数理解析研究所講究録
979巻 1997年 57-74

非可換L^2-空間における完全正値写像について

岩手大人社　三浦　康秀 (Yasuhide MIURA)

0. Introduction

In the theory of operator algebras a notion of a selfdual cone is highly instrumental in studying a non-commutative order in a Hilbert space. Many authors have studied the problem how an algebraic structure of a von Neumann algebra is determined by the underlying Hilbert space. In **[C]** A. Connes introduced the orientation in a facially homogeneous selfdual cone and constructed a von Neumann algebra related to the selfdual cone. B. Iochum **[I2]** studied the (not necessarily orientable) homogeneous selfdual cones and showed the relationship between these cones and the Jordan Banach algebras. It is important to investigate a positive map on a selfdual cone and we have many results of the positive map (for example **[Y1]**, **[Y2]**, **[I3]**). A geometric interpretation was given by B. Iochum **[I1]** to an algebraic notion of a conditional expectation of a von Neumann algebra by using an orientation property in a selfdual cone.

On the other hand, L. M. Schmitt and G. Wittstock **[SW]** characterized a matrix ordered standard form of a von Neumann algebra by using a projection property in the family of selfdual cones instead of orientation. Matrix ordered spaces were first introduced by M. D. Choi and E. G. Effros [**CE**] as the appropriate objects to which completely positive maps apply and enabled us to handle non-commutative order. The author [**M1**] considered the relationship between a completely positive projection on $L^2(M)$ and a normal conditional expectation on a σ-finite von Neumann algebra M.

The purpose of this note is to consider the relationship between the completely positive maps—-especially completely positive projections and completely positive isometries—-on $L^2(M)$ and the corresponding maps on M. In Section 2 we deal with the completely positive projections on a matrix ordered standard form of a (not necessarily σ-finite) von Neumann algebra and show that each of a completely positive projection and a conditional expectation induces the other. In Section 3 we deal with the completely positive isometries on the matrix ordered standard form and investigate the relationship between those maps and isomorphisms of von Neumann algebras.

We shall use the lecture note of Takesaki [**T2**] as references of the standard results of the modular theory of operator algebras. We shall also use the notation as introduced in [**SW**] for matrix ordered standard forms.

1. Preliminaries

We begin with some basic definitions and results concerning matrix ordered Hilbert spaces. For details and proofs we refer to [**SW**]. Let $M_{n,m}$ and M_n be the spaces of all complex $n \times m$ and $n \times n$ matrices respectively. We write st:$\alpha \mapsto \alpha^*$ for the natural involution on $M_{n,m}$. Let H be a complex Hilbert space. We write $H_n = H \otimes M_n (= M_n(H))$ for the tensor product of the Hilbert spaces. Let H^+ be a selfdual cone in H. For any natural number n, we denote a selfdual cone in H_n by H_n^+. We call $(H, H_n^+, n \in \mathbb{N})$ a matrix ordered Hilbert space if $\alpha \in M_{n,m}$ then $\alpha H_m^+ \alpha^* \subset H_n^+$. Let $J = J_{H^+}$ be the induced involution on H. We then have a natural involution

$$J_{n,m} = J \otimes \mathrm{st} : H \otimes M_{n,m} \to H \otimes H_{m,n}$$

defined by $[\xi_{i,j}] \mapsto [J\xi_{j,i}]$ and we write J_n for $J_{n,n}$. If $(H, H_n^+, n \in \mathbb{N})$ a matrix ordered Hilbert space, then $J_n = J_{H_n^+}$.

Let $(H^{(1)}, H_n^{(1)+}, n \in \mathbb{N})$ and $(H^{(2)}, H_n^{(2)+}$ be matrix ordered Hilbert spaces. A linear map ρ of $H^{(1)}$ into $H^{(2)}$ is said to be n-positive, if $\rho_n = \rho \otimes 1_n$ maps $H_n^{(1)+}$ into $H_n^{(2)+}$, where 1_n denotes the identity on the $n \times n$ matrices M_n. If ρ is n-positive for all $n \in \mathbb{N}$, then ρ is said to be completely positive.

