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作用素不等式とσ-normal C^*-環について
(On operator inequalities and σ-normal C^*-algebras)

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Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the Banach $\star$-algebra of all bounded linear operators on $\mathcal{H}$.

A C^*-algebra $\mathcal{A}$ is a $\star$-subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed under the norm topology. *Gelfand*, *Naimark* and *Kaplansky* gave an elegant axiomatic characterization of C^*-algebras in the following form:

Let $\mathcal{A}$ be any Banach algebra with an involution. If the norm on $\mathcal{A}$ satisfies that

$$||x^*x|| = ||x||^2$$

for each $x \in \mathcal{A}$, then there exist a Hilbert space $\mathcal{H}$ and a $\star$-isomorphism from $\mathcal{A}$ onto a C^*-algebra on $\mathcal{H}$.

A von Neumann algebra $\mathcal{A}$ is a $\star$-subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed under the weak operator topology and contains the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$. Then, the following theorem is well-known:

Theorem A Let $\{A_\alpha\}$ be any increasing net of hermitian operators in $\mathcal{A}$ majorized by an hermitian operator $B \in \mathcal{A}$. Then there exists a strong limit $A \in \mathcal{A}$ of $\{A_\alpha\}$ in such a way that

$$A = \sup_\alpha A_\alpha$$

in the ordered space $\mathcal{A}_h$ of all hermitian elements in $\mathcal{A}$.

Moreover, for any given $A \in \mathcal{A}$, put $P = P_{\overline{range(A^*)}}$, where $P_{\mathcal{M}}$ is the orthogonal projection onto the subspace $\mathcal{M}$ and we have that

$$((A^*A)^{\frac{1}{2}} + \frac{1}{n}1_{\mathcal{H}})^{-1}(A^*A)^{\frac{1}{2}} \nearrow P \quad (n \to \infty)$$

and for each n,

$$((A^*A)^{\frac{1}{2}} + \frac{1}{n}1_{\mathcal{H}})^{-1}(A^*A)^{\frac{1}{2}} \in \mathcal{A}.$$

Hence, it follows that $P \in \mathcal{A}$ and $PA^* = A^*$. This P can be characterized as follows:

P is the smallest projection $E \in \mathcal{A}$ that satisfies $EA^* = A^*$.

It follows also that for each $A \in \mathcal{A}$, denoting the right annihilator of $\{A\}$ by $\mathcal{R}a(\{A\})$,

$$\mathcal{R}a(\{A\}) = (1_{\mathcal{H}} - P)\mathcal{A}.$$

In his book [3], S.K. Berberian says the following: "**Von Neumann algebras are blessed with an excess of structures–algebraic, geometric, topological–so much,**

that one can easily obscure, through proof by overkill, what makes a particular theorem work.”

In connection with this philosohpy, along lines with an abstact treatment of the theory of von Neumann algebras not directly concerned with the representation of the algebras on Hilbert spaces, C.E. Rickart introduced a concept of Bp^*-algebras, which is now called **Rickart C^*-algebras** as follows:

Definition(see [3] for details) A Rickart C^*-algebra is a C^*-algebra $\mathcal{A}$ such that, for each $x \in \mathcal{A}$,

$$\mathcal{R}a(\{x\}) : (= \{y \in \mathcal{A} \mid xy = 0\}) = g\mathcal{A}$$

with g a projection in $\mathcal{A}$.

Remark Such a projection g is uniquely determined. It follows also that

$$\mathcal{L}a(\{x\}) : (= \{y \in \mathcal{A} \mid yx = 0\}) = \mathcal{R}a(\{x^*\})^* = (h\mathcal{A})^* = \mathcal{A}h$$

for a suitable projection $h \in \mathcal{A}$.

By the preceding argument just before the Definition, it follows that every von Neumann algebra is a Rikart C^*-algebra, however, the converse is not true in general as the following examples show:

(1) Let X be a perfect locally compact separable metric space and let $\mathcal{B}(X)$ be the C^*-algebra of all complex-valued bounded Borel functions on X. Then, $\mathcal{B}(X)$ is a Rickart C^*-algebra which is not a von Neumann algebra.

