. For this purpose we introduce two inference rules: the *padding rule* and the *height rule*. In both rules the lowersequent is identical with the uppersequent. Let S_u $[S_l]$ denote the uppersequent [the lowersequent], resp. Also let $h(S)$ denote the *height* of a sequent (in a proof P).

$$\frac{\Gamma}{\Gamma}\ (pad)_b$$

with $o(S_l)=o(S_u)+b$.

$$\frac{\Gamma}{\Gamma}\ (hgt)$$

with $h(S_u)=h(S_l)+1$ and $o(S_l)=D_1o(S_u)$. D_1a denotes an ordinal term defined in [B2].
Using these rules we can unwind gaps between D and $r(D)$ so that
$o(r(P))=o(P)[1]$ holds.

1 Fundamental sequences

The following is the fundamental sequences given in [A] and a slight variant in Buchhloz [B2]. Let q be a natural number.

Definition 1 (Buchholz [B2]) The term structure $(T(q),\cdot[\cdot])$

1. Inductive definition of the sets $PT(q)$ and $T(q)$

(T0) $PT(q)\subseteq T(q)$

(T1) $0\in T(q)$

(T2) $a\in T(q)\,\&\,0\leq u\leq q\Rightarrow D_ua\in PT(q)$

(T3) $a_0,\dots,a_k\in PT(q)(k>0)\Rightarrow(a_0,\dots,a_k)\in T(q)$

2. For $a_0,\dots,a_k\in PT(q)$ and $k\in\{-1,0\}$, we set

$$(a_0,\dots,a_k)=\begin{cases}0 & k=-1\\ a_0 & \text{otherwise}\end{cases}$$

[1] This observation is also stated by M. Hamano and M. Okada [H-O].

3. $a+0=0+a=a$; $(a_0,\ldots,a_k)+(b_0,\ldots,b_m)=(a_0,\ldots,a_k,b_0,\ldots,b_m)$; $a\cdot 0=0$; $a\cdot(n+1)=a\cdot n+a$
4. $\omega =_{df} \{0,1,1+1,\ldots\}\subset T(q)$ with $1=D_0 0$
5. For $u\leq q$,
   $T_u(q)=\{(D_{u_0}a_0,\ldots,D_{u_k}a_k): k\geq -1,\ a_0,\ldots,a_k\in T(q),\ u_0,\ldots,u_k\leq u\}$
6. $dom(a)$ and $a[z]$ for $a\in T(q)$ and $z\in dom(a)$

   ([ ].1) $dom(0)=\emptyset$

   ([ ].2) $dom(1)=\{0\}$; $1[0]=0$

   ([ ].3) $dom(D_{u+1}0)=T_u(q)$; $(D_{u+1}0)[z]=z$

   ([ ].4) Let $a=D_v b$ with $b\neq 0$.

   (a) If $b=b_0+1$, then $dom(a)=\omega$; $a[z]=(D_v b_0)\cdot(n+1)$

   (b) If $dom(b)\in\{\omega\}\cup\{T_u(q): u<v\}$, then $dom(a)=dom(b)$; $a[z]=D_v b[z]$

   (c) If $dom(b)\in\{T_u(q): u\geq v\}$, then $dom(a)=\omega$; $a[n]=D_v b[b_n]$ with $b_0=1$, $b_{m+1}=D_u b[b_m]$

   ([ ].5) $a=(a_0,\ldots,a_k)\, k>0$: $dom(a)=dom(a_k)$; $a[z]=(a_0,\ldots,a_{k-1})+a_k[z]$

Let $OT(q)\subset T(q)$ denote the set of *ordinal terms* in [B1]. For example $OT(1)$ corresponds to the Howard ordinal $\psi_0\varepsilon_{\Omega+1}$. In [B1] Buchholz shows that the proof theoretic ordinal of the theory ID_q for q-fold iterated inductive definitions is given by the ordinal $\psi_0\varepsilon_{\Omega_q+1}$, i.e., the order type of $OT_0(q)$.

Proposition 1 (Buchholz [B1])

$$a,z\in OT(q)\,\&\, z\in dom(a)\ \Rightarrow\ a[z]\in OT(q)$$

Coventions.

1. $\Omega_q =_{df} D_q 0$
2. $0[n]=0$; $(a+1)[n]=a$ for $n\in\omega$
3. $a[n]^0=a$; $a[n]^{m+1}=(a[n]^m)[n]$
4. $D_u^0 a=a$; $D_u^{k+1}a=D_u(D_u^k a)$

ERA denotes the *Elementary Recursive Arithmetic*.
Let $(PI)_q$ denote the following inference rule:

$$\frac{A(0,p)\quad \alpha\neq 0\wedge A(\alpha[1],r(p))\supset A(\alpha,p)}{A(\alpha,p)}\ (PI)_q$$

where α denotes a variable ranging over $OT(q)$, and A $[r]$ is an elementary recursive relation$\in\mathcal{E}^3_*$ [function$\in\mathcal{E}^3$], resp.

