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Characterization Theorems for Multiplicative Fragment of Intuitionistic Non-Commutative Linear Logic (Preliminary Report)*†

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Abstract

In [7], we showed that a proof net of MNCLL(Multiplicative fragment of Non-Commutative Linear Logic) can be characterized by means of the notion of strong planity of a marked Danos-Regnier graph, as well as the notion of a certain long-trip condition, called the stack-condition, of a marked Danos-Regnier graph, the latter of which is related to Abrusci's balanced long-trip condition ([1]). In this note, we shall also apply our methods to Intuitionistic Linear Logic, and obtain characterization theorems for Intuitionistic Multiplicative Non-Commutative Linear Logic, in terms of signed Danos-Regnier graphs.

1 Non-Commutative Proof Nets for Intuitionistic System.

In this note, we denote Multiplicative Commutative Linear Logic by MLL. It is well-known that the proof nets of MLL are characterized by a simple and elegant graph-theoretic condition, saying that any Danos-Regnier graph is a proof net of MLL if and only

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if it is acyclic and connected under any choice of par-link switching (cf. Danos-Regnier [2]). This condition is sometimes called as the (Danos-Regnier) switching condition. This characterization is a simplified version of a famous result of Girard [3], which is called the long-trip condition.

In [7], we introduced a system of Non-Commutative Linear Logic MNCLL, which is logically equivalent to the multiplicative fragment of Cyclic Linear Logic introduced by Yetter [10]; and we gave several correctness conditions of the proof nets for system MNCLL, including the strong planarity and the stack condition. In this note, we extend our results to the intuitionistic version of Multiplicative Non-Commutative Linear Logic IMNCLL. We introduce a so-called L-proof net, a notion of intuitionistic non-commutative proof net; this is induced from Roorda's formulation [8] of Intuitionistic Multiplicative Non-Commutative Linear Logic, where each conclusion node has a polarity + or −. Then we introduce the notions of L-strong planarity and the stack condition and show that our characterization theorem holds for this system.

We denote a sequence of formulas by a capital Greek letter, such as $\Delta, \Gamma, \Sigma, \cdots$.

Definition 1.1 *We define the system L (Roorda [8]).*
Axioms:
$A \Rightarrow A$, *where* A *is an atomic formula.*
Rules of inference:

$$\frac{\Sigma \Rightarrow A \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, \Sigma, A\backslash B, \Delta \Rightarrow C}\ (\backslash 2) \qquad \frac{A, \Sigma \Rightarrow B}{\Sigma \Rightarrow A\backslash B}\ (\backslash 1)$$

$$\frac{\Sigma \Rightarrow A \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, B/A, \Sigma, \Delta \Rightarrow C}\ (/\ 2) \qquad \frac{\Sigma, A \Rightarrow B}{\Sigma \Rightarrow B/A}\ (/\ 1)$$

$$\frac{\Sigma \Rightarrow A \quad \Gamma \Rightarrow B}{\Sigma, \Gamma \Rightarrow A \cdot B}\ (\cdot 2) \qquad \frac{\Sigma, A, B, \Gamma \Rightarrow C}{\Sigma, A \cdot B, \Gamma \Rightarrow C}\ (\cdot 1)$$

$$\frac{\Sigma \Rightarrow A \quad \Gamma, A, \Delta \Rightarrow C}{\Gamma, \Sigma, \Delta \Rightarrow C}\ (Cut).$$

We note that the system L becomes Lambek Calculus [4], if the inference rules $(\backslash 1)$ and $(/\ 1)$ are applied only when the antecedent Σ is non-empty.

Definition 1.2 *We define a signed formula, or a formula with polarity inductively as follows:*
(1) If A *is an atomic formula, then* A^+ *and* A^- *are atomic signed formulas, (2) if* A *and* B *are signed formulas, then so are* $(A \otimes B)^+$, $(A \otimes B)^-$, $(A \wp B)^+$ *and* $(A \wp B)^-$.

Definition 1.3 *For each signed formula A, we define its dual formula A^* inductively as follows:*
(1) If B^+ is an atomic signed formula, then $(B^+)^* = B^-$, (2) If B^- is an atomic signed formula, then $(B^-)^* = B^+$, (3) $((B \otimes C)^+)^* = (B^* \wp C^*)^-$, (4) $((B \otimes C)^-)^* = (B^* \wp C^*)^+$, (5) $((B \wp C)^+)^* = (B^* \otimes C^*)^-$, and (6) $((B \wp C)^-)^* = (B^* \otimes C^*)^+$.

As a result of the above definition, we can prove by induction on the complexity of a signed formula that $(A^*)^* = A$, for any signed formula A.