Let (M, H, J, H^+) be a standard form of a von Neumann algebra. Let $H_n^+ (H_1^+ = H^+, n \in \mathbb{N})$ be a family of selfdual cones in H_n. We call $(M, H, H_n^+, n \in \mathbb{N})$ a matrix ordered standard form, if for every $a \in M \otimes M_{n,m}$

$$aJ_{n,m}aJ_m(H_m^+) \subset H_n^+$$

holds. Let φ be a faithful normal semi-finite weight on M, and (π_φ, H_φ) be a GNS-representation of M by φ. Put

$$(H_\varphi)_n^+ = \overline{\text{co}}\{[\pi_\varphi(a_i)J_\varphi\pi_\varphi(a_j)J_\varphi\xi]_{i,j=1}^n | a_1, \cdots, a_n \in M, \xi \in H_\varphi^+\}.$$

Then $(\pi_\varphi(M), H_\varphi, (H_\varphi)_n^+, n \in \mathbb{N})$ is a matrix ordered standard form. Conversely, let $(H, H_n^+, n \in \mathbb{N})$ be a matrix ordered Hilbert space. Put

$$\mathcal{M} = \{x \in B(H) | \{\text{diag}(x, 1, \cdots, 1)\Xi\text{diag}(x, 1, \cdots, 1)^J\} \in H_n^+$$
$$\text{for every } \Xi \in H_n^+ \text{ and all } n \in \mathbb{N}\},$$

where $\text{diag}(x_1, x_2, \cdots, x_n)$ denotes the $n \times n$ matrix with entries $a_{ij} = \delta_{ij}x_i (x_i \in B(H))$ and $\{x\xi y^J\} = \frac{1}{2}(xJyJ\xi + JyJx\xi)$. It is shown that if the completed face $(F_{\{\xi\}})^{\perp\perp}$ generated by $\xi \in H_n^+$ is projectable for all $\xi \in H_n^+, n \in \mathbb{N}$, then $(\mathcal{M}, H, H_n^+, n \in \mathbb{N})$ is a matrix ordered standard form.

2. Completely positive projections

We shall first show that a conditional expectation induces a completely positive projection. To prove it, we need a lemma.

Lemma 2.1. *Let M and φ be as in Section 1. Then there exists a completely positive isometry u of H onto H_φ.*

Proof. By [**H2**, Theorem 2.3] there exists an isometry u of H onto H_φ such that

$$\pi_\varphi(x) = uxu^{-1} \ (\forall x \in M), \ J_\varphi = uJu^{-1}, \ {H_\varphi}^+ = uH^+.$$

If $[\xi_{ij}]_{i,j=1}^n \in {H_n}^+(\xi_{ij} \in H)$, then for any $x_1, \cdots, x_n \in M$and $\zeta \in {H_\varphi}^+$ we have

$$\begin{aligned}(u_n[\xi_{ij}], [\pi_\varphi(x_i)J_\varphi\pi_\varphi(x_j)J_\varphi\zeta]) &= \sum_{i,j=1}^n (\pi_\varphi({x_i}^*)J_\varphi\pi_\varphi({x_j}^*)J_\varphi u\xi_{ij}, \zeta) \\ &= \sum_{i,j=1}^n (u{x_i}^*J{x_j}^*J\xi_{ij}, \zeta) \\ &= (u[{x_1}^*, \cdots, {x_n}^*]J_{1,n}[{x_1}^*, \cdots, {x_n}^*]J_n[\xi_{ij}], \zeta) \\ &\geq 0.\end{aligned}$$

It follows that $u_n{H_n}^+ \subset (H_\varphi)_n^+$. □

Proposition 2.2. *Let $(M, H, {H_n}^+, n \in \mathbb{N})$ be a matrix ordered standard form of a von Neumann algebra M, and L be a von Neumann subalgebra of M. If ε is a normal conditional expectation of M onto L with respect to a faithful normal semi-finite weight φ on M, then there exists a completely positive projection e on H satisfying the following conditions:*

i) $L = M \cap \{e\}'$.

ii) $(L|eH, eH, e_nH_n{}^+, n \in \mathbb{N})$ *is a matrix ordered standard form.*

iii) eH^+ *is a separating set for* M.