For a theory T let $Con^{(n)}(T)$ denote the iterated consistency of T:

$$Con^{(0)}(T)\Leftrightarrow\forall x(0=0);\ Con^{(n+1)}(T)\Leftrightarrow Con(T+Con^{(n)}(T))$$

Now our theorems are stated as follows:

Theorem 1 *For each natural number q,*

1. *Over ERA, $\{Con^{(n)}(ID_q): n<\omega\}$ is equivalent to $(PI)_{q+1}$.*
2. *Over ERA, the 1-consistency $RFN_{\Sigma_1}(ID_q)$ of ID_q is equivalent to $\forall n\exists m\{(D_0 D^n_{q+1}(\Omega_{q+1}\cdot n))[1]^m=0\}$.*

For provably total recursive functions we have, e.g.,

Proposition 2 *For each provably total recursive function f in PA, there exist k and d such that $\forall n[f(n)\leq d\cdot\mu m\{(D_0 D_1^k(\Omega_1(k+n)))[1]^m=0\}]$.*

This is seen from a slight modification of the proof of the theorem and so we omit a proof.

Remark. U. Schmerl [S] gives a proof of a variant of the Theorem 1.1 ($q=0$), i.e., for PA via Girard's Hierarchy Comparison Theorem. In [S] the base theory (for us ERA) contains the fast growing functions $F_\alpha\,(\alpha<\varepsilon_0)$ and/or the slow growing functions $G_\alpha\,(\alpha<\psi_0\varepsilon_{\Omega+1})$ and their defing equations. Hence it seems that Schmerl's result is incomparable to ours.

2 Proof of Theorem

Fix a natural number q. We prove the Theorem 1.1. The Theorem 1.2 is proved similarly.

First consider the easy half: The rule $(PI)_{q+1}$ is a derived rule in $ERA + \{Con^{(n)}(ID_q) : n < \omega\}$.

Let $prov_T$ denote a standard proof predicate for a theory T and "B" the gödel number of an expression B. This follows from the following fact which is shown in [A]:

Proposition 3 *For some elementary recursive function f we have*

$$ERA \vdash \forall a \in T_0(q+1)\{prov_{ID_q}(f(a), "\forall n \exists m(\dot{a}[n]^m = 0)")\}$$

Next consider the other half. Let $\forall x B(x)$ be a Π^0_1 sentence. $ID_q + \forall x B(x)$ denote the theory obtained from ID_q by adding extra axioms $B(t)$ for an arbitrary term t. It suffices to show, in $ERA + (PI)_{q+1}$, $Con(ID_q + \forall x B(x))$ under the assumption '$\forall x B(x)$ is true'. Our proof is an adaption from Gentzen's and Takeuti's reduction in [T].

First ID_q is embedded in a first order theory NID_{q+1}. In the latter theory the universe ω of ID_q is replaced by a constant N and this constant is treated as if it were a Π^1_1 formula. Then as in [T] the inference rules for the constant N are analysed by using a *substitution rule*. Also as mentioned above we introduce two new rules, the padding rule and the height rule to unwind gaps in Gentzen-Takeuti reduction. Now details follow.

The *language* L of ID_q consists of

1. function constants 0 and the successor $'$,
2. arithmetic predicate constants are lower elementary recursive relations $R \in \mathcal{L}^2_*$ and their negations $\neg R$,
3. the least fixed points $\{P_u\}_{1 \le u \le q}$ for a fixed positive operator form $\mathcal{A}(X^+, Y, n)$ and
4. logical symbols $\wedge, \vee, \forall, \exists$.

The *negation* $\neg A$ of a formula A is defined by using de Morgan's law and the elimination of double negations. A prime formula $R(t_1, \dots, t_n)$ or its negation $\neg R(t_1, \dots, t_n)$ with an arithmetic predicate R is an *a.p.f.*(arithmetic prime formula).

The *axioms* in ID_q are axioms for function and arithmetic predicate constants, the induction axiom (IA) and axioms $(P.1)$, $(P.2)$ of the least fixed points $\{P_u\}_{1 \le u \le q}$ for arbitrary formula F:

(IA) $F(0) \wedge \forall x(F(x) \supset F(x')) \supset \forall x F(x)$

$(P.1)$ $\mathcal{A}_u(P_u) \subseteq P_u$

$(P.2)$ $\mathcal{A}_u(F) \subseteq F \supset P_u \subseteq F$
where $\mathcal{A}_u(X) = \{n : \mathcal{A}(X, \sum_{1 \le v < u} P_v, n)\}$.

The *language* L_N of NID_{q+1} consists of $L \cup \{N\} \cup \{X_i : i < \omega\}$ with a unary predicate constant N and a list of unary predicates X_i. These unary predicates are denoted X, Y, *etc.* We sometimes write P_0 for the constant N. For a predicate constant $H \in \{P_u : u \le q\} \cup \{X_i : i < \omega\}$, we write $t \in H$ for $H(t)$ and $t \notin H$ for $\neg H(t)$. A formula is said to be an *E formula* if it is either an a.p.f. or a formula in one of the following shapes; $A \vee B$, $\exists x A$ or $t \notin H$ with $H \in \{P_u : u \le q\} \cup \{X_i : i < \omega\}$. A formula is an *A formula* if its negation is an E formula. For a formula A in L let A^N denote the result of restricting all quantifiers in A to N. For each $u \le q$ let $\mathcal{N}_u(X, t)$ denote the formula:

$$\mathcal{N}_0(X, t) \equiv 0 \in X \wedge \forall x(x \in X \supset x' \in X) \supset t \in X;$$

$$\mathcal{N}_u(X, t) \equiv \mathcal{A}^N_u(X) \subseteq X \supset t \in X \ (u \neq 0)$$

NID_{q+1} is fomulated in Tait's calculus, i.e., one sided sequent calculus. Finite sets of formulae is called a *sequent*. Sequents are denoted by Γ, Δ, *etc.*

Axioms in NID_{q+1} are:

logical axiom $\Gamma, \neg A, A$
where A is either an a.p.f. or a formula of the shape $t \in X$.

arithmetic axiom 1. Γ, Δ_R
where Δ_R consists of a.p.f'.s and corresponds to the definition of a lower elementary relation R.

2. Γ, A for a true closed a.p.f. A.

3. Γ, Δ_0
   where there exists a sequent Δ_1 so that $\Delta = \Delta_0 \cup \Delta_1$ is an instance of a defining axiom for R in 1 and Δ_1 consists solely of false closed a.p.f.'s.

Inference rules in NID_{q+1} are:
$(\wedge), (\vee), (\forall), (\exists), (cut), (weak)$ and $(P_u), (\neg P_u)$ for $(u \leq q)$.

1. $(\wedge), (\vee), (\forall), (\exists)$: In these rules the principal formula is contained in the uppersequent. For example

$$\frac{\Gamma, \exists x A(x), A(t)}{\Gamma, \exists x A(x)} \ (\exists)$$

2. In the rule (cut)

$$\frac{\Gamma, \neg A \quad A, \Delta}{\Gamma, \Delta} \ (cut)$$

   the *cut formula* A is an E formula.

3. $(weak)$ is the *weakening*:

$$\frac{\Gamma}{\Delta} \ (weak)$$

   with $\Gamma \subseteq \Delta$.

4. (P_u):

$$\frac{\Gamma, t \in P_u, \mathcal{N}_u(X, t)}{\Gamma, t \in P_u} \ (P_u)$$

   where X is the *eigenvariavle*, i.e., does not occur in the lowersequent.

5. $(\neg P_u)$:

$$\frac{\Gamma, t \notin P_u, \neg \mathcal{N}_u(F, t)}{\Gamma, t \notin P_u} \ (\neg P_u)$$

   for an arbitrary formula F in the language L_N.

Lemma 1 *For any sentence A in L,*

$$ID_q \vdash A \Rightarrow NID_{q+1} \vdash A^N$$

Proof. It suffices to show that the following sequents are provable in
NID_{q+1}:

$t \notin P_u, t \in P_u$: This is proved by induction on $u \leq q$. By IH(=Induction Hypothesis) we have $\neg \mathcal{N}_u(X, t), \mathcal{N}_u(X, t)$. Rules (P_u) and $(\neg P_u)$ yields $t \notin P_u, t \in P_u$.

$(IA)^N$: For a given formula $F(x)$ assume $a \in N$, $F(0)$ and $\forall x \in N(F(x) \supset F(x'))$. We have to show $F(a)$. Let $G(x)$ denote the fomula $x \in N \wedge F(x)$. Then we see $\forall x(G(x) \supset G(x'))$ from $x \in N \supset x' \in N$. The latter follows from the rules $(\neg N) = (\neg P_0)$ and (N). Also we have $G(0)$. On the other hand we have $\mathcal{N}_0(G, a)$ by the rule $(\neg N)$ and $a \in N$. Thus we get $G(a)$ and hence $F(a)$.