We now define a system *IMNCLL*, which is later shown to be a one-sided version of system L. There is a one-one correspondence between the rules of inference in the two systems. Thus we name each rule in IMNCLL with that of the correponding rule in L.

Definition 1.4 *We define system* IMNCLL.
Axioms.
$\vdash A^*,\ A$, *where A is a signed formula.*
Rules of inference:

$$\frac{\vdash \Gamma, A^+ \quad \vdash B^-, \Delta}{\vdash \Gamma, (A^+ \otimes B^-)^-, \Delta}(\backslash 2) \qquad \frac{\vdash \Gamma, B^+; A^-, \Delta}{\vdash \Gamma, (A^- \wp B^+)^+, \Delta}(\backslash 1)$$

$$\frac{\vdash \Gamma, B^- \quad \vdash A^+, \Delta}{\vdash \Gamma, (B^- \otimes A^+)^-, \Delta}(/2) \qquad \frac{\vdash \Gamma, A^-, B^+, \Delta}{\vdash \Gamma, (B^+ \wp A^-)^+, \Delta}(/1)$$

$$\frac{\vdash \Gamma, A^+ \quad \vdash B^+, \Delta}{\vdash \Gamma, (B^+ \otimes A^+)^+, \Delta}(\cdot 2) \qquad \frac{\vdash \Gamma, A^-, B^-, \Delta}{\vdash \Gamma, (A^- \wp B^-)^-, \Delta}(\cdot 1)$$

$$\frac{\vdash \Gamma, A \quad \vdash A^*, \Delta}{\vdash \Gamma, \Delta}(Cut).$$

Proposition 1.5 *The system IMNCLL admits the cut-elimination.*

We call a signed formula a *+-formula*, if the outermost sign of the formula is +.

Lemma 1.6 *Any derivation in IMNCLL has precisely one terminal +-formula.*

Proof. We prove this by induction on the length of the derivation in IMNCLL. If the derivation consists only of an axiom, then the claim clearly holds. We argue according to the last inference rule added to the derivation. We only discuss for case (/2); and similar arguments work for other cases: Let us assume that the last inference is

$$\frac{\vdash \Gamma, B^- \quad \vdash A^+, \Delta}{\vdash \Gamma, (B^- \otimes A^+)^-, \Delta}(/2).$$

Then we have provable sequents Γ, B^- and A^+, Δ. By the induction hypothesis, each of them has precisely one +-formula; in other words, there is no +-formula in Δ, and there exists precisely one +-formula in Γ. Thus we conclude that the sequent $\Gamma, (B^- \otimes A^+)^-, \Delta$ satisfies the condition as well. □

As we mentioned above, there is a one-one correspondence between the rules of inference in systems IMNCLL and L. Now we make the correspondence between the sequents of IMNCLL and those of L in such a way that;

$$B_1, \cdots, B_n \Rightarrow A \text{ corresponds to } \vdash B_1^-, \cdots, B_n^-, A^+.$$

Moreover we identify the order of the formulas in a sequent of IMNCLL up to the cyclic shifts: So $B_1^-, \cdots, B_n^-, A^+$ is identified with $B_i^-, \cdots, B_n^-, A^+, B_1^-, \cdots, B_{i-1}^-$.

Theorem 1.7 *(Implicitly in Roorda [8]) The system L is equivalent to IMNCLL.*

Proof. We note that there is a one-one correspondence between the rules of inference in systems IMNCLL and L as well as sequents of the systems. Thus we prove that any derivation in L has a derivation in IMNCLL, by induction on the length of the derivation in L. If the derivation consists only of an axiom, then the claim clearly holds. Now let us assume that the last applied inference rule is

$$\frac{\Sigma \Rightarrow A \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, \Sigma, A\backslash B, \Delta \Rightarrow C} (\backslash 2).$$

By induction hypothesis, there are derivations in IMNCLL for sequents $\Sigma \Rightarrow A$ and $\Gamma, B, \Delta \Rightarrow C$, whose terminal edges are Σ^-, A^+ and $\Gamma^-, B^-, \Delta^-, C^+$, respectively. By the cyclic shift, the second sequent becomes $B^-, \Delta^-, C^+, \Gamma^-$. By the inference rule $(\backslash 2)$ in IMNCLL, we obtain a new derivation in IMNCLL for $\Sigma^-, (A^+ \otimes B^-)^-, \Delta^-, C^+, \Gamma^-$. This corresponds to the terminal edge $\Gamma, \Sigma, A\backslash B, \Delta \Rightarrow C$. Similar arguments work for the cases of other inference rules in L.