Proof. By Lemma 2.1 we may consider $(M, H, H_n{}^+, n \in \mathbb{N})$ as $(\pi_\varphi(M), H_\varphi, (H_\varphi)_n$ $\mathbb{N})$. Let e be a projection on H_φ defined by

$$e\eta_\varphi(x) = \eta_\varphi(\varepsilon(x)), x \in \mathfrak{A}_\varphi.$$

It suffices by [**T2**, Theorem] to prove iii). Choose an arbitrary element x in M. Suppose that $\pi_\varphi(x)\xi = 0$ for all $\xi \in e\mathfrak{A}_\varphi \subset \mathfrak{A}_\varphi$. For every $\eta \in \mathfrak{A}'_\varphi$ we have

$$\pi_\varphi(x)\pi(\xi)\eta = \pi_\varphi(x)\pi'(\eta)\xi = \pi'(\eta)\pi_\varphi(x)\xi = 0.$$

Let $\{y_i\}$ be a net in $\pi(\mathfrak{A}_\varphi)$ which converges strongly to 1. Since ε is normal, $\varepsilon(y_i) \to \varepsilon(1) = 1$. This implies the existence of a net in $\pi(e\mathfrak{A}_\varphi)$ converging strongly to 1. Hence $x = 0$. It follows that eH_φ is a separating set for $\pi_\varphi(M)$. This means that eH_φ is a cyclic set for $\pi_\varphi(M)' = J_\varphi\pi_\varphi(M)J_\varphi$. Since $eJ_\varphi = J_\varphi e$ and the span of $eH_\varphi{}^+$ is eH_φ, iii) holds. □

We shall next consider the converse of the above proposition. We need two lemmata.

Lemma 2.3. *Suppose that* $(M, H, H_n^+, n \in \mathbb{N})$ *is a matrix ordered standard form of a von Neumann algebra* M. *If* e *is a completely positive projection on* H*,then there exists a von Neumann algebra* N *such that* $(N, eH, e_n H_n^+, n \in \mathbb{N})$ *is a matrix ordered standard form.*

Proof. One easily sees that $(eH, e_n H_n^+, n \in \mathbb{N})$ is a matrix ordered Hilbert space. By [**I2**, Proposition II.1.6, Proposition II.1.3 i)] $e_n H_n^+$ is regular. Therefore, the completed face $(F_{\{\xi\}})^{\perp\perp}$ generated by ξ is projectable for every $\xi \in e_n H_n^+, n \in \mathbb{N}$. There then exists the von Neumann algebra N by [**SW**, Theorem 4.3]. □

Lemma 2.4. *Let* $(M, H, H_n{}^+, n \in \mathbb{N})$ *be a matrix ordered standard form of a von Neumann algebra* M, *and* e *be a 2-positive projection on* H *such that* eH^+ *is a separating set for* M. *Assume that* (N, eH, J_{eH^+}, eH^+) *and* $(M_2(N), e_2H_2, J_{e_2H_2^+}, e_2H_2{}^+)$ *are standard forms of von Neumann algebras* N *and* $M_2(N)$, *respectively. If we put* $L = M \cap \{e\}'$, *then* $L|eH = eM|eH = N$. *Furthermore, there exists an orthogonal system* $\{\xi_i; i \in \mathbf{I}\}$ *in* eH^+ *such that* φ *and* $\varphi|L$ *defined by* $\varphi(a) = \sum_{i\in\mathbf{I}} \omega_{\xi_i}(a)$ $(a \in M^+)$ *are faithful normal semi-finite weights on* M *and* L, *respectively.*

Proof. The first part of this proof is due to [**M1**, Lemma 2]. We put $K = eH, K^+ = eH^+, K_2 = e_2H_2^+$ and $K_2^+ = e_2H_2^+$. By assumption, one easily sees that $eJ|K = J_{K^+}, e_2J_2|K_2 = J_{K_2^+}$. Take a derivation $\delta \in D(H_2^+)$. By [**C**, Lemma 5.3] $e_2\delta_2$ belongs to $D(K_2^+)$. Since

$(M_2(M), H_2, J_2, H_2^+)$ is a standard form, for each $X = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \in$ $M_2(M)$ there exists by [**I2**, Theorem VI.1.2 ii)] $Y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} \in$ $M_2(N)$ satisfying

$$e_2(X + J_2XJ_2)\Xi = (Y + J_{K_2^+}YJ_{K_2^+})\Xi\ , \quad \forall \Xi \in K_2.$$

By setting $\Xi = \begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix}$ with $\xi \in K$ we have

$$\begin{bmatrix} 0 & ex\xi \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Jy_2J\xi & (y_1 + Jy_4J)\xi \\ 0 & y_3\xi \end{bmatrix},$$

so that $y_2 = y_3 = 0$ and $ex\xi = y_1\xi + Jy_4J\xi$. Moreover, if we set $\Xi = \begin{bmatrix} 0 & 0 \\ 0 & \xi \end{bmatrix}$ with $\xi \in K$ then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (y_4 + Jy_4J)\xi \end{bmatrix}.$$

It follows that $ex\xi = (y_1 - y_4)\xi, \xi \in K$. Hence $eM|K \subset N$.