$(P.1)^N$: Assume $\mathcal{A}_u^N(P_u, a)$. We have to show $a \in P_u$. By the rule (P_u) it suffices to show $\mathcal{N}_u(X, a)$. Assume $\mathcal{A}_u^N(X) \subseteq X$. We show $a \in X$.
Claim. $P_u \subseteq X$
Proof of the Claim. Assume $x \in P_u$. By the rule $(\neg P_u)$ we have $\mathcal{N}_u(X, x)$. The assumption $\mathcal{A}_u^N(X) \subseteq X$ yields $x \in X$. □
From this Claim and the positivity of X in $\mathcal{A}$ we see $\mathcal{A}_u^N(X, a)$. Thus again by the assumption $\mathcal{A}_u^N(X) \subseteq X$ we conclude $a \in X$.

$(P.2)^N$: For a given formula F assume $\mathcal{A}_u^N(F) \subseteq F$ and $a \in N \cap P_u$. We show $F(a)$. This follows from the rule $(\neg P_u)$.

□

Definition 2 *The length* $\mid A \mid$ of a formula A in L_N

1. $| A |= 0$ for a prime formula A. Specifically $| (\neg)H(t) |= 0$ for any predicate H.

2. $| QxA |=| F | +1$ for $Q \in \{\forall, \exists\}$

3. $| A_0 \circ A_1 |= \max\{| A_i | +1 : i = 0,1\}$ for $\circ \in \{\wedge, \vee\}$.

A formula $A \in Pos_u$ iff 1. if a predicate P_v occurs positively in A, then $v \leq u$, and 2. if a predicate P_v occurs negatively in A, then $v < u$.
Observe that $\mathcal{N}_u(X,t) \in Pos_u$ and $\neg P_u(t) \notin Pos_v$ & $Pos_v \subseteq Pos_u$ for $v \leq u$.

Let $\forall x B(x)$ denote a fixed *true* Π^0_1 sentence with an a.p.f. B. The system $NID_{q+1} + \forall x B(x)$ is obtained from NID_{q+1} by adding the axiom

$$(B)\ \Gamma, B(t)$$

for an arbitrary term t and three inference rules; the padding rule $(pad)_b$ $(b \in OT(q+1))$, the height rule (hgt) mentioned in the introduction and the *substitution* rule $(sub)_u$ $(u \leq q)$.

$$\frac{\Gamma(X)}{\Gamma(F)}\ (sub)_u$$

where 1. $\Gamma(X) \subseteq Pos_u$, 2. X is the eigenvariable, i.e., does not occur the lowersequent, 3. F is an arbitrary formula in L_N and 4. $\Gamma(F)$ denotes the result of substituting F for X in $\Gamma(X)$.

Let P be a proof (in $NID_{q+1} + \forall x B(x)$) and Γ a sequent in P. We define the *height* $h(\Gamma) = h(\Gamma; P)$ of Γ in P as follows:

1. $h(\Gamma) = 0$ if Γ is either the endsequent of P or the uppersequent of a rule (sub).

2. $h(\Gamma) = h(\Delta) + 1$ if Γ is the uppersequent of an (hgt) whose lowersequent is Δ.

3. $h(\Gamma) = h(\Delta)$ if Γ is the uppersequent of a rule other than (sub) and (hgt) and Δ is the lowersequent.

Again let P be a proof (in $NID_{q+1} + \forall x B(x)$). Let o denote an assignment of an ordinal term $o(\Gamma) = o(\Gamma; P) \in OT(q+1)$ to each sequent Γ in P. If the assignment $o : \Gamma \mapsto o(\Gamma)$ enjoys the following conditions, then we say that o is an *ordinal assignment* for P.

1. $o(\Gamma) \neq 0$ for each axiom Γ.
   Assume that Γ is the lowersequent of a rule J and Γ_0 and Γ_1 denote the uppersequents of J.

2. $o(\Gamma) = o(\Gamma_0)$ if J is one of $(\forall), (weak), (P_u)$.

3. $o(\Gamma) = o(\Gamma_0) = o(\Gamma_1)$ if J is $(\wedge)$.
   N.B. We require ordinals assigned to uppersequents of a $(\wedge)$ are equal.

4. $o(\Gamma) = o(\Gamma_0) + b$ for some nonzero $0 \neq b \in OT(q+1)$ if J is either $(\exists)$ or $(\vee)$.
   In this case we write, e.g., $(\vee)_b$ for the rule $(\vee)$.

5. $o(\Gamma) = o(\Gamma_0) + b$ if J is $(pad)_b$.