(2) Let $\mathcal{A}$ be a unital non-atomic C^*-algebra and let $\mathcal{B}(\mathcal{A})$ be the Borel envelope of $\mathcal{A}$, then $\mathcal{B}(\mathcal{A})$ is a Rickart C^*-algebra which is not a von Neumann algebra.

(3) Let $\mathcal{A}$ be a unital C^*-algebra and let $\hat{\mathcal{A}}$ be the regular σ-completion of $\mathcal{A}$. Then $\hat{\mathcal{A}}$ is a **singular** Rickart C^*-algebra.

Rickart C^*-algebras have very nice order properties, in particular, they have plenty of projections in such a way that **the set $Proj(\mathcal{A})$ of all projections in a Rickart C^*-algebra $\mathcal{A}$ is a σ-complete lattice with respect to the natural order of projections** (see Lemma 4).

A C^*-algebra $\mathcal{A}$ is said to be **monotone** (resp. **monotone σ-) complete**, if every increasing net (resp. sequence) of elements in the ordered space A of all hermitian elements of $\mathcal{A}$ has a supremum in A. It is straightforward to verify that every monotone complete C^*-algebra is an AW^*-algebra. For type I AW^*-algebras, the converse is known to be true, however, for a general AW^*-algebras, this question is still open, although an impressive attack on the problem was made by E. Christensen and G.K.Pedersen who showed that properly infinite AW^*-algebras are monotone σ-complete [4].

A parallel question in Rickart C^*-algebra theory is whether or not every Rickart C^*-algebra is mnotone σ-complete.

A C^*-algebra $\mathcal{A}$ is said to be **σ-normal** (resp. **normal**) if every increasing sequence (resp. net) of projections in $\mathcal{A}$ has a supremum in A.

P. Ara and D. Goldstein recently showed in [2] that every Rickart C^*-algebra is σ-normal. Their proof uses an ingenious, deep analysis of the structure of Rickart C^*-algebras and regular rings, together with a result by Christensen and Pedersen, which says that every properly infinite Rickart C^*-algebra is monotone σ-complete.

In this talk, I would like to give you a proof of a theorem (see, below, Theorem) which is a nominally more general result on the one hand and a simple alternative proof of the σ-normality of Rickart C^*-algebras on the other (see, for details, [8]). In fact, in the course of my proof, **under the assumption that Rickart C^*-algebras are UMF-algebras**(see Lemma 6), I use the following simple properties on C^*-algebras only:

Let $\mathcal{A}$ be a unital C^*-algebra. Then $||x||^2 = ||x^*x||$ for any $x \in \mathcal{A}$. If $a \leq b$ in A, then $x^*ax \leq x^*bx$ for every $x \in \mathcal{A}$.

The main part of this talk will be published in the Bulletin of the London Mathematical Society.

Throughout this note, all C^*-algebras are unital. Let $\mathcal{A}$ be a C^*-algebra and let $Proj(\mathcal{A})$ be the ordered subset (of A) of all projections in $\mathcal{A}$.

A C^*-algebra $\mathcal{A}$ is said to be a **UMF-algebra (uniform majorization-factorization algebra)**, if, for any pair a, $b \in \mathcal{A}$, whenever $a^*a \leq b^*b$, there exists $c \in \mathcal{A}$ such that $||c|| \leq 1$ and $a = cb$ (see for details, [1]). Note that Rickart C^*-algebras (and so von Neumann algebras) are UMF-algebras(see Lemma 6). Concerning another non-trivial ones, for examples, we note that a corona C^*-algebra $M(\mathcal{A})/\mathcal{A}$ is a UMF-algebra for any separable non-unital C^*-algebra $\mathcal{A}$, where $\mathcal{M}(\mathcal{A})$ is the multiplier algebra of $\mathcal{A}$. In particular, the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and ℓ^∞/c_0 are UMF-algebras.

Theorem *Let $\mathcal{A}$ be a UMF-C^*-algebra. Let $\{e_n\}$ be an increasing sequence of elements in $Proj(\mathcal{A})$ and let e_0 be in $Proj(\mathcal{A})$. Suppose that whenever $x \in \mathcal{A}$ satisfies $xe_n = 0$ for all n, $xe_0 = 0$. Then, for any $a \in A$, $e_n \leq a$ for all n implies that $e_0 \leq a$.*

A Proof of Theorem.