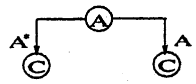
Secondly we show that any derivation in IMNCLL has a corresponding derivation in L, by induction on the length of the derivation in IMNCLL. If the derivation consists only of an axiom, then the claim clearly holds. Let us assume that the last inference rule is

$$\frac{\vdash \Gamma, B^- \quad \vdash A^+, \Delta}{\vdash \Gamma, (B^- \otimes A^+)^-, \Delta} (/2).$$

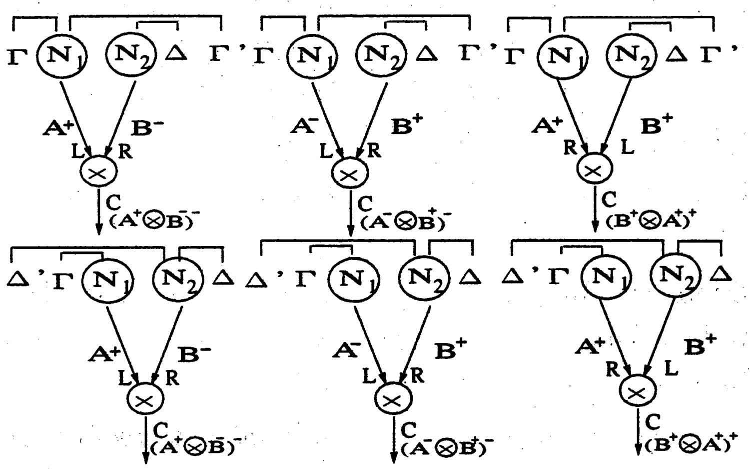
Then we have provable sequents Γ, B^- and A^+, Δ. By Lemma 1.6, each of them has precisely one +-formula. Let us denote Δ and Γ as Σ^- and Δ^-, C^+, Γ^-, respectively. Hence the sequents obtained above become A^+, Σ^- and Δ^-, C^+, Γ^-,

B^-, respectively. By induction hypothesis, they correspond to sequents $\Sigma \Rightarrow A$ and $\Gamma, B, \Delta \Rightarrow C$ in L. By the rule $(/2)$ in L, we obtain terminal edges $\Gamma, B/A, \Sigma, \Delta \Rightarrow C$, and this corresponds to the last sequent in the derivation in IMNCLL, which is $\Delta^-, C^+, \Gamma^-, (B^- \otimes A^+)^-, \Sigma^-$. □

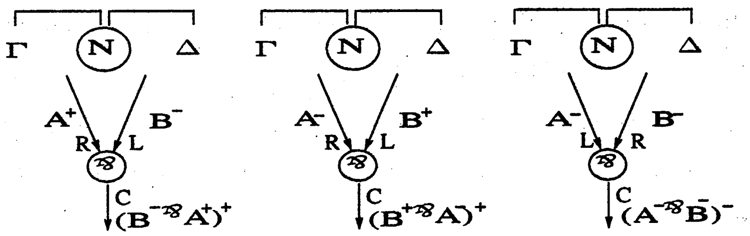
Definition 1.8 *We define* L-proof nets *by induction on the derivation in IMNCLL.*
Axiom. *We draw an axiom-link in an L-proof net as the axiom-link in MNCLL, with A and A^*, where A is a formula.*



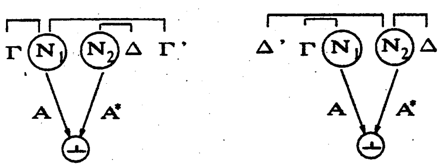
Tensor. *Now we draw a tensor-link as the following 6 types:*



Par. *Now we draw a par-link as the following 3 types:*



Cut. *We draw a cut-link as the following 2 types:*



Proposition 1.9 *The system of L-proof nets admits the cut-elimination.*

An L-proof net has the inductive structure inherited from the system L.

Definition 1.10 *A sequence $A_{i+1}, \cdots, A_n, A_1, \cdots, A_i$ $(i \leq n)$ is called as a cyclic shift of $A_1, \cdots, A_n$.*

Lemma 1.11 *Let G be an L-proof net with terminal edges Σ. Then for any cyclic shift Σ' of Σ, there exists an L-proof net with terminal edges Σ'.*

Proof. It follows since we identify the order of the formulas in a sequent in IMNCLL up to the cyclic shifts. □

By *a plane proof net* of MLL, we mean a (commutative) proof net without crossings in the graph drawing.

Definition 1.12 *A directed Danos-Regnier graph (or D-R graph) is a directed graph, which consists of axiom-links, cut-links, tensor-links, par-links and conclusion nodes: An axiom-link has two out-edges; a cut-link has two in-edges; each of a tensor-link and a par-link has two in-edges and one out-edge.*

Definition 1.13 *An edge in a D-R graph connected to a conclusion node is called a terminal edge.*

We will follow Danos and Regnier's convention to denote a formula by an edge and a logical connective by a link in a D-R graph. The following characterization theorem for proof nets of MLL is due to Danos and Regnier.