6. $o(\Gamma) = o(\Gamma_0) + \Omega_{1+u}$ if J is $(\neg P_u)$.

7. $o(\Gamma) = o(\Gamma_0) + o(\Gamma_1)$ if J is (cut).

8. $o(\Gamma) = D_{q+1}o(\Gamma_0)$ if J is (hgt).

9. $o(\Gamma) = D_u o(\Gamma_0)$ if J is $(sub)_u$ $(u \leq q)$.

For an ordinal assignment o for a proof P we set $o(P) = o(\Gamma_{end})$ with the endsequent Γ_{end} of P.

Remark.

1. The padding rule $(pad)_b$ is implicit in the literature, e.g., in [B2].

2. The substitution rule (sub) comes from [T] but Buchholz mentions a substitution operation $Nt \mapsto Ft$ in the proof of Lemma 4.12 in [B2].

Let P be a proof in $NID_{q+1} + \forall x B(x)$ and o an ordinal assignment for P. We say that (P, o) is a *proof with the o.a.* (=ordinal assignment) o if the following conditions are fulfilled:

(p0) The endsequent of P is the empty sequent.

(p1) The final part of P is an empty $(sub)_0$ followed by a nonempty series $\{(pad)_{b_i}\}_{i\leq n}$ of paddings with $dom(b_i) \in \{\emptyset, \{0\}, \omega\}$:

$$\begin{array}{ll} & \vdots \\ & \text{---}\ (sub)_0 \\ & \text{---}\ (pad)_{b_0} \\ & \vdots \\ P & \text{---}\ (pad)_{b_n} \end{array}$$

(p2) For any (cut) in P,

$$\frac{\Gamma, \neg A \quad A, \Delta}{\Gamma, \Delta}\ (cut)$$

$| A |\leq h(A, \Delta; P) = h(\Gamma, \neg A; P)$.

(p3) For any $(\neg P_u)$ in P,

$$\frac{\Gamma, t \notin P_u, \neg\mathcal{N}_u(F,t)}{\Gamma, t \notin P_u}\ (\neg P_u)$$

$| \neg\mathcal{N}_u(F,t) |\leq h(\Gamma, t \notin P_u, \neg\mathcal{N}_u(F,t); P)$.

Proposition 4 *Assume $ID_q + \forall x B(x)$ is inconsistent. Then there exists a proof P with an o.a. o.*

Proof. By Lemma 1 pick a proof P_0 in NID_{q+1} ending with the empty sequent. P_0 contains none of rules $(sub), (pad), (hgt)$. Below the endsequent of P_0 attach some (hgt)'s to enjoy the conditions **(p2)** and **(p3)**. After that attach further a $(sub)_0$ and a $(pad)_0$ to ensure the condition **(p1)**. Let P denote the resulting proof in $NID_{q+1} + \forall x B(x)$ of the empty sequent. For each sequent Γ in P_0 set the ordinal $o(\Gamma) = \Omega_{q+1} \cdot n$ for some $n < \omega$. Then the whole proof P has an ordinal $o(P) = D_0 D^k_{q+1}(\Omega \cdot k)$ for some k. □

Thus assuming that $\forall x B(x)$ is true, it suffices to show the following lemma.

Lemma 2 *Let (P, o) be a proof with an o.a. o. Then there exists a proof $(P', o) = r(P, o)$ with an o.a. o such that*

$$o(P') = o(P)[1]$$

It remains to prove the Lemma 2.

Let P be a proof (not neccessrily ending with the empty sequent). The *main branch* of P is a series $\{\Gamma_i\}_{i\leq n}$ of sequents in P such that:

1. Γ_0 is the endsequent of P.
2. For each $i < n$ Γ_{i+1} is the *right* uppersequent of a rule J_i so that Γ_i is the lowersequent of J_i and J_i is one of the rules $(cut), (weak), (hgt), (sub)$ and $(pad)_0$.
3. Either Γ_n is an axiom or Γ_n is the lowersequent of one of the rules $(\vee), (\exists), (\neg P_u)$ and $(pad)_b$ with $b \neq 0$.

The sequent Γ_n is called the *top* (of the main branch) of the proof P.

Let P be a proof with an o.a. and Γ a sequent in P. The *u-resolvent* of Γ is the uppermost substitution rule $(sub)_v$ below Γ with $v \leq u$. Note that such a substitution rule always exists by the condition **(p 1)**.

Let Φ denote the top of the proof P with the o.a. o. Put $\alpha = o(P)$. Observe that we can assume Φ contains no first order free variable.

