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# JORDAN-HÖLDER TYPE THEOREM IN NORMAL INTERMEDIATE SUBFACTOR LATTICES FOR DEPTH TWO INCLUSIONS OF AFD $\mathrm{II}_1$ FACTORS

TAMOTSU TERUYA (照屋 保)

ABSTRACT. Let $N \subset M$ be a depth 2 inclusion of AFD $\mathrm{II}_1$ factors with finite Jones index. Let $K$ and $L$ be normal intermediate subfactors of $N \subset M$. If $K \cap L = N$ and $M$ is generated by $K$ and $L$, then we can represent $M, K, L, N$ as $M = P \otimes R, K = Q \otimes R, L = P \otimes S$, and $N = Q \otimes S$ for some inclusins $P \supset Q$ and $R \supset S$. Using this characterization, we shall prove Jordan-Hölder type theorem in normal intermediate subfactor lattices for depth 2 inclusions of AFD $\mathrm{II}_1$ factors.

## 1. INTRODUCTION

Let $N \subset M$ be an irreducible inclusion of type $\mathrm{II}_1$ factors with finite index. In [9], the auther introduced the notion of normality for intermediate subfactors of $N \subset M$ as follows:

**Definition 1.1.** Let $K$ be an intermediate subfactor of the inclusion $N \subset M$. Let $N \subset M \subset M_1 \subset M_2$ be the Jones tower for $N \subset M$ and $K_1$ the basic extension for $K \subset M$. Then $K$ is a *normal intermediate subfactor* of the inclusion $N \subset M$ if $e_K \in \mathcal{Z}(N' \cap M_1)$ and $e_{K_1} \in \mathcal{Z}(M' \cap M_2)$, where $e_K$ and $e_{K_1}$ are the Jones projections for $K \subset M$ and $K_1 \subset M_1$, respectively.

With the above notation, if the depth of $N \subset M$ is 2, then $N' \cap M_1$ and $M' \cap M_2$ are a dual pair of Hopf $C^*$-algebras. and $K' \cap K_1$ is a $*$-subalgebra and a left coideal of $N' \cap M_1$( see [1]). Then $K$ is a normal intermediate subfactor of $N \subset M$ if and only if $K' \cap K_1$ is a subHopf algebra and the left and right adjoint action of $N' \cap M_1$ leave $K' \cap K_1$ invariant (see [3]).

Watatani[10] studied intermediate subfactor lattices $\mathcal{L}(N \subset M)$ and relations between modular identity and commuting and co-commuting (nondegenerate) square conditions. The author[9] proved if the depth of $N \subset M$ is 2, then the set $\mathcal{N}(N \subset M)$ of all normal intermediate subfactors of $N \subset M$ is a sublattice of $\mathcal{L}(N \subset M)$ and a modular lattice.

Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD $\mathrm{II}_1$ factors with finite index. Our purpose is to show Jordan-Hölder type theorem in normal intermediate subfactor lattices for $N \subset M$. To be more precise, we prove that if $M = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n = N$ and $M = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_m = N$ are maximal chains of $\mathcal{N}(N \subset M)$, then $m = n$ and the inclusions $A_{i-1} \supset A_i$ are isomorphic to the inclusions $B_{j-1} \supset B_j$ in some order. To show this , we characterize tensor products of depth 2 inclusions of AFD $\mathrm{II}_1$ factors with finite index as follows: Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD $\mathrm{II}_1$ factors with finite index. Let $K$ and $L$ be normal intermediate subfactors for $N \subset M$. If $K \cap L = N$ and $M$ is generated by $K$ and $L$, then we can represent $M, K, L, N$ as $M = P \otimes R, K = Q \otimes R, L = P \otimes S$ and $N = Q \otimes S$ for some inclusions $P \supset Q$ and $R \supset S$.

## 2. A characterization of tensor products of depth 2 inclusions

Let $N \subset M$ be an irreducible, depth 2 inclusion of $\mathrm{II}_1$ factors with $[M:N] < \infty$ and $\mathcal{N}(N \subset M)$ the all normal intermediate subfactors of $N \subset M$. Suppose that $K, L \in \mathcal{N}(N \subset M)$ and $M$ is generated by $K$ and $L$, and $N = K \cap L$. Then

$$\begin{array}{ccc} K & \subset & M \\ \cup & & \cup \\ N & \subset & L \end{array}$$

is commuting and co-commuting (nondegenerate) square (see [6, 8]). Let $K_1 = \langle K, e_K^M \rangle$ and $L_1 = \langle L, e_L^M \rangle$ be the basic extension with the Jones projections $e_K^M$ and $e_L^M$ for $K \subset M$ and $L \subset M$, respectively. Then it is well known that

$$\begin{array}{ccc} M & \subset & K_1 \\ \cup & & \cup \\ L & \subset & \langle L, e_K^M \rangle \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \subset & L_1 \\ \cup & & \cup \\ K & \subset & \langle K, e_L^M \rangle \end{array}$$

are also nondegenerate commuting squares.

**Lemma 2.1.** *With the above notation, $L \subset K_1$ and $K \subset L_1$ are irreducible, depth 2 inclusions. Moreover, $M$ and $\langle L, e_K^M \rangle$ are normal intermediate subfactors of $L \subset K_1$ and, $M$ and $\langle K, e_L^M \rangle$ are normal intermediate subfactors of $K \subset L_1$*

*Proof.* Since $L \subset M$ and $M \subset K_1$ are depth 2 inclusion by [9], the depth of $L \subset K_1$ is 2 by [7]. Similarly, $K \subset L_1$ is a depth 2 inclusion. It is easy to see that $L \subset K_1$ and $K \subset L_1$ are irreducible inclusions. □

**Lemma 2.2.** *With the above notation, we have*

$$K' \cap K_1 = \langle K, e_L^M \rangle' \cap M_1 = N' \cap \langle L, e_K^M \rangle$$

$$L' \cap L_1 = \langle L, e_K^M \rangle' \cap M_1 = N' \cap \langle K, e_L^M \rangle.$$

*Proof.* By Lemma 2.1 and [8], we have $[M : K] = [L : N] = [L_1 : \langle K, e_L^M \rangle]$. Therefore we have

$$\dim_{\mathbb{C}}(K' \cap K_1) = \dim_{\mathbb{C}}(\langle K, e_L^M \rangle' \cap M_1) = \dim_{\mathbb{C}}(N' \cap \langle L, e_K^M \rangle).$$

Let $x$ be an element of $K' \cap K_1$. Since $e_L^M$ is an element of the center of $N' \cap M_1$ and $K' \cap K_1 \subset N' \cap M_1$, $x$ and $e_L^M$ are commutative and hence $x \in \langle K, e_L^M \rangle' \cap M_1$. So we have $K' \cap K_1 \subset \langle K, e_L^M \rangle' \cap M_1$. By $\dim_{\mathbb{C}}(K' \cap K_1) = \dim_{\mathbb{C}}(\langle K, e_L^M \rangle' \cap M_1)$, we have $K' \cap K_1 = \langle K, e_L^M \rangle' \cap M_1$.

Since $M_1$ is the basic extension of $K_1$ by $\langle L, e_K^M \rangle$ with the Jones projection $e_L^M$, we have $\langle L, e_K^M \rangle = \{e_L^M\}' \cap K_1$. Since $e_L^M$ is an element of the center of $N' \cap M_1 (\supset K' \cap K_1)$, if $x$ is an element of $K' \cap K_1$, then $x \in \{e_L^M\}' \cap K_1 = N' \cap \langle L, e_K^M \rangle$. And hence $K' \cap K_1 \subset N' \cap \langle L, e_K^M \rangle$. And $K' \cap K_1 = N' \cap \langle L, e_K^M \rangle$ by $\dim_{\mathbb{C}}(K' \cap K_1) = \dim_{\mathbb{C}}(N' \cap \langle L, e_K^M \rangle)$. Similarly, we have $L' \cap L_1 = \langle L, e_K^M \rangle' \cap M_1 = N' \cap \langle K, e_L^M \rangle$. □

**Theorem 2.3.** *Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD $II_1$ factors with $[M : N] < \infty$. If $K$ and $L$ are normal intermediate subfactors of $N \subset M$ such that $K \cap L = N$ and $M$ is generated by $K$ and $L$, then we can represent $M, N, K, L$ as $M = P \otimes R, N = Q \otimes S, K = Q \otimes R$ and $L = P \otimes S$*

*Proof.* $N \subset M$ has the generating property, i.e., there exists a tunnel $M = N_0 \supset N = N_1 \supset N_2 \supset \cdots \supset N_i \supset \cdots$ such that

$$M = \overline{\bigcup_{i=1}^{\infty}(M \cap N_i')}^{weak} \supset N = \overline{\bigcup_{i=1}^{\infty}(N \cap N_i')}^{weak}$$

(see for example [4, 5]). Let

$$\begin{array}{cccc} A_{00} \supset & A_{01} \supset & A_{02} \supset & \cdots \\ \cup & \cup & \cup & \\ A_{10} \supset & A_{11} \supset & A_{12} \supset & \cdots \\ \cup & \cup & \cup & \\ A_{20} \supset & A_{21} \supset & A_{22} \supset & \cdots \\ \cup & \cup & \cup & \\ \vdots & \vdots & \vdots & \end{array}$$

be the commuting and co-commuting squares such that the initial commuting square is

$$\begin{array}{ccc} M & \supset & L \\ \cup & & \cup \\ K & \supset & N \end{array}$$

and $A_{ii} = N_i$ for $i = 1, 2, \ldots$ as in [8]. Note that for the square

$$\begin{array}{ccc} A_{kl} & \supset & A_{k,l+1} \\ \cup & & \cup \\ A_{k+1,l} & \supset & A_{k+1,l+1}, \end{array}$$

$A_{kl} \supset A_{k+1,l+1}$ is again irreducible, depth 2 and, $A_{k,l+1}$ and $A_{k+1,l}$ are normal intermediate subfactors of $A_{kl} \supset A_{k+1,l+1}$. We put

$$P = \overline{\bigcup_{i=1}^{\infty} (A_{00} \cap A'_{i0})}^{weak} \supset Q = \overline{\bigcup_{i=1}^{\infty} (A_{10} \cap A'_{i0})}^{weak}$$

$$R = \overline{\bigcup_{i=1}^{\infty} (A_{00} \cap A'_{0i})}^{weak} \supset S = \overline{\bigcup_{i=1}^{\infty} (A_{01} \cap A'_{0i})}^{weak} .$$

Then we can see $M = P \otimes R, N = Q \otimes S, K = Q \otimes R$ and $L = P \otimes S$ by Lemma 2.2 and [2]. □

## 3. Jordan-Hölder type theorem

In this section, we shall prove Jordan-Hölder type theorem for depth 2 inclusions of AFD $\mathrm{II}_1$ factors.

**Theorem 3.1.** *Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD $II_1$ factor. If $K$ and $L$ are normal intermediate subfactors of $N \subset M$, then $K \subset K \vee L$ and $K \cap L \subset L$ are conjugate.*

*Proof.* Since the set $\mathcal{N}(N \subset M)$ of all normal intermediate subfactors of $N \subset M$ is a sublattice of $\mathcal{L}(N \subset M)$, $K \vee L$ and $K \cap L$ are elements of $\mathcal{N}(N \subset M)$. Therefore $N \subset K \vee L$ and $N \subset K \cap L$ are depth 2 inclusion by [9, Theorem 4.6]. Moreover $K \cap L$ is a normal intermediate subfactor of $N \subset K \vee L$ by [9, Proposition 3.7]. So we have $K \cap L \subset K \vee L$ is depth 2 inclusion by [9, Theorem 4.6]. By theorem 2.3, there exist inclusions $P \supset Q$ and $R \supset S$ such that $K \vee L = P \otimes R, K = P \otimes S, L = Q \otimes R$ and $K \cap L = Q \otimes S$. So we can see both $K \vee L \subset K$ and $L \supset K \cap L$ are conjugate to $R \subset S$. □

**Theorem 3.2.** *Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD $II_1$ factors with $[M : N] < \infty$. Let $K, \tilde{K}, L, \tilde{L}$ be normal intermediate subfactors of $N \subset M$ with $K \supset \tilde{K}$ and $L \supset \tilde{L}$. Then the pairs $\tilde{K} \vee (K \cap L) \supset \tilde{K} \vee (K \cap \tilde{L})$ and $\tilde{L} \vee (K \cap L) \supset \tilde{L} \vee (\tilde{K} \cap L)$ are conjugate.*

*Proof.* Since $\tilde{K} \vee (K \cap L) = (\tilde{K} \vee (K \cap \tilde{L})) \vee (K \cap L)$, the pairs $\tilde{K} \vee (K \cap L) \supset \tilde{K} \vee (K \cap \tilde{L})$ and $K \cap L \supset (K \cap L) \cap (\tilde{K} \vee (K \cap \tilde{L}))$ are conjugate by the previous theorem. Similarly, the pair $\tilde{L} \vee (K \cap L) \supset \tilde{L} \vee (\tilde{K} \cap L)$ and $K \cap L \supset (K \cap L) \cap (\tilde{L} \vee (\tilde{K} \cap L))$. are conjugate. Since $\mathcal{N}(N \subset M)$ is a modular lattice by [9], we hve $(K \cap L) \cap (\tilde{K} \vee (K \cap \tilde{L})) = ((K \cap L) \cap \tilde{L}) \vee (K \cap \tilde{L})) = (K \cap \tilde{L}) \vee (K \cap \tilde{L})$. Similarly, we hvave

$(K \cap L) \cap (\tilde{L} \vee (\tilde{K} \cap L)) = (K \cap \tilde{L}) \vee (K \cap \tilde{L})$. We have thus proved the theorem. □

In a lattice $L$, a finite chain $x = x_0 \supseteq x_1 \supseteq \cdots \supseteq x_d = y$ is maximal if $x_i \supsetneqq x_{i+1}$ and $x_i \supseteq a \supseteq x_{i+1}$ implies $x = a$ or $x_{i+1} = a$ for $i = 1, 2, \ldots, d-1$.

**Theorem 3.3.** *Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD $II_1$ factors with $[M : N] < \infty$. If $M = A_0 \supset A_1 \supset \cdots \supset A_n = N$ and $M = B_0 \supset B_1 \supset \cdots \supset B_m$ are two maximal chain of $\mathcal{N}(N \subset M)$, then $m = n$ and the inclusions $A_{i-1} \supset A_i$ are isomorphic to the inclusions $B_{j-1} \supset Bj$ in some order.*

*Proof.* Put

$$A_{ij} = A_i \vee (A_{i-1} \cap B_j)$$

and

$$B_{ji} = B_j \vee (A_i \cap B_{j-1}).$$

Then $A_{i,j-1} \supset A_{ij}$ is isomorphic to $B_{j,i-1} \supset B_{ji}$ by Theorem 3.2. Since $A_0 \supset A_1 \supset \cdots \supset A_s$ is maximal chain, for any $i (i = 1, 2, \ldots, s)$, there uniquely exists $j$ such that $A_{i-1} = A_{i,j-1} \supset A_{ij} = A_i$. Then $B_{j-1} = B_{j,i-1} \supsetneqq B_{ji} = B_j$. And hence $A_{i-1} \supset A_i$ is isomorphic to $B_{j-1} \supset B_j$. □

**Example 3.4.** Let $G$ be a semi direct group $B \rtimes A$ of finite groups $A$ and $B$. Let

$$M = P \rtimes_\gamma B \supset N = P^{(A,\gamma)} = \{x \in P | \gamma_a(x) = x, \forall a \in A\},$$

where $\gamma$ is an outer action of $G$ on $\mathrm{II}_1$ factor $P$. Then the depth of $N \subset M$ is 2 (see for example [7]). Let $A_0 = A \supsetneqq A_1 \supsetneqq \cdots \supsetneqq A_r = \{e\}$ and $B_0 = B \supsetneqq B_1 \supsetneqq$

$\cdots \supsetneq B_s = \{e\}$ be normal subgroups of $G$ such that if $H$ is a normal subgroup of $G$ with $A_{i-1} \supsetneq H \supset A_i$ or $B_{j-1} \supsetneq H \supset B_j$, then $H = A_i$ or $H = B_j$. Then $M = P \rtimes_\gamma B_0 \supset P \rtimes_\gamma B_1 \supset \cdots P = P^{(A_r,\gamma)} \supset P^{(A_{r-1},\gamma)} \supset \cdots P^{(A_0,\gamma)} = N$ is a maximal chain of $\mathcal{N}(N \subset M)$ by [9]. Therefore if $M = C_0 \supset C_1 \supset \cdots C_n = N$ a maximal chain of $\mathcal{N}(N \subset M)$, then $n = r + s$ and the inclusins $C_{k-1} \supset C_k$ are isomorphic to $R \rtimes F \supset P$ or $R \supset R^F$ for some $\mathrm{II}_1$ factor and some finite group $F$.

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