We shall next prove that $N \subset L|K$. Note that in a standard form (M, H, J, H^+) the map $q \mapsto qJqJH^+$ is an order isomorphism of the set of all projections in M onto the set of all closed faces in H^+ (see [**SW**, Proposition 3.4], [**I2**, Corollary VI.2.3]). Hence, if p is a projection in N, then $\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} J_{K_2^+} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} J_{K_2^+}K_2^+$, which will denoted by F, is a closed face in K_2^+ and $P_F = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} J_{K_2^+} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} J_{K_2^+}$. There then exists a projection $P = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$ in $M_2(M)$ such that $P_{<F>} = PJ_2PJ_2$, where $P_{<F>}$ denotes a projection on the closed linear span of the face

$< F >$ generated by F in H_2^+. It follows from [**I2**, Lemma II.1.7] that $P_F\Xi = e_2 P_{<F>}\Xi$ for all $\Xi \in K_2$. By setting $\Xi = \begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix}$ we have $p\xi = eaJcJ\xi$ for all $\xi \in K$. On the other hand, since $e_2 P_F e_2 \leq P_{<F>}$, we have for all $\xi \in K$

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & \xi \end{bmatrix} &= \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} J_{K_2^+} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} J_{K_2^+} \begin{bmatrix} 0 & 0 \\ 0 & \xi \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} J_2 \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} J_2 \begin{bmatrix} 0 & 0 \\ 0 & \xi \end{bmatrix} \\ &= \begin{bmatrix} bJbJ\xi & bJcJ\xi \\ cJbJ\xi & cJcJ\xi \end{bmatrix}. \end{aligned}$$

We then have $b\xi = 0$ by [**SW**, Corollary 3.3]. It follows that $b = 0$ because K_2 is a separating set for M. Since $\xi = cJcJ\xi = c\xi$, we have $c = 1$. Therefore, $p\xi = ea\xi$ for all $\xi \in K$. Since $e_2 P_{<F>} = P_{<F>} e_2$ by [**I2**, Lemma II.1.7], i.e.,

$$\begin{bmatrix} eaJaJ & ea \\ eJaJ & e \end{bmatrix} = \begin{bmatrix} aJaJe & ae \\ JaJe & e \end{bmatrix},$$

we have $ea = ae$. Therefore, $L|K = eM|K = N$.

Recall that for $\xi, \eta \in H^+$, $\xi \perp \eta$ if and only if $p(\xi) \perp p(\eta)$, where $p(\xi)$ denotes the support projection of a vector functional ω_ξ on M. By Zorn's lemma there exists a maximal family $\{\xi_i : i \in \mathbf{I}\} \subset eH^+$ such that $\{p(\xi_i)\}$ is mutually orthogonal. By maximality we have $\sum_{i\in\mathbf{I}} p(\xi_i) = 1$. Then φ is a faithful normal semi-finite weight on M. In fact, for any finite subset $\mathbf{J}$ of $\mathbf{I}$ we put

$$\varphi_{\mathbf{J}}(a) = \sum_{i\in\mathbf{J}} \omega_{\xi_i}(a), a \in M^+.$$

Then $\varphi(a) = \lim_{\mathbf{J}} \varphi_{\mathbf{J}}(a), a \in M^+$, so that φ is a normal weight on M because $\{\varphi_{\mathbf{J}}\}$ is monotone increasing. Put $e_{\mathbf{J}} = \mathrm{supp}(\varphi_{\mathbf{J}}) = \sum_{i \in \mathbf{J}} p(\xi_i)$. Then $\varphi(a) = \varphi_{\mathbf{J}}(a), a \in e_{\mathbf{J}} M^+ e_{\mathbf{J}}$. Hence φ is semi-finite. If $\varphi(a) = 0, a \geq 0$, then $\omega_{\xi_i}(a) = 0$ for all $i \in \mathbf{I}$. This implies $a^{1/2} p(\xi_i) = 0$. Since $\sum_{i \in \mathbf{I}} p(\xi_i) = 1$, we have $a = 0$. Thus φ is faithful.

