数理解析研究所講究録
982巻 1997年 196-208

Quantum Stochastic Process as Continuous Flow of Fock Space Operators

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Introduction

During the last decade quantum stochastic calculus on Fock space has developed into a new field of mathematics with various applications to quantum physics by many authors, see the excellent textbooks Meyer [9], Parthasarathy [16] and references cited therein. In short, their discussion is in principle based on a quantum version of Itô theory and a crucial role has been played by three basic quantum stochastic processes $\{A_t\}$, $\{A_t^*\}$ and $\{\Lambda_t\}$, which are called respectively the annihilation process, the creation process and the number (gauge) process. In fact, quantum stochastic integrals of Itô type are introduced by means of Riemann–Lebesgue integrals against the infinitesimal increments dA_t, dA_t^* and $d\Lambda_t$ just as in the classical case where a stochastic integral of Itô type is defined using the infinitesimal increment of Brownian motion dB_t.

On the other hand, the (classical) Itô theory has developed considerably together with distribution theory on an infinite dimensional space. Among others Hida's white noise calculus [2] allows to formulate the white noise as a smooth flow (with respect to the time parameter) not as a generalized stochastic process in the sense of Gelfand and Itô. To be precise, let

$$(E) \subset L^2(E^*, \mu) \subset (E)^*$$

be the standard white noise triplet or Hida–Kubo–Takenaka space [6], where (E^*, μ) is the Gaussian space associated with the Gelfand triple

$$E = \mathcal{S}(\mathbb{R}) \subset H = L^2(\mathbb{R}, dt) \subset E^* = \mathcal{S}'(\mathbb{R}). \tag{0.1}$$

Then the Brownian motion $\{B_t\}$, originally a continuous flow in $L^2(E^*, \mu)$, becomes a smooth one in $(E)^*$ with the derivative

$$W_t = \frac{d}{dt} B_t \qquad \text{or equivalently} \qquad dB_t = W_t dt.$$

Then the smooth flow $t \mapsto W_t \in (E)^*$ is called the white noise process. It is the clue of the white noise approach to (classical) stochastic analysis (see e.g., [2], [3], [6], [7]) that the infinitesimal increment of Brownian motion is replaced with the smooth flow of white noise times the time increment dt.

Through the celebrated Wiener–Itô–Segal isomorphism between $L^2(E^*, \mu)$ and the Boson Fock space over $H_{\mathbb{C}}$ we can discuss a quantum analogue of white noise calculus, that is, white noise approach to quantum stochastic calculus. In fact, the basic processes $\{A_t\}$, $\{A_t^*\}$ and $\{\Lambda_t\}$ are smooth flows of operators on white noise functions and we have

$$dA_t = a_t\, dt, \qquad dA_t^* = a_t^*\, dt, \qquad d\Lambda_t = a_t^* a_t\, dt,$$

where a_t and a_t^* are the pointwisely defined annihilation and creation operators, respectively. Accordingly, quantum stochastic integrals of Itô type introduced by Hudson–Parthasarathy [5] can be discussed in terms of white noise distribution theory on the analogy of the classical case. This new approach was launched out by Huang [4] and Obata [11], [12], [13], [14], [15] with technical background [10].

The present short note is intended as a supplement to the above mentioned papers. We examine the canonical expression of a quantum stochastic process more in detail and discuss the quantum Itô formula. It is our hope that Hida's idea of white noise works well also for quantum stochastic calculus.

General Notation Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be locally convex spaces.

$\mathfrak{X}_{\mathbb{C}}$: the complexification of $\mathfrak{X}$ when it is a real space.

$\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$: the space of continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$; equipped with the topology of bounded convergence.

$\mathfrak{X}^*$: the space of continuous linear functionals on $\mathfrak{X}$; equipped with the strong dual topology. This is a particular case of $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$.

$\mathfrak{X} \otimes \mathfrak{Y}$: the Hilbert space tensor product when both $\mathfrak{X}, \mathfrak{Y}$ are Hilbert spaces.

$\mathfrak{X} \otimes_\pi \mathfrak{Y}$: the completed π-tensor product. When there is no danger of confusion, $\otimes_\pi$ is denoted by $\otimes$ for simplicity.