Theorem 1.14 *(Danos and Regnier [2]) A D-R graph is a proof net of MLL, if and only if it is always acyclic and connected under any choice of par-switchings (see [2] for the notion of par-switchings).*

We call the condition that a D-R graph is always acyclic and connected under any choice of par-switchings, as *the switching condition.*

Definition 1.15 *A marked D-R graph is a D-R graph, where each of a tensor-link and a par-link has two in-edges labeled L (left) and R (right), respectively, and one out-edge labeled C (conclusion).*

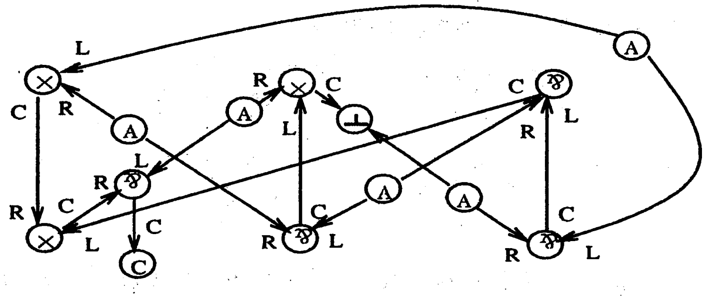


Fig. 2. An example of marked D-R graphs.

2 Intuitionistic Non-Commutative Proof Net Implies L-Strong Planity.

In this section, we introduce a notion of signed D-R graphs. Then we give a notion of L-strongly planity, which is shown to characterize L-proof nets in terms of signed D-R graphs. Our main theorem in this section is that non-commutative proof nets are quivalent to L-strongly planar singed D-R graphs.

Definition 2.1 *To each link of degree 3, we can assign a triple of signs* (s_1, s_2, s_3), *where* s_1 *is the sign of L-edge,* s_2 *R-edge,* s_3 *C-edge. A signed D-R graph is a marked D-R graph, in which each edge is labeled either* $+$ *or* $-$; *every axiom-link and cut-link consists of a pair of formulas of opposite signs; every par-link in the graph is assigned* $(-,+,+)$, $(+,-,+)$ *or* $(-,-,-)$; *every tensor-link in the graph is assigned* $(+,-,-)$, $(-,+,-)$, *or* $(+,+,+)$.

Definition 2.2 *The links with C-edge labeled with* $-$ *are called a* $-$-link, *and The links with C-edge labeled with* $+$ *are called a* $+$-link.

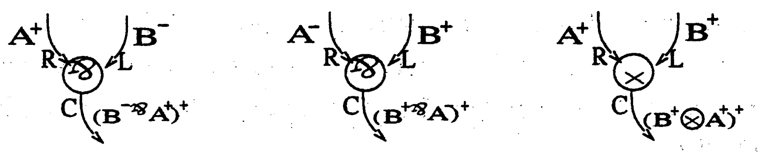


Fig. 7. Figures for +-links.

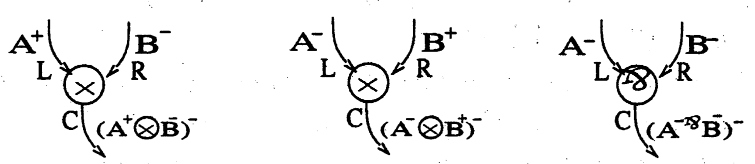


Fig. 8. Figures for −-links.

Definition 2.3 *(1) A signed D-R graph G is said to be L-strongly planar with terminal edges* $A_1, \cdots, A_n$, *if there exists a closure* $\bar{G}$ *of the graph G, which has a plane drawing drawing with one terminal edge* $A_1 \wp \cdots \wp A_n$, *in which (1.1) there exists a precisely one* $+$*-formula; (1.2) all the* $-$*-links are uniformly directed; (1.3) all the* $+$*-links are uniformly directed; and (1.4) the* $-$*-links and the* $+$*-links are reversely directed. (2) A signed D-R graph G is said to be L-strongly planar, if it is L-strongly planar with some terminal edges* Σ.

As a matter of simplicity, we assume that the signed D-R graph G is a plane L-directed graph drawing, in which the $-$-links are clockwisely ordered, and the $+$-links are counter-clockwisely ordered, respectively.