Suppose that $a \in A$ satisfies that $e_n \leq a$ for all n. Then, for every positive real number ϵ, $a + \epsilon \geq e_n$ for all n. Thus we have that

$$(a+\epsilon)^{-\frac{1}{2}}e_n(a+\epsilon)^{-\frac{1}{2}} \leq 1$$

for all n and hence we get that

$$||(a+\epsilon)^{-\frac{1}{2}}e_n|| = ||(a+\epsilon)^{-\frac{1}{2}}e_n(a+\epsilon)^{-\frac{1}{2}}||^{\frac{1}{2}} \leq 1$$

for all n. Define

$$x(k) = \sum_{n=1}^{k} \frac{1}{2^n}(a+\epsilon)^{-\frac{1}{2}}e_n$$

for $k = 1, 2, \cdots$, and it follows that $||x(k) - x(l)|| \to 0$ $(k,\ l \to \infty)$. Since $\mathcal{A}$ is complete by the norm topology, we can find $x \in \mathcal{A}$ such that $||x - x(k)|| \to 0$ $(k \to \infty)$. Formally, we write

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n}(a+\epsilon)^{-\frac{1}{2}}e_n.$$

By the same way, define

$$y(k) = \sum_{n=1}^{k} \frac{1}{2^n}e_n$$

for $k = 1, 2, \cdots$, and we can get an element $y \in \mathcal{A}$ such that

$$y = \sum_{n=1}^{\infty} \frac{1}{2^n}e_n$$

and $||y(k) - y|| \to 0$ $(k \to \infty)$. Since $e_k \geq e_n$ for each $n \leq k$, it follows that

$$x(k) = (a+\epsilon)^{-\frac{1}{2}}e_k y(k)$$

for each k and so we have that $x(k)^*x(k) \leq y(k)^*y(k)$ for each k. If k goes to infinity, then it follows that $x^*x \leq y^*y$. Since $\mathcal{A}$ is a UMF-algebra, one can find $c \in \mathcal{A}$ with $||c|| \leq 1$ such that $x = cy$, that is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n}(a+\epsilon)^{-\frac{1}{2}}e_n = \sum_{n=1}^{\infty} \frac{1}{2^n}ce_n.$$

Multiplying by e_1 on the right in the above equality, we get that

$$\left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right)(a+\epsilon)^{-\frac{1}{2}}e_1 = \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right)ce_1,$$

that is, $(a+\epsilon)^{-\frac{1}{2}}e_1 = ce_1$ and so we have

$$\sum_{n=2}^{\infty} \frac{1}{2^n}(a+\epsilon)^{-\frac{1}{2}}e_n = \sum_{n=2}^{\infty} \frac{1}{2^n}ce_n.$$

Again multiplying by e_2 on the right in the above equality, we have $(a+\epsilon)^{-\frac{1}{2}}e_2 = ce_2$. By repeating these arguments, it follows that $(a+\epsilon)^{-\frac{1}{2}}e_n = ce_n$ for all n. Hence, by our assumption, we get that $(a+\epsilon)^{-\frac{1}{2}}e_0 = ce_0$. Since $||c|| \leq 1$, we have that

$$(a+\epsilon)^{-\frac{1}{2}}e_0(a+\epsilon)^{-\frac{1}{2}} = ce_0c^* \leq 1,$$

and so $e_0 \leq a + \epsilon$ for all ϵ follows, that is, we get that $e_0 \leq a$. This completes the proof.

Corollary ([2]) *Let $\mathcal{A}$ be a* Rickart *C^*-algebra. Then, $\mathcal{A}$ is σ-normal.*

Before going into the proof of Corollary, we shall make a survey of results on Rickart C^*-algebras for the sake of completeness ([1], [2] and [3]).

Lemma 1 (See [3]) *Let $\mathcal{A}$ be a* Rickart *C^*-algebra. Then $\mathcal{A}$ is unital.*

In fact, by our definition, the right annihilator $\mathcal{R}a(\{0\}) = e\mathcal{A}$ for some $e \in Proj\mathcal{A}$. Since $\mathcal{R}a(\{0\}) = \mathcal{A}$, it follows that $\mathcal{A} = e\mathcal{A}$ and so e is the unit for $\mathcal{A}$ because $\mathcal{A}$ is a $*$-algebra.