Case 1. Φ is the lowersequent of a rule $(p)_b$ at which the ordinal b is padded. This means that either $(p)_b = (pad)_b$ with $b \neq 0$ or $(p)_b = (\vee)_b, (\exists)_b$ with $b > 1$.:

$$\begin{array}{ll} & \overline{\Phi}\ +b,\ (p)_b \\ P & \vdots \end{array}$$

Case 1.1. Either the top Φ is the endsequent (, then the last rule is $(p)_b = (pad)_b$ with $b \neq 0$), and/or $dom(b) = \omega$: $dom(\alpha) = dom(b)$. Replace the rule $(p)_b$ by $(p)_{b[1]}$. Note that $b[1] \neq 0$ if $b > 1$.

$$\begin{array}{ll} & \overline{\Phi}\ +b[1],\ (p)_{b[1]} \\ P' & \vdots \end{array}$$

Case 1.2. Otherwise:
Case 1.21. $dom(b) = T_u(q+1)$: Let I be the u-resolvent of Φ and Γ the lowersequent of J. $o(\Gamma)$ is of the form $D_v a$ for some $v \leq u$ and a with $dom(a) = T_u(q+1)$. We have $(D_v a)[1] = D_v a[a_1]$ with $a_1 = D_u a[1]$. Replace the $(p)_b$ by $(p)_{b[a_1]}$.
Case 1.22. $dom(b) = \{0\}$, i.e., $b = b_0 + 1$ for some b_0: Let J denote the uppermost (sub) or (hgt) below Φ and Γ the lowersequent of J. $o(\Gamma)$ is of the form $D_v(a + b_0 + 1)$:

$$\begin{array}{ll} & \overline{\Phi}\ +b_0+1 \\ & \vdots \\ & \overline{\Gamma}\ +D_v(a+b_0+1)\,J \\ P & \vdots \end{array}$$

Replace the $(p)_{b_0+1}$ by $(p)_{b_0}$ and insert a new $(pad)_c$ immediately below J with $c = D_v(a + b_0)$:

$$\begin{array}{ll} & \overline{\Phi}\ +b_0 \\ & \vdots \\ & \overline{\Gamma}\ +D_v(a+b_0)\,J \\ & \overline{\Gamma}\ +c\cdot 2\,(pad)_c \\ P' & \vdots \end{array}$$

Case 2. Φ is an axiom and contains a true a.p.f. A: $\Phi = A, \Delta_0$.

$$P \quad \dfrac{\begin{array}{cc} & A,\Delta_0 \\ \vdots\, a & \vdots\, b \\ \Gamma, \neg A & A, \Delta \end{array}}{\Gamma, \Delta}\ a+b \\ \vdots$$

where $a = o(\Gamma, \neg A), b = o(A, \Delta)$ $(b \neq 0)$. Eliminate the false a.p.f. $\neg A$ and insert a $(weak)$ and an appropriate (pad) as in **Case 1** to get $o(P') = \alpha[1]$.

$$P' \quad \begin{array}{c} \vdots\, a \\ \dfrac{\Gamma}{\Gamma, \Delta}\ (weak) \\ \vdots\ (pad) \end{array}$$

Case 3. Φ is a logical axiom: $\Phi = \neg X(t), X(t), \Delta_0$. Put
$X^+(t) = X(t), X^-(t) = \neg X(t), (\neg X(t))^- = X(t), (\neg X(t))^+ = X(t)$.

$$P \quad \dfrac{\begin{array}{cc} \vdots\, a & \vdots\, b \\ \Gamma, \tilde{X}^{\mp}(t) & \tilde{X}^{\pm}(t), \Delta \end{array}}{\Gamma, \Delta} \\ \vdots$$

where $\tilde{X}^{\mp}(t) \in \Delta$ and $\tilde{X}$ denotes either X or a formula F by a (sub). As in **Case 2** insert a $(weak)$ and a (pad).

$$P' \quad \begin{array}{c} \vdots\, a \\ \dfrac{\Gamma, \tilde{X}^{\mp}(t)}{\Gamma, \Delta}\ (weak) \\ \vdots\ (pad) \end{array}$$

Case 4. Φ is the lowersequent of a $(\vee)_1$.
Case 5. Φ is the lowersequent of an $(\exists)_1$.
Consider the **Case 4**. Let J denote a (cut) at which the descendent of the principal formula $A_0 \vee A_1$ of the $(\vee)_1$ vanishes.
Case 4.1. There exists an (hgt) or a (sub) between Φ and J: Let I denote the uppermost one among such

rules.

$$
\begin{array}{ccc}
 & \dfrac{A_i, A_0 \vee A_1, \Delta_0}{A_0 \vee A_1, \Delta_0}\,(\vee)_1\ +1 & \\
 & \vdots & \\
 & \overline{A'_0 \vee A'_1, \Delta'}\ I & \\
\vdots & \vdots & \\
\multicolumn{2}{c}{\dfrac{\Gamma, \neg\tilde{A}_0 \wedge \neg\tilde{A}_1 \qquad \tilde{A}_0 \vee \tilde{A}_1, \Delta}{\Gamma, \Delta}\ J} & \\
P & \vdots &
\end{array}
$$

where 1) $i = 0, 1$, 2) $\tilde{A}_i$ is either A_i or $A_i[X := F]$ and 3) $A'_i \in \{A_i, \tilde{A}_i\}$. For the lowersequent of I $o(A'_0 \vee A'_1, \Delta') = D_u(c+1)$ for some $u \leq q+1$ and a c.
Lower the $(\vee)_1$ under I and change $+1$ into $+D_u c$:

$$
\begin{array}{ccc}
 & A_i, A_0 \vee A_1, \Delta_0 & \\
 & \vdots & \\
 & \overline{A'_i, A'_0 \vee A'_1, \Delta'}\ I & \\
 & \dfrac{}{A'_0 \vee A'_1, \Delta'}\,(\vee)_{D_u c} & \\
\vdots & \vdots & \\
\multicolumn{2}{c}{\dfrac{\Gamma, \neg\tilde{A}_0 \wedge \neg\tilde{A}_1 \qquad \tilde{A}_0 \vee \tilde{A}_1, \Delta}{\Gamma, \Delta}} & \\
P' & \vdots &
\end{array}
$$

Case 4.2. Otherwise: Let I denote the uppermost (hgt) below the $(cut)\,J$. Such an (hgt) exists since $|\,\tilde{A}_0 \vee \tilde{A}_1\,| > 0$. Put $h = h(\tilde{A}_0 \vee \tilde{A}_1, \Delta) - 1$. Then $h = h(\Lambda) \geq \max\{|\,\tilde{A}_0\,|, |\,\tilde{A}_1\,|\}$ for the lowersequent Λ of the $(hgt)\,I$.

$$
\begin{array}{ccc}
 & \dfrac{A_i, A_0 \vee A_1, \Delta_0}{A_0 \vee A_1, \Delta_0}\,(\vee)_1\ +1 & \\
\vdots\ a & \vdots\ b+1 & \\
\multicolumn{2}{c}{\dfrac{\Gamma, \neg\tilde{A}_0 \wedge \neg\tilde{A}_1 \qquad \tilde{A}_0 \vee \tilde{A}_1, \Delta}{\Gamma, \Delta}\ J,\ a+b+1} & \\
 & \vdots\ c+1 & \\
 & \dfrac{}{\Lambda}\ (hgt)\,I,\ D_{q+1}(c+1) & \\
P & \vdots &
\end{array}
$$

where $a = o(\Gamma, \neg\tilde{A}_0 \wedge \neg\tilde{A}_1)$, $b + 1 = o(\tilde{A}_0 \vee \tilde{A}_1, \Delta)$ and $o(\Lambda) = D_{q+1}(c+1)$ for some c.
Assuming $\neg\tilde{A}_i$ is an E formula, let P' be the following:

$$
\begin{array}{cc}
\begin{array}{c}
\quad A_i, A_0 \vee A_1, \Delta_0 \\
\vdots\ a \qquad\qquad \vdots\ b \\
\dfrac{\Gamma, \neg\tilde{A}_0 \wedge \neg\tilde{A}_1 \qquad \tilde{A}_0 \vee \tilde{A}_1, \Delta.\tilde{A}_i}{\Gamma, \Delta, \tilde{A}_i}\ a+b \\
\vdots\ c \\
\dfrac{}{\Lambda, \tilde{A}_i}\ D_{q+1}c
\end{array}
&
\begin{array}{c}
\dfrac{\dfrac{\neg\tilde{A}_i, \Gamma}{\neg\tilde{A}_i, \Gamma, \Delta}\,(weak),\ a}{\neg\tilde{A}_i, \Gamma, \Delta}\,(pad)_b,\ a+b \\
\vdots \\
\dfrac{}{\neg\tilde{A}_i, \Lambda}\ D_{q+1}c
\end{array}
\\
\multicolumn{2}{c}{\dfrac{\qquad\qquad\qquad\qquad\qquad\qquad}{\Lambda}\ (cut),\ (D_{q+1}c)\cdot 2} \\
P' \qquad\qquad & \vdots
\end{array}
$$

Here the subproof ending with $\neg\tilde{A}_i, \Gamma$ is obtained from the subproof of P ending with the left uppersequent $\Gamma, \neg\tilde{A}_0 \vee \neg\tilde{A}_1$ of the $(cut)\,J$ by inversion. Observe that we still have $a = o(\neg\tilde{A}_i, \Gamma; P')$ under the same ordinal assignment since the lowersequent and the uppersequents of a rule $(\wedge)$ have the same assigned ordinal.
Case 6. Φ is the lowersequent of a $(\neg P_u)$.
Let J denote the (cut) at which the descendent of the principal formula of the $(\neg P_u)$ vanishes and I the u-resolvent of $\Phi = t\neg P_u, \Delta_0$. Here note that there is no $(sub)_v (v \leq u)$ between the $(\neg P_u)$ and J by the restriction: the uppersequent of a $(sub)_v \subseteq Pos_v$, i.e., $\neg P_u(t) \notin Pos_v$. Therefore the u resolvent I is below J. Also by the definition there is no $(sub)_v (v \leq u)$ between J and I.

$$P \qquad \begin{array}{c} \cfrac{\begin{array}{c}\vdots\ a\\ \Gamma, t \in P_u\end{array} \qquad \begin{array}{c}\cfrac{t \notin P_u, \neg\mathcal{N}_u(F,t), \Delta_0}{t \notin P_u, \Delta_0}\ (\neg P_u),\ +\Omega_{u+1}\\ \vdots\ b\\ t \notin P_u, \Delta\end{array}}{\Gamma, \Delta}\ J,\ a+b \\ \vdots\ c \\ \cfrac{\Lambda}{\tilde{\Lambda}}\ (sub)_u\, I,\ D_v c \end{array}$$

where $a = o(\Gamma, t \in P_u), b = o(t \notin P_u, \Delta), c = o(\Lambda)$ and $o(t \notin P_u, \Delta_0) = b_0 + \Omega_{u+1}$ with $b_0 = o(t \notin P_u, \neg\mathcal{N}_u(F,t), \Delta_0)$. We have $dom(b) = dom(c) = dom(\Omega_{u+1}) = T_u(q+1)$.
Put $z = D_u c[1]$. Let P' be the following:

$$P' \qquad \begin{array}{c} \cfrac{\begin{array}{c}\vdots\ a\\ \Gamma, t \in P_u\end{array} \qquad \begin{array}{c}\cfrac{t \notin P_u, \Delta_0 \neg\mathcal{N}_u(F,t) \qquad \begin{array}{c}\vdots\ a\\ \cfrac{\cfrac{\Gamma, \mathcal{N}_u(X,t)}{\mathcal{N}_u(X,t), \Gamma, \Delta}\ (weak)}{\mathcal{N}_u(X,t), \Gamma, \Delta}\ (pad)_{b[1]},\ b[1]\\ \vdots\\ \cfrac{\mathcal{N}_u(X,t), \Lambda}{\mathcal{N}_u(F,t), \Lambda}\ (sub)_u,\ z = D_u c[1]\end{array}}{t \notin P_u, \Delta_0, \Lambda}\ (cut),\ b_0 + z\\ \vdots\ b[z]\\ t \notin P_u, \Delta, \Lambda\end{array}}{\Gamma, \Delta, \Lambda} \\ \vdots\ c[z] \\ \cfrac{\Lambda, \Lambda}{\tilde{\Lambda}}\ D_v c[z] \end{array}$$

where the subproof ending with $\Gamma, \mathcal{N}_u(X,t)$ is obtained from the subproof in P ending with the left uppersequent $\Gamma, t \in P_u$ of the $(cut)\, J$ by inversion. Note that $\mathcal{N}_u(F,t)$ is an E formula. We have $o(\Gamma, \mathcal{N}_u(X,t); P') = a = o(\Gamma, t \in P_u; P)$ and hence $o(P_0) = o(\mathcal{N}_u(X,t), \Lambda; P') = c[1]$. Thus $o(\mathcal{N}_u(F,t), \Lambda; P') = D_u c[1] = z$ and $o(\tilde{\Lambda}; P') = D_v c[z] = (D_v c)[1]$ with $D_v c = o(\tilde{\Lambda}; P)$. Therefore $o(P') = \alpha[1]$.

This completes a proof of the Lemma 2 and hence of the Theorem 1.

Remark. As in [A] we have for each $n < \omega$

$$I\Sigma_k \vdash \forall d \exists m \{(D_0 D_1^{k-1}(\Omega \cdot n))[d]^m = 0\}$$

From this we can expect to sharpen the Theorem 1 for fragments, e.g., for $I\Sigma_k$ but we have no proof of the following:
Show that

$$I\Sigma_k \not\vdash \forall n \exists m \{(D_0 D_1^{k-1}(\Omega \cdot n))[1]^m = 0\}$$

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