Theorem 2.4 *Let G be an L-proof net with terminal edges Σ. Then it is an L-strongly planar signed D-R graph G with terminal edges Σ satisfying the switching condition.*

Proof. Let the L-proof net have terminal nodes Σ. We can construct by induction on the structure of the L-proof net, a plane L-directed graph drawing, in which the $-$-links are clockwisely ordered, and the $+$-links are counter-clockwisely ordered, respectively, as in Theorem 3.7 in [7] . □

3 Stack Condition Implies Intuitionistic Non- Commutative Proof Net.

In this section, we give the notion of the stack condition, and show that it characterizes the L-proof nets.

The notion of a *stack condition* is defined by a *special trip*, which is a long trip with restrictions. The stack condition is originally obtained for MNCLL [7], which is obtained from an attempt to analyze the relationship between the strong planity [7] and the long trip condition introduced by Abrusci [1]. We modify this stack condition in order to characterize IMNCLL.

Definition 3.1 *For a given signed D-R graph G with an edge A,*
(1) T is a point of G, iff T is $A\downarrow$ or $A\uparrow$,
(2) we call a sequence $T_1, \cdots, T_n$ of points of G a one-way special trip from $A\uparrow$ (or $A\downarrow$) in G, iff the sequence is portion of the long trip in G from $T_1 = A\uparrow$ to $T_n = A\downarrow$ (or $T_1 = A\downarrow$ to $T_n = A\uparrow$, respectively), with the following switching:
(2.1) every $+$-$\otimes$-link $(+,+,+)$ is switched on "L" ("left"),
(2.2) every $-$-$\otimes$-link $(+,-,-)$ or $(-,+,-)$ is switched on "R" ("right"),
(2.3) every $+$-$\wp$-link $(-,+,+)$ or $(+,-,+)$ is switched on "R" ("right"),
(2.4) every $-$-$\wp$-link $(-,-,-)$ is switched on "L" ("left").

Let G be a signed D-R graph satisfying the switching condition. By Theorem 1.14, graph G is a proof net of MLL. We say an edge is a *critical node* (a critical vertex of Abrusci [1]), if it is a terminal edge or a R-edge of a par-link.

Definition 3.2 *(Definition of a stack.) Let S be a stack consisting of the ordered pairs defined above. $Top(S)$ represents the top element in the stack S. An action Pop pops up the top element in the stack S, which is denoted as $Pop(S)$. An action $Push(A, S)$ pushes a new element A on the top of the stack S.*

We define an algorithm with stack S which is later used for the correctness criteria.

Definition 3.3 *(Definition of a stack algorithm.) A stack algorithm is defined inductively on a special long trip.*
(Initial State.) $S \equiv \phi$.
If we visit:
(Case 1.) $B \downarrow$ *followed by* $B \uparrow$, $Push(B, S)$.
(Case 2.) $B \downarrow$ *followed by* $B \wp C \downarrow$, *Pop, if* $Top(S) = C$ *and* $\wp$ *is* $-$*-link, or the algorithm fails and the content of the stack is discarded otherwise.*
(Case 3.) $C \downarrow$ *followed by* $B \wp C \downarrow$, *Pop, if* $Top(S) = B$ *and* $\wp$ *is* $+$*-link, or the algorithm fails and the content of the stack is discarded otherwise.*
(Default.) S is unchanged in all the other cases.

Definition 3.4 *Let* $\Sigma = A_1, \cdots, A_n$. *Let G be a D-R graph satisfying the switching condition, and consider a special trip on G starting from* $A_n \downarrow$. *We say that graph G with terminal edges Σ satisfies the stack condition, if the content of the stack S is* $A_1, A_2, \cdots A_n$ *at the end of the trip.*

Remark. If graph G with terminal edges $\Sigma = A_1, \cdots, A_n$ satisfies the stack condition, any special trip on G starting with $A_i \downarrow$ $(i \neq n)$ results in a *cyclic shift* $A_{i+1}, \cdots, A_n, A_1, A_2, \cdots A_i$ of $A_1, A_2, \cdots A_n$ in S at the end of the trip.

Finally we show that the stack condition implies the L-proof nets.

Lemma 3.5 *Let G be a signed D-R graph with terminal edges Σ satisfying both the switching condition and the stack condition. Then for any special trip* $T_1, \cdots, T_n$, *with* $T_1 = D \downarrow$ *with a terminal edge D in Σ, the content of stack S at the end of the trip is a cyclic shift of Σ in which D is the rightmost formula.*

Proof. By the property of the special trips. □

Lemma 3.6 *Let G be a signed D-R graph with terminal edges Σ satisfying both the switching condition and the stack condition. Then for any cyclic shift Σ' of Σ, G with terminal edges Σ' satisfies the stack condition.*

Proof. By Lemma 3.5. □

Definition 3.7 *An edge A is said to be connected to an edge B, if there is a path connecting the edges A and B.*

Theorem 3.8 *Let G be a signed D-R graph with terminal edges Σ satisfying both the switching condition and the stack condition. Then it is an L-proof net with terminal edges Σ.*

Proof. Because the signed D-R graph G satisfies the switching condition, by Theorem 1.14, G is a proof net of MLL. Thus we may assume the inductive structure of proof net G. We prove by induction on the inductive structure of proof net G.
Axiom. Clear.
Par. We only consider for $(+,-,+)$-link; and the similar arguments work for the other par links. Let Σ be $\Gamma,(A^-\wp B^+)^+,\Delta$. Let G' be a signed D-R graph obtained by removing the par-link between A^- and B^+. We show the stack condition on G' with terminal edges $\Gamma,B^+,A^-;\Delta$ follows. Let C be the rightmost formula in Δ. By the stack condition of G, a special trip $T_1,\cdots,T_n$ on G starting $T_1 = C\downarrow$ gives the content of S equal to $\Gamma,(A^-\wp B^+)^+,\Delta$ at the end of the trip. We construct a special trip $T'_1,\cdots,T'_m$ on G', such that the content of S is Γ,B^+,A^-,Δ at the end of the trip. We follow the same trip up to $B^+\downarrow$; let $T_i = B^+\downarrow$. We define $T'_j = T_j$ $(j\leq i)$. We define the rest of the trip as $T'_j = T_{j+2}$ $(i+1\leq j\leq m)$.

Because the trips are exactly the same up to T_i and T'_i, and *Pop* is excuted at $T_{i+1} = (A^-\wp B^+)^+\downarrow$, $Top(S) = A^-$ at $T'_i = B^+\downarrow$. Hence the content of S at $T'_{i+1} = B^+\uparrow$ is B^+,A^-,Δ. Since the rest of the trips are again exactly the same, the claim holds. The rest of the proof follows from the induction hypothesis applied to G'.
Tensor. We may assume there is no par-link in Σ, whose C-edge is a terminal one. By Splitting Lemma [3], we moreover may assume the tensor-link is added last. We only consider for the $(+,+,+)$-link; and the similar arguments work for the other tensor links. By the stack condition of G, and Lemma 3.6, we assume a special trip $T_1,\cdots,T_n$ on G starting $T_1 = (B^+\otimes A^+)^+\downarrow$ gives the content of S equal to Σ', where Σ' is a cyclic shift of Σ and $(B^+\otimes A^+)^+$ is the rightmost formula in Σ'. Let G_{B^+} and G_{A^+} be signed D-R graphs obtained from signed D-R graph G by removing the tensor-link between B^+ and A^+, whose edges are connected to edge B^+, and are connected to edge A^+, respectively. Hence G_{B^+} and G_{A^+} are only connected at $(B^+\otimes A^+)^+$ in G. Because of the property of the special trip, $T_2 = (B^+\otimes A^+)^+\uparrow$, $T_3 = A^+\uparrow$; and there exist an integer $i<n$, and formulas D and C, such that each T_j $(3\leq j\leq i)$ is a point in the subgraph G_{A^+} and $T_i = D\downarrow$, and $T_{i+1} = A^+\downarrow$, $T_{i+2} = B^+\uparrow$, $T_{i+3} = C\uparrow$, each T_j $(i+3\leq j\leq n-1)$ is a point in the subgraph G_{B^+} and $T_n = B^+\downarrow$. Therefore there exist Γ and Δ such that $\Sigma\equiv\Delta,\Gamma,(B^+\otimes A^+)^+$, where Γ are the terminal edges in G^+_A and Δ are the terminal

edges in G_B^+. Moreover, the part of the special trip $A^+ \downarrow, T_3, \cdots, T_i$ gives a special trip on a signed D-R graph G_{A^+} such that the content of S is Γ, G_{A^+} at the end of the trip, and the part of the special trip $B^+ \downarrow, T_{i+2}, \cdots, T_{n-1}$ gives a special trip on a signed D-R graph G_{B^+} such that the content of S is Δ, B^+ at the end of the trip. Thus both graphs G_{A^+} and G_{B^+} satisfy the stack condition. By induction hypothesis, both G_{A^+} and G_{B^+} are L-proof nets with terminal edges Γ, A^+ and Δ, B^+, respectively. Hence there exists an L-proof net with terminal edges $\Delta, \Gamma, (B^+ \otimes A^+)^+$. By Lemma 1.11, we obtain an L-proof net with terminal edges Σ.

Cut. Similar to the case of tensor. □

4 L-Strong Planity Implies Stack Condition.

In order to establish the equivalence between the L-proof nets and the two characterizations, we then prove that the L-strong planity implies the stack condition.

Definition 4.1 *An edge A is said to be unilaterally connected to an edge B, if there is a directed path from the edge A to the edge B.*

Lemma 4.2 *Let G be a signed D-R graph satisfying the switching condition. Then signed D-R graph G is L-strongly planar with terminal edges Σ iff it is L-strongly planar graph with terminal edges Σ' for any cyclic shift Σ' of Σ.*

Proof. Let $\Sigma = A_1, \cdots, A_n$. As same as Lemma 3.9 in [7] , we can construct a closure of G, as graph drawing with terminal edge $A_n \wp (A_1 \wp \cdots \wp A_{n-1})$ such that the —-links are clockwisely directed and +-links are counter-clockwisely directed. □

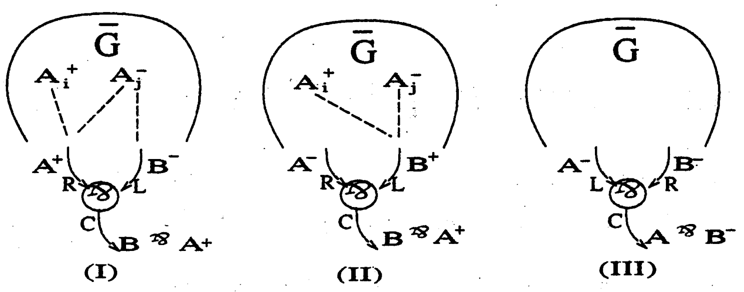
Lemma 4.3 *Let A^- be an edge in a signed D-R graph and be an associative par instance of Σ. Then any edge in Σ is signed $-$.*

Proof. By induction on the number of elements in Σ. □

Lemma 4.4 *Assume an L-strongly planar signed D-R graph G with terminal edges $A_1, \cdots, A_n$ satisfying the switching condition. If $1 \le i < j \le n$, then in a closure $\bar{G}$ of G, the following hold:*
(1) for the edges A_i^+ and A_j^-, there exists a par-link such that the edge A_i^+ is unilaterally connected its R-edge and the edge A_j^- is unilaterally connected to its L-edge.
(2) for the edges A_i^- and A_j^+, there exists a par-link such that the edge A_i^- is unilaterally connected to its R-edge and the edge A_j^+ is unilaterally connected to its L-edge,
(3) for the edges A_i^- and A_j^-, there exists a par-link such that the edge A_i^- is unilaterally connected its L-edge and the edge A_j^- is unilaterally connected to its R-edge.

Proof. we only discuss on (1), but arguments for (2) and (3) are similar. We may assume that closure $\bar{G}$ of G is a plane signed D-R graph drawing with a single terminal edge which is an associative par instance of $A_1, \cdots, A_n$. Moreover in the graph $\bar{G}$, we may assume that the links with C-edge labeled with $-$ are clockwisely ordered, and that the links with C-edge labeled with $+$ are counter-clockwisely ordered, respectively.

We prove the lemma by induction on the number of formulas in the associative par instance, whose in-edges, the edges A_i^+ and A_j^- are unilaterally connected to. We have three types of par-links:



By Lemma 4.3, the possible par-link is either type (I) or (II). If the par-link is type (I), we argue as in Lemma 6.2 in [7]. If the par-link is type (II), then by Lemma 4.3, both A_i^+ and $A_j{}^-$ are connected to B^+. By induction hypothesis, the claim holds. □.

Lemma 4.5 *A signed D-R graph satisfies the switching condition, then there is only one +-formula connected to a conclusion node.*

Proof. By induction on the number of links. □

Proposition 4.6 *Assume that an L-strongly planar signed D-R graph G with terminal edges Σ satisfies the switching condition, and no terminal edge in G is a C-edge of a par-link. Then there is a splitting formula $A \otimes B$ in Σ.*

Proof. The argument goes as Theorem 1.14.

Lemma 4.7 *Assume that an L-strongly planar signed D-R graph G with terminal edges $\Gamma, A \otimes B$, satisfies the switching condition, and that $A \otimes B$ is a splitting formula. Let G_B be a graph obtained from G by removing the tensor link between A and B, whose edges are connected to edge B.*
(1) Assume that $A \otimes B$ is signed $-$, and that an edge D_B is the rightmost edge in Γ, which belongs to graph G_B, then any edge in Γ left to D_B belongs to G_B as well.
(2) Assume that $A \otimes B$ is signed $+$, and that an edge D_A is the rightmost edge in Γ, which belongs to graph G_A, then any edge in Γ left to D_A belongs to G_A as well.

Proof. The proof essentially goes as Lemma 6.3 in [7] with a help of Lemma 4.4, except that we argue separately according to the signs of A, D_B and D_L. However, Lemma 4.5 reduces the number of cases we have to argue. □

Lemma 4.8 *Assume that an L-strongly planar signed D-R graph G with terminal edges* Σ *satisfies the switching condition, and that* $A \otimes B$ *is a splitting formula. Let* G_A *and* G_B *be signed D-R graphs obtained from G by removing the tensor-link between A and B, whose edges are connected to edge A, and are connected to edge B, respectively.*
(1) If $A \otimes B$ *is signed* $-$*, then there are sequences* Γ *and* Δ *of terminal edges in G, such that (1.1) the edges in* Γ *belong to* G_A *and the edges in* Δ *belong to* G_B*, (1.2) signed D-R graphs* G_A *with terminal edges* Γ, A *and* G_B *with terminal edges* B, Δ *are L-strongly planar, (1.3)* $\Gamma, A \otimes B^-, \Delta$ *is a cyclic shift of* Σ*.*
(2) If $A \otimes B$ *is signed* $+$*, then there are sequences* Γ *and* Δ *of terminal edges in G, such that (2.1) the edges in* Γ *belong to* G_B *and the edges in* Δ *belong to* G_A*, (2.2) signed D-R graphs* G_B *with terminal edges* Γ, B *and* G_A *with terminal edges* A, Δ *are L-strongly planar, (2.3)* $\Gamma, A \otimes B^+, \Delta$ *is a cyclic shift of* Σ*.*

Proof. The proof essentially goes as Lemma 6.4 in [7] with a help of Lemmas 4.2 and 4.7. We note that if $A \otimes B$ is signed $-$, then A and B are oppositely signed; and if $A \otimes B$ is signed $+$, then A and B are both signed $+$. Use Lemma 4.5. □

Lemma 4.9 *Assume that an L-strongly planar signed D-R graph G with terminal edges* D, Γ *satisfies the switching condition, and that* $\perp$ *in G is a splitting formula. Let* G_A *be a graph obtained from G by removing the cut-link between A and* A^**, whose edges are connected to edge A. Assume that an edge* D_A *is the rightmost edge in* Γ*, which belongs to graph* G_A*, then any edge in* Γ *left to* D_A *belongs to* G_A *as well.*

Proof. The proof essentially goes as Lemma 6.5 in [7] with a help of Lemma 4.4. □

Lemma 4.10 *Assume that an L-strongly planar signed D-R graph G with terminal edges* Σ *satisfies the switching condition, and that* $\perp$ *is a splitting formula. Let* G_A *and* G_{A^*} *be signed D-R graphs obtained from G by removing the cut-link between A and* A^**, whose edges are connected to edge A, and are connected to edge* A^**, respectively.*
Then there are sequences Γ *and* Δ *of terminal edges in G, such that (1) the edges in* Γ *belong to* G_A *and the edges in* Δ *belong to* G_{A^*}*, (2) signed D-R graphs* G_A *with terminal edges* Γ, A *and* G_{A^*} *with terminal edges* A^*, Δ *are L-strongly planar. (3)* Γ, Δ *is a cyclic shift of* Σ*.*

Proof. We note that A and A^* are oppositely signed. The argument essentially goes as Lemma 6.6 in [7] with a help of Lemmas 4.2 and 4.9. □

Theorem 4.11 *Assume that a signed D-R graph G satisfies the switching condition. If G is L-strongly planar with terminal edges Σ, then G with Σ satisfies the stack condition.*

Proof. We prove by induction on the number of links in the signed D-R graph G. We use Proposition 4.6 to keep the inductive step, and show the removal of any par-link or any tensor-link preserves the stack condition by Lemmas 4.2, 4.8 and 4.10. The argument goes as in Theorem 6.7 in [7] . □

Theorem 4.12 *(Characterization theorem with respect to the signed D-R graph for L) A signed D-R graph represents an L-proof net iff (1) it satisfies the switching condition and it is L-strongly planar, iff (2) it satisfies the switching condition and the stack condition.*

Proof. By Theorems 2.4 and 4.11. □

Roorda's characterization of the proof nets for Lambek Calculus is written as the condition on λ-terms assigned to formulas, and not quite geometric [8]. Our question is whether the additional condition of the non-empty antecedent on the inference rules ($\backslash$ 1) and (/ 1) of Lambek Calculus (see the remark after Definition 1.1) can be interpreted as some geometrical property of signed D-R graphs. One would think that we can simply add the condition that there are strictly more than one $-$-signed terminal edges in the signed D-R graph: But this does not mean that any smaller signed D-R graph obtained by splitting the original signed D-R graph always preserves the same property. Hence, the following remains an open question: What is a geometric characterization of proof nets for Lambek Calculus in terms of signed D-R graphs?

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