From now onward, we shall denote this e by 1.

Lemma 2 (See [3]) *Let $\mathcal{A}$ be a* Rickart *C^*-algebra and $x \in \mathcal{A}$. Then, there are a unique pair of projections $e,\ f \in \mathcal{A}$ such that*

(1) $xe = x$,

(2) *for any $y \in \mathcal{A}$, $xy = 0$ if, and only if, $ey = 0$,*

(3) $fx = x$,

and

(4) *for any $y \in \mathcal{A}$, $yx = 0$ if, and only if, $yf = 0$.*

In fact, since $\mathcal{R}a(\{x\}) = (1-e)\mathcal{A}$ and the left annihilator $\mathcal{L}a(\{x\}) = \mathcal{A}(1-f)$, Lemma 2 follows easily.

From now onward, we shall denote e by $RP(x)$ and f by $LP(x)$ respectively.

Lemma 3 (See [3]) *Let $\mathcal{A}$ be a* Rickart *C^*-algebra and suppose that a family of $\{e_i\}$ projections in $\mathcal{A}$ has a supremum e in $Proj(\mathcal{A})$. Then, for any $x \in \mathcal{A}$, whenever $xe_i = 0$ for all i, we have $xe = 0$.*

In fact, if $xe_i = 0$ for all i, then $e_i \leq 1 - RP(x)$ for all i and so we have $e \leq 1 - RP(x)$ by our definition, that is, $xe = 0$ follows.

For any pair $e,\ f$ of projections in $\mathcal{A}$, put $e \vee f = f + RP(e(1-f))$ and put $e \wedge f = e - LP(e(1-f))$. Then by these operations, $Proj(\mathcal{A})$ becomes a lattice with respect to the natural order.

Lemma 4 (See [3]) *Let $\mathcal{A}$ be a* Rickart *C^*-algebra. Then, the lattice $Proj(\mathcal{A})$ is σ-complete.*

First of all, we shall show that if every orthogonal sequence of projections in $\mathcal{A}$ has a supremum in $Proj(\mathcal{A})$, then every sequence of projections has a supremum in $Proj(\mathcal{A})$. In fact, let $\{e_n\}$ be any given sequence of projections. Put $f_1 = e_1$ and put

$$f_n = e_1 \vee e_2 \vee \cdots \vee e_n - (e_1 \vee e_2 \vee \cdots \vee e_{n-1}) \ (n \geq 2).$$

Then $\{f_n\}$ is orthogonal. Let $f_0 = \bigvee_{n=1}^{\infty} f_n$. We shall show that $f_0 = \bigvee_{n=1}^{\infty} e_n$. Note that $f_0 \geq e_1$. Suppose that $f_0 \geq e_i \ i = 1, 2, \cdots, n$. Then note that $f_0 \geq e_\vee e_2 \vee \cdots \vee e_{n+1} - (e_1 \vee e_2 \vee \cdots \vee e_n)$ and that $f_0 \geq e_1 \vee e_2 \vee \cdots \vee e_n$. Hence, $f_0 \geq e_1 \vee e_2 \vee \cdots \vee e_{n+1}$ follows. So we are done.

Next let $\{p_n\}$ be any orthogonal family of projections in $\mathcal{A}$. Define as before

$$x_n = \sum_{k=1}^{n} \frac{1}{2^k} p_k \ (n = 1, 2, \cdots)$$

and we have $||x_m - x_n|| \to 0$ as $m, \ n \to \infty$ as before and so there is $x_0 \in \mathcal{A}$ such that $||x_n - x_0|| \to 0$ as $n \to \infty$. Let $e = RP(x_0)$. If $f \in Proj(\mathcal{A})$ satisfies that $p_n \leq f$ for all n, then $x_n f = x_n$ for all n and so we have that $xf = x$, that is, $e \leq f$ follows. It remains to show that $p_n \leq e$ for all n. For any fixed index m, $p_m x_n = 2^{-m} p_m$ for all $n \geq m$. Therefore, $e_m x_0 = 2^{-m} p_m$ for all m, that is, $p_m = 2^m e_m x_0$ for all m. Since $x_0 e = x_0$, it follows that $p_m e = p_m$ for all m. So we are done.

A C^*-algebra $\mathcal{B}$ has *polar decomposition property* (PD) if, for each $x \in \mathcal{B}$, there exists a unique projection $RP(x) \in \mathcal{B}$ which satisfies properties (1) and (2) in Lemma 2 and there exists a partial isometry $w \in \mathcal{B}$ such that $x = w|x|$ and $w^*w = RP(|x|)$.

Lemma 5 ([6]) *Let $\mathcal{B}$ be a unital C^*-algebra such that $\mathcal{B}$ and the 2×2 matrix algebra $\mathcal{M}_2(\mathcal{B})$ over $\mathcal{B}$ have* (PD). *Then $\mathcal{B}$ is a UMF-C^*-algebra.*

Take any $a, \ b \in \mathcal{B}$ with $a^*a \leq b^*b$. Let

$$x = \begin{pmatrix} a & 0 \\ (b^*b - a^*a)^{\frac{1}{2}} & 0 \end{pmatrix}.$$

Then $x \in \mathcal{M}_2(\mathcal{B})$. Let $x = w|x|$ be a polar decomposition of x in $\mathcal{M}_2(\mathcal{B})$. Note that

$$x^*x = \begin{pmatrix} a^* & (b^*b - a^*a)^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ (b^*b - a^*a)^{\frac{1}{2}} & 0 \end{pmatrix} = \begin{pmatrix} b^*b & 0 \\ 0 & 0 \end{pmatrix}$$

and hence it follows that

$$|x| = \begin{pmatrix} |b| & 0 \\ 0 & 0 \end{pmatrix}.$$

Put

$$w = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}.$$

Then

$$w^*w = \begin{pmatrix} w_1^*w_1 + w_3^*w_3 & w_1^*w_2 + w_3^*w_4 \\ w_2^*w_1 + w_4^*w_3 & w_2^*w_2 + w_4^*w_4 \end{pmatrix} = \begin{pmatrix} RP(|b|) & 0 \\ 0 & 0 \end{pmatrix} = RP(|x|).$$

Hence we have that

$$w_2^*w_2 + w_4^*w_4 = 0$$

and so $w_2 = w_4 = 0$ follows, which impleis that

$$w = \begin{pmatrix} w_1 & 0 \\ w_3 & 0 \end{pmatrix}.$$

Since $w_1^*w_1 + w_3^*w_3$ is a projection, we get that $||w|| \leq 1$. Hence it follows that

$$\begin{pmatrix} a & 0 \\ (b^*b - a^*a)^{\frac{1}{2}} & 0 \end{pmatrix} = \begin{pmatrix} w_1 & 0 \\ w_3 & 0 \end{pmatrix} \begin{pmatrix} |b| & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} w_1|b| & 0 \\ w_3|b| & 0 \end{pmatrix},$$

that is, $a = w_1|b| = w_1u^*b$, where $b = u|b|$ is a polar decomposition of b in $\mathcal{B}$. Put $c = w_1u^*$ and we get the desired factorization $a = cb$ and $||c|| \leq 1$. Hence $\mathcal{B}$ is a UMF-C^*-algebra.

By an ingenious, deep analysis of the structure of Rickart C^*-algebras and regular rings, Ara and Goldstein showed in [1] that for every Rickart C^*-algebra $\mathcal{A}$, $\mathcal{M}_n(\mathcal{A})$ is a Rickart C^*-algebra with (PD) for each n. Hence we have the following:

Lemma 6 ([1]) *Every* Rickart *C^*-algebra is a UMF-C^*-algebra.*

A Proof of Corollary

In fact, let $\{e_n\}$ be any increasing sequence in $Proj(\mathcal{A})$. Then it has a supremum e_0 in $Proj(\mathcal{A})$ in such a way that for any $x \in \mathcal{A}$, $e_nx = 0$ for all n implies that $xe_0 = 0$ by Lemma 3 and Lemma 4. Since $\mathcal{A}$ is a UMF-algebra by Lemma 6, e_0 is just the supremum of $\{e_n\}$ in A by the theorem.

Remark. A parallel problem in AW^*-algebra theory was considered by J.D.M. Wright and the author [10], [7] and [9] (see also [5]). The above corollary gives us also a simple alternative proof of the normality of σ-finite AW^*-algebras (see for details, [10], [5], [7] and [9]).

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