We shall next show that $\varphi|L$ is a faithful normal semi-finite weight on L. Put $\varphi_0(x^\circ) = \sum_{i \in \mathbf{I}} \omega_{\xi_i}(x^\circ), x^\circ \in N^+$. Since

$$\sum_{i \in \mathbf{I}} N'\xi_i = \sum_{i \in \mathbf{I}} J_{eH^+} N \xi_i = e(\sum_{i \in \mathbf{I}} JM\xi_i) = e(\sum_{i \in \mathbf{I}} M'\xi_i) = 1_{eH},$$

φ_0 is a faithful normal semi-finite weight on N. Since eH is a separating set for M, the map $x \in L \mapsto x|eH \in N$ is an onto $*$-isomorphism. Using the equality $\varphi(x) = \varphi_0(x|eH), x \in L$, we see that the set $\{x \in L | \varphi(x) < \infty\}$ is strongly dense in L. This completes the proof. □

Theorem 2.5. 1) *Let $(M, H, H_n{}^+, n \in \mathbb{N})$ be a matrix ordered standard form of a von Neumann algebra M, and e be a projection on H with $eH = K$ such that eH^+ is a separating set for M. Then the following three conditions are equivalent:*

i) *e is completely positive.*

ii) *For every $n \in \mathbb{N}$, $e_n H_n^+$ is a selfdual cone in K_n and $e_n H_n^+ = H_n^+ \cap K_n$.*

iii) *e is 2-positive and there exists a family of selfdual cones $K_n{}^+$ in K_n with $K^+ = eH^+$ and $K_2{}^+ = e_2 H_2{}^+$ such that $(K, K_n{}^+, n \in \mathbb{N})$*

is a matrix ordered Hilbert space and any completed face $(F_{\{\xi\}})^{\perp\perp}$ *in* ${K_n}^+$ *is projectable for every* $\xi \in {K_n}^+, n \in \mathbb{N}$.

2) *Under the condition* 1), *if* $L = M \cap \{e\}'$, *then* $(L|eH, eH, e{H_n}^+, n \in \mathbb{N})$ *is a matrix ordered standard form. In addition, there exists a faithful normal conditional expectation* ε *with respect to the faithful normal semifinite weight* φ *on* M *as defined in Lemma 2.4. Furthermore, we have* $L|eH = eM|eH$.

Proof. 1) i)$\Leftrightarrow$ ii): If $e_n H_n^+ \subset H_n^+$, then $H_n^+ \cap K_n$ is a selfdual cone in K_n. In fact, let $\xi \in K_n$ belong to a dual cone of $H_n^+ \cap K_n$, then $(\xi, \eta) = (\xi, e_n \eta) \geq 0$ for all $\eta \in H_n^+$. Hence $\xi \in H_n^+ \cap K_n$. Since $H_n^+ \cap K_n \subset e_n H_n^+$ and each cone of both sides is selfdual, they are equal. ii)$\Rightarrow$ i) is trivial.

i)$\Rightarrow$ iii): We apply Lemma 3.3,

2) Let M, K and e as in assumption of 1), and let iii) hold. By [**SW**, Theorem 4.3] there exists a von Neumann algebra N such that $(N, K, {K_n}^+, n \in \mathbb{N})$ is a matrix ordered standard form. For any x in M there exists uniquely by Lemma 2.4 $\alpha(x)$ in L such that $ex\xi = \alpha(x)\xi$ for all ξ in K, since eH^+ is a separating set for M. We may consider by Lemma 2.1 (M, H, J, H^+) as $(\pi_\varphi(M), H_\varphi, J_\varphi, {H_\varphi}^+)$. Then $\mathfrak{B} = \{\eta_\varphi(x) \in \mathfrak{A} | x \in L\}$ is a left Hilbert subalgebra of $\mathfrak{A}_\varphi$ with completion eH_φ. Let x be an arbitrary element in $\pi(\mathfrak{A}_\varphi)$, and let $\{\xi_i\}$ be a net in