1 Standard Triplet of White Noise Functions

We adopt the standard notation following [10]. The Gelfand triple (0.1) is constructed from the differential operator

$$A = 1 + t^2 - \frac{d^2}{dt^2}$$

in the standard manner. Namely, E is the C^∞-domain of A equipped with the topology defined by the norms $|\xi|_p = |A^p \xi|_H$, $p \in \mathbb{R}$. The constant numbers

$$\begin{aligned} \delta &= \text{Hilbert–Schmidt norm of } A^{-1} < \infty, \\ \rho &= \text{operator norm of } A^{-1} = (\inf \operatorname{Spec}(A))^{-1} = \frac{1}{2}, \end{aligned}$$

and the obvious inequalities

$$0 < \rho < 1; \qquad |\xi|_p \le \rho^q |\xi|_{p+q}, \qquad \xi \in E, \quad p \in \mathbb{R}, \quad q \ge 0, \tag{1.1}$$

are used throughout with no special notice. The probability space (E^*, μ), where μ is the standard Gaussian measure, is called the *Gaussian space.* For $\xi \in E_{\mathbb{C}}$ we define an *exponential vector* by

$$\phi_\xi(x) = \exp\left(\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right), \qquad x \in E^*.$$

In particular, ϕ_0 is called the *vacuum.*

Let $\Gamma(H_{\mathbb{C}})$ be the Boson Fock space over $H_{\mathbb{C}}$, the space of $\mathbb{C}$-valued L^2-functions on $\mathbb{R}$. An element in $\Gamma(H_{\mathbb{C}})$ is a sequence $(f_n)_{n=0}^\infty$ such that $f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$ and $\sum_{n=0}^\infty n! \,|\, f_n \,|_0^2 < \infty$. There is a unitary isomorphism between $L^2(E^*, \mu)$ and $\Gamma(H_{\mathbb{C}})$ uniquely determined by the correspondence $\phi_\xi \leftrightarrow (\xi^{\otimes n}/n!)$. This is the celebrated *Wiener–Itô–Segal isomorphism.* For a general $\phi \in L^2(E^*, \mu)$ we write the correspondence as $\phi \sim (f_n)$. In fact, we can reconstruct a function on E^* from (f_n) by means of Hermite polynomials, or more precisely, renormalized tensor products $:x^{\otimes n}:$, for full details see [3] or [10].

The second quantized operator of A is denoted by $\Gamma(A)$. For $\phi \sim (f_n)$ its action is given by $\Gamma(A)\phi \sim (A^{\otimes n} f_n)$. It is known that $\Gamma(A)$ is a positive selfadjoint operator on $L^2(E^*, \mu)$ with Hilbert–Schmidt inverse. We thereby obtain a complex Gelfand triple:

$$(E) \subset L^2(E^*, \mu) \cong \Gamma(H_{\mathbb{C}}) \subset (E)^*,$$

which is called the *standard white noise triplet* or the *Hida–Kubo–Takenaka space* [6]. Elements in (E) and $(E)^*$ are called a *test (white noise) function* and a *generalized (white noise) function*, respectively. We denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the canonical bilinear form on $(E)^* \times (E)$. For the defining norms $\|\cdot\|_p$ of (E) we have

$$\|\phi\|_p^2 = \|\Gamma(A)^p \phi\|_0^2 = \sum_{n=0}^\infty n! \,|\, (A^{\otimes n})^p f_n \,|_0^2 = \sum_{n=0}^\infty n! \,|\, f_n \,|_p^2, \qquad p \in \mathbb{R}, \tag{1.2}$$

where $\phi \sim (f_n)$. Obviously, $\phi \in (E)$ if and only if $f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$ for all n and $\sum_{n=0}^\infty n! \,|\, f_n \,|_p^2 < \infty$ for all $p \geq 0$. If $F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{\mathrm{sym}}$ and $\sum_{n=0}^\infty n! \,|\, F_n \,|_{-p}^2 < \infty$ for some $p \geq 0$, then there exists a unique $\Phi \in (E)^*$ such that

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^\infty n! \langle F_n, f_n \rangle, \qquad \phi \sim (f_n) \in (E).$$

Conversely, every $\Phi \in (E)^*$ is of this type. In that case we write $\Phi \sim (F_n)$ and call it the *Wiener-Itô expansion* of Φ.

For each $y \in E_{\mathbb{C}}^*$ there exists a unique operator $D_y \in \mathcal{L}((E), (E))$ such that

$$D_y \phi_\xi = \langle y, \xi \rangle \phi_\xi, \qquad \xi \in E_{\mathbb{C}}.$$

This is called the *annihilation operator.* In particular,

$$a_t = D_{\delta_t}, \qquad t \in \mathbb{R},$$

is called the *annihilation operator at a point* t or *Hida's differential operator.* (In some literature ∂_t is used for a_t.) The adjoint a_t^* belongs to $\mathcal{L}((E)^*, (E)^*)$ and is called the *creation* *operator at a point* t. These operators are *not* operