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GALOIS CORRESPONDENCE OF COMPACT GROUP ACTIONS

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§0. INTRODUCTION

In this note, we will sketch the main idea of the proofs of our results on a Galois correspondence in operator algebras in our collaboration [ILP] with R. Longo and S. Popa.

Let M be a von Neumann algebra and $\alpha : G \longrightarrow Aut(M)$ an action of a locally compact group G on M. We denote by M^G the fixed point subalgebra of M under α. For a closed subgroup $H \subset G$ we denote by M^H the fixed point subalgebra of M under α restricted to H. We say that the Galois correspondence holds for $M^G \subset M$ if the map $H \mapsto M^H$ gives 1-1 correspondence between the set of all intermediate von Neumann subalgebras of $M^G \subset M$ and the set of all closed subgroups of G. In a similar way, we say that the Galois correspondence holds for $M \subset M \rtimes_\alpha G$ if the map $H \mapsto M \rtimes_\alpha H$ gives 1-1 correspondence between the set of all intermediate von Neumann subalgebras of $M \subset M \rtimes_\alpha G$ and the set of all closed subgroups of G.

There are a substantial number of articles about Galois correspondences in operator algebras, and a comprehensive history of the subject may be found in the monograph [NTa]. We just mention the following three papers [NT][Ch][HS] among others. In [NT], N. Nakamura and Z. Takeda initiated the study of the subject, and they obtained the complete Galois correspondence of outer actions of finite groups on II_1 factors. In [Ch], H. Choda tried to generalize Nakamura and Takeda's result to outer actions of discrete groups on general factors, and he obtained the Galois correspondence under some condition. In [HS], U. Haagerup and E. Størmer studied very particular actions, the modular automorphism groups of periodic weights on type III_λ $(0 < \lambda <)$ factors and obtained the Galois correspondence. One should notice that the heart of the matter in the proofs of the above mentioned results is the existence of the conditional expectations onto a given intermediate subfactor, which is highly non trivial except the case of II_1 factors. We generalize these results simultaneously.

§1. Main results

Although we could unify the statements of our main results in the language of Kac algebras (see Theorem 1.3 below), we state them separately in order to provide readers with a good perspective.

In what follows, every factor has separable pre-dual and every locally compact group is separable.

Theorem 1.1. *Let M be a factor and $\alpha : \Gamma \longrightarrow Aut(N)$ be an outer action of a discrete group Γ. Then the Galois correspondence holds for $N \subset N \rtimes_\alpha \Gamma$.*

We call that a faithful action of a compact group on a factor is minimal if the relative commutant of the fixed point algebra is trivial.

Theorem 1.2. *Let M be a factor and $\alpha : G \longrightarrow Aut(M)$ be a minimal action of a compact group G. The Galois correspondence holds for $M^G \subset M$.*

We generalize and unify the above two statements in the category of compact Kac algebras.

Let $\mathcal{A}$ be a compact Kac algebra with a coproduct δ. An action of $\mathcal{A}$ on a von Neumann algebra M is a normal *-homomorphism $\pi : M \longrightarrow M \otimes \mathcal{A}$ satisfying $(\pi \otimes id_{\mathcal{A}}) \cdot \pi = (id_M \otimes \delta) \cdot \pi$. The fixed point algebra M^π is defined to be

$$M^\pi = \{x \in M; \pi(x) = x \otimes 1\}.$$

π is called minimal if the relative commutant of M^π in M is trivial and the following holds:

$$\overline{\{(\omega \otimes id_{\mathcal{A}})(M);\ \omega \in M_*\}}^{w} = \mathcal{A}.$$

A unital von Neumann subalgebra $\mathcal{B}$ of a Kac algebra $\mathcal{A}$ is called a left coideal von Neumann subalgebra if $\mathcal{B}$ satisfies

$$\delta(\mathcal{B}) \subset \mathcal{A} \otimes \mathcal{B}.$$

For a left coideal von Neumann subalgebra $\mathcal{B}$, we set

$$M(\mathcal{B}) = \{x \in M;\ \pi(x) \in M \otimes \mathcal{B}\}.$$

Theorem 3.3. *Let M be a factor and $\pi : M \longrightarrow M \otimes \mathcal{A}$ be a minimal action of a compact Kac algebra $\mathcal{A}$. Then, the Galois correspondence holds for $M_\pi \subset M$; i.e the map $\mathcal{B} \mapsto M(\mathcal{B})$ gives 1-1 correspondence between the set of all left coideal von Neumann subalgebras of $\mathcal{A}$ and the set of all intermediate subfactors of $M^\pi \subset M$.*

If Γ is a discrete group and $\mathcal{R}(\Gamma)$ is the group von Neumann algebra of Γ, then every left coideal von Neumann subalgebra of $\mathcal{R}(\Gamma)$ is in the form $\mathcal{R}(H)$ for some subgroup H. If G is a compact group, then every left coideal von Neumann subalgebra $L^\infty(G)$ is in the form $L^\infty(G/H)$ for some closed subgroup $H \subset G$. Thus, Theorem 1.3 generalizes Theorem 1.1 and Theorem 1.2.

§2. THE PROOF IN DISCRETE GROUP CASE.

There are two technical points about the proof of the main results. One is the averaging argument based on the existence of a simple injective subfactor, which we will explain in the case of discrete groups in the rest of this note. The other is analysis of general inclusions of factors of discrete type, which allows us to generalize the averaging argument to the general case.

Let N be a factor and $\alpha : \Gamma \longrightarrow Aut(N)$ an outer action of a discrete group Γ on N. The crossed product $N \rtimes_\alpha \Gamma$ is the von Neumann algebra generated by N and a unitary representation $\{\lambda_g\}_{g \in \Gamma}$ implementing α_g. Let $E : M \rtimes_\alpha \Gamma \to M$ be the canonical conditional expectation. For $x \in N \rtimes \Gamma$, x_g denotes the Fourier coefficient of x at g, i.e. $x_g = E(x\lambda_g^*)$. Although $\sum_g x_g \lambda_g$ does not converges in any decent operator algebra topology in general, the formal expression $x = \sum_g x_g \lambda_g$ can be justified in almost all the cases.

Let L be an intermediate subfactor of $N \subset N \rtimes_\alpha \Gamma$ and set

$$H = \{h \in \Gamma;\ \lambda_h \in L\}.$$

To prove Theorem 1.1, it suffices to show that $L = N \rtimes_\alpha H$, or more precisely, it suffices to show that for every $x \in L$, $x_g = 0$ unless $g \in H$. For this we need the averaging argument. We may assume that N is infinite because it is easy to show the above statement for finite N by using the conditional expectation onto L.

A subfactor R of a factor N is called simple if R and $J_N R J_N$ generate $B(L^2(N))$. On of the remarkable features of a simple subfactor R is that R determines automorphisms; i.e. if $\alpha, \beta \in Aut(N)$, $\alpha|_R = \beta|_R$ implies $\alpha = \beta$. Indeed, suppose $\theta \in Aut(N)$ and $\theta|_R = id_R$. Let u be the canonical implementation of θ. Then since u commutes with J_N, u belongs to

$$R' \cap J_N R' J_R = (R \vee J_N R J_R)' = \mathbf{C},$$

which shows $\theta = id$, and consequently $\alpha \cdot \beta^{-1} = id$. By using this property, it is easy to show that if ν is an outer automorphism of N, then

$$\{a \in N;\ ax = \theta(x)a, \quad \forall x \in R\} = \mathbf{C}. \tag{1.1}$$

In [L], R. Longo shows that every infinite factor with separable pre-dual has a simple injective subfactor.

Proof of Theorem 1.1. Now, let R be a simple injective subfactor of N. Suppose $x_g \neq 0$ for some $x \in L$. We would like to show $g \in H$. By replacing x with yxz for some $y, z \in N$, we may assume $x_g = 1$. Set

$$\mathcal{C} = \overline{conv\{ux\alpha_{g^{-1}}(u^*);\ u \in U(R)\}}^{w},$$

and define an action of the unitary group $U(R)$ on $\mathcal{C}$ by $\tau_u(a) = ua\alpha_{g^{-1}}(u^*)$, $u \in U(R)$, $a \in \mathcal{C}$. Since R is injective, it is AFD due to A. Connes' deep result. Thus there exists a fixed point $a \in \mathcal{C}$ under the $U(R)$ action. Direct computation shows that $a_g = 1$ and that for all $k \in \Gamma$ a_k satisfies

$$xa_k = a_k \alpha_{kg^{-1}}(x), \quad \forall x \in R.$$

Thanks to (1.1), this implies that $a_k = 0$ except $k = g$, which means $a = \lambda_g$. Thus $g \in H$. $\square$

In the general case, one can regards the larger factor as a Roberts type crossed product of the smaller factor and a system of endomorphisms on it. One can generalize the above averaging argument to endomorphisms after some effort.

REFERENCES

[Ch] H. Choda, *A Galois correspondence in von Neumann algebras*, Tohoku math. J. **30** (1978), 491–504.

[ILP] M. Izumi, R. Longo, S. Popa, *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras*, preprint (1996).

[L] R. Longo, *Simple injective subfactors*, Adv. Math. **63** (1987), 152-171.

[NT] N. Nakamura, Z. Takeda, *A Galois theory for finite factors*, Proc. Japan Acad. **36** (1960), 258-260.

[NTa] Y. Nakagami, M. Takesaki, *Duality for crossed product of von Neumann algebras*, Lecture Notes in Math., vol. 731, Springer-Verlag, Berlin-Hidelberg-New York, 1978.