$\mathfrak{B}'$ such that $\{\pi'(\xi_i)\}$ converges strongly to e. For any element ζ in $\mathfrak{A}'$ we have

$$\begin{aligned}(eS\eta_\varphi(x), \zeta) &= (\eta_\varphi(x^*), e\zeta) = \lim_i(\pi'(\xi_i)\eta_\varphi(x^*), e\zeta) \\ &= \lim_i(x^*\xi_i, e\zeta) = \lim_i(\xi_i, \alpha(x)e\zeta) \\ &= \lim_i(\xi_i, \pi'(\zeta)\eta_\varphi(\alpha(x))) = \lim_i(\pi'(F\zeta)\xi_i, \eta_\varphi(\alpha(x))) \\ &= (F\zeta, \eta_\varphi(\alpha(x))),\end{aligned}$$

using the fact that the invariance

$$\varphi(\alpha(x)) = \sum_{i\in\mathbf{I}}(\alpha(x)\xi_i, \xi_i) = \sum_{i\in\mathbf{I}}(x\xi_i, \xi_i) = \varphi(x), \ \ x \in M^+$$

implies

$$\varphi(\alpha(x)^*\alpha(x)) \leq \varphi(x^*x) < \infty, x \in \mathfrak{A}_\varphi.$$

Hence $eS\eta_\varphi(x) = S\eta_\varphi(\alpha(x)),\ x \in \pi(\mathfrak{A}_\varphi)$. In addition, since

$$\eta_\varphi(\alpha(x)) = \lim_i \alpha(x)\xi_i = \lim_i ex\xi_i = e\eta_\varphi(x),$$

it follows that eS coincides with Se on $\mathfrak{A}_\varphi$. Hence $eS = Se$, so that $e\Delta_\varphi = \Delta_\varphi e$, and L is invariant under ${\Delta_\varphi}^{it}(\forall t \in \mathbb{R})$. We see from the theorem of Takesaki [**T2**, Theorem] the existence of the conditional expectation ε.

1) iii)$\Rightarrow$ i): We apply Proposition 2.2. This completes the proof. □

We remark in Theorem 2.5 2) that the conditional expectation ε is uniquely determined under the condition that a faithful normal semi-finite weight φ is represented by the cone eH^+.

3. Completely positive isometries

Let $(M, H, H_n^+, n \in \mathbb{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ be matrix ordered standard forms of von Neumann algebras. Then Lemma 2.1 shows the following fact: If ρ is a $*$-isomorphism of M onto $\hat{M}$, then there exists a completely positive isometry u of H onto $\hat{H}$ such that $\rho(x) = uxu^{-1}, x \in M$.

Theorem 3.1. *Let $(M, H, H_n^+, n \in \mathbb{N})$ be a matrix ordered standard form of a von Neumann algebra, and $(\hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ be a matrix ordered Hilbert space. If u is a completely positive isometry from H onto $\hat{H}$, then there exists a von Neumann algebra $\hat{M}$ of which $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ is a matrix ordered standard form. In addition, we have $uMu^{-1} = \hat{M}$.*

Proof. We shall first show that if G is an arbitrary completed face in $\hat{H}_n^+, n \in \mathbb{N}$, then G is projectable. Since $u_n H_n^+ = \hat{H}_n^+$, G is writen as $G = u_n F$ for some completed face F in H_n^+. By assumption, F is projectable. It follows that $u_n P_F u_n^{-1} \hat{H}_n^+ \subset u_n F$, where P_F denotes a projection of H_n^+ onto the closed linear span $[F]$ of F. Therefore, it suffices to prove that $P_{u_n F} = u_n P_F u_n^{-1}$. Indeed, since $u_n P_F u_n^{-1}$ is a projection and F is a selfdual cone in $[F]$, the above equality holds. There then exists by [**SW**, Theorem 4.3] the von Neumann algebra $\hat{M}$ such that $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ is a matrix ordered standard form.

Choose an element $x \in M$. We then obtain for all $\Xi = \begin{bmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{bmatrix} \in \hat{H}_n^+$

$$\{\mathrm{diag}(uxu^{-1}, 1, \cdots, 1)\Xi\mathrm{diag}(uxu^{-1}, 1, \cdots, 1)^{\hat{J}}\} =$$

$$= \frac{1}{2}\left(\begin{bmatrix} uxu^{-1}\hat{J}uxu^{-1}\hat{J}\xi_{11} & uxu^{-1}\xi_{12} & \ldots & uxu^{-1}\xi_{1n} \\ \hat{J}uxu^{-1}\hat{J}\xi_{21} & \xi_{22} & \ldots & \xi_{2n} \\ \vdots & \vdots & & \vdots \\ \hat{J}uxu^{-1}\hat{J}\xi_{n1} & \xi_{n2} & \ldots & \xi_{nn} \end{bmatrix}\right.$$
$$+ \left.\begin{bmatrix} \hat{J}uxu^{-1}\hat{J}uxu^{-1}\xi_{11} & uxu^{-1}\xi_{12} & \ldots & uxu^{-1}\xi_{1n} \\ \hat{J}uxu^{-1}\hat{J}\xi_{21} & \xi_{22} & \ldots & \xi_{2n} \\ \vdots & \vdots & & \vdots \\ \hat{J}uxu^{-1}\hat{J}\xi_{n1} & \xi_{n2} & \ldots & \xi_{nn} \end{bmatrix}\right)$$
$$= \begin{bmatrix} uxJxJu^{-1}\xi_{11} & uxu^{-1}\xi_{12} & \ldots & uxu^{-1}\xi_{1n} \\ uJxJu^{-1}\xi_{21} & \xi_{22} & \ldots & \xi_{2n} \\ \vdots & \vdots & & \vdots \\ uJxJu^{-1}\xi_{n1} & \xi_{n2} & \ldots & \xi_{nn} \end{bmatrix}$$
$$= u_n\mathrm{diag}(x, 1, \cdots, 1)J_n\mathrm{diag}((x, 1, \cdots, 1)J_n u_n^{-1}\Xi,$$

which belongs to $\hat{H}_n^+$ because u_n is a completely positive map. This implies $uxu^{-1} \in \hat{M}$, i.e., $uMu^{-1} \subset \hat{M}$. Taking the implementation by $\hat{J}$, we obtain the converse inclusion. □

Let $(H, H_n^+, n \in \mathbb{N})$ be a matrix ordered Hilbert space. We shall write $L(H^+)$ for the 1-positive bounded maps on H. Put

$$U(H^+) = \{u \in L(H^+) | u \text{ is a unitary}\}$$

and

$$CPU(H^+) = \{u \in L(H^+)|u \text{ is a completely positive unitary}\}.$$

Moreover, let $(M, H, H_n^+, n \in \mathbb{N})$ be a matrix ordered standard form of a von Neumann algebra. Put

$$CPU^\circ(H^+) = \{uJuJ|u \text{ is a unitary in } M\}.$$

One easily sees that $CPU(H^+)$ is a topological group under the strong operator topology. Since H_n^+ is generated by the elements $[a_iJa_jJ\xi]_{i,j=1}^n(a_1, \cdots, M, \xi \in H^+)$, $uJuJ$ is completely positive. One then sees that $CPU^\circ(H^+) \subset CPU(H^+)$. In the following proposition we shall show that there exists a one-to-one correspondence between $CPU(H^+)$ (resp. $CPU^\circ(H^+)$) and a group of the automorphisms $\mathrm{Aut}(M)$ of M (resp. the inner automorphisms $\mathrm{Int}(M)$).

Proposition 3.2. *Keep the notation above. If we put $\alpha_u(x) = uxu^{-1}, x \in M$, then the map: $u \mapsto \alpha_u$ is a homeomorphism of $CPU(H^+)$ ont* $\mathrm{Aut}(M)$. *In addition, $CPU^\circ(H^+)$ is homeomorphic to* $\mathrm{Int}(M)$.

Before going into the proof of the above proposition, we shall state the following lemma.

Lemma 3.3. *Let (M, H, J, H^+) be a standard form of a von Neumann algebra. If $u \in U(H^+)$ belongs to M or M', ten $u = 1$.*

Proof. By symmetry it suffices to prove in the case $u \in M'$. Take an arbitrary element $\xi \in H$. Then ξ is written as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4)$ such that $\xi_1 \perp \xi_2$ and $\xi_3 \perp \xi_4$, $\xi_i \in H^+$. Since $u\xi = JuJ\xi$, $u = JuJ$. Hence $u \in M \cap M'$ and $u = u^*$. In addition, since $s(\xi_1) \perp s(\xi_2)$ and $s(\xi_3) \perp s(\xi_4)$, where $s(\xi)$ denotes the support projection of a vector functional ω_ξ on M, and $uH^+ = H^+$, we have

$$(u\xi, \xi) = \sum_{i=1}^{4} (u\xi_i, \xi_i) \geq 0.$$

Hence u is a positive operator, and so $u = 1$. □

Proof of Proposition 3.2. If $\alpha_u = \alpha_v$ for $u, v \in CPU(H^+)$, then $v^{-1}ux = xv^{-1}u$ for all $x \in M$, i.e., $v^{-1}u \in M'$. Hence Lemma 3.3 shows that $u = v$. It follows from Theorem 3.1 that $CPU(H^+)$ and $\mathrm{Aut}(M)$ are isomorphic. By [**H2**, Proposition 3.5] $CPU(H^+)$ is homeomorphic to $\mathrm{Aut}(M)$.

It is now clear that $CPU^\circ(H^+)$ is isomorphic to $\mathrm{Int}M)$. □

In the above proposition, if $uJuJ = vJvJ$ for unitaries $u, v \in M$, then $v^*u = Jvu^*J \in M'$. Then there exists a unitary w in the center of M such that $u = vw$.

Proposition 3.4. *Let* $(M, H, H_n^+, n \in \mathbb{N})$ *and* $(N, K, K_n^+, n \in \mathbb{N})$ *are matrix ordered standard forms of von Neumann algebras. If* $H = K$ *and* $H_1^+ = K_1^+$*, then there exists a projection* p *on* H *satisfying the* *following conditions:*

i) $p, 1-p$ *are completely positive.*

ii) $p_n H_n^+ \subset K_n^+$ *and* $(1-p_n)H_n^{+\prime} \subset K_n^+$ *for every* $n \geq 2$*, where* $H_n^{+\prime}$ *denotes the set of all transposed elements of* H_n^+*.*

Proof. By [**H1**, Theorem 5.10] there exists a central projection p in M such that $N = pM + (1-p)M'$. We then have $p = pJpJ$ and $1-p = (1-p)J(1-p)J$, which are completely positive maps. Therefore, i) holds. Since the corresponding family of selfdual cones to the matrix ordered standard form of the commutant M' coincides with $H_n^{+\prime}, n \in \mathbb{N}$, ii) holds. □

References

[**CE**] M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156–209.

[**C**] A. Connes, *Caractérisation des espaces vectoriels ordonnées sous-jacents aux algèbres de von Neumann*, Ann. Inst. Fourier **24** (1974), 121–155.

[**H1**] U. Haagerup, *The standard form of von Neumann algebras*, Thesis, University of Copenhagen, 1973.

[**H2**] ———, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), 271–283.

[**I1**] B. Iochum, *Cônes autopolaires dans les espaces de Hilbert*, Thèse, Univ. de Provence Centre de Saint-Charles, 1975.

[**I2**] ———, *Cônes autopolaires et algèbres de Jordan*, Lecture Notes in Mathematics, 1049, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.

[**I3**] ———, *Positive maps on self-dual cones*, Proc. Amer. Math. Soc. **110** (1990), 755–766.

[M1] Y. Miura, *Completely positive projections on a Hilbert space*, Proc. Amer. Math. Soc. **124** (1996), 2475–2478.

[M2] ———, *On a completely positive projection on a non-commutative L^2-space*, preprint.

[SW] L. M. Schmitt and G. Wittstock, *Characterization of matrix-ordered standard forms of W^*-algebras*, Math. Scand. **51** (1982), 241–260.

[T1] M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Lecture Notes in Mathematics, 128, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[T2] ———, *Conditional expectations in von Neumann algebras*, J. Funct. Anal. **9** (1972), 306–321.

[Y1] S. Yamamuro, *Absolute values in orthogonally decomposable spaces*, Bull. Austral. Math. Soc. **31** (1985), 215–233.

[Y2] ———, *Homomorphisms on an orthogonally decomposable Hilbert space*, Bull. Austral. Math. Soc. **40** (1989), 333–336.

DEPARTMENT OF MATHEMATICS
FACULTY OF HUMANITIES AND SOCIAL SCIENCES
IWATE UNIVERSITY
MORIOKA, 020
JAPAN
E-mail address: ymiura@msv.cc.iwate-u.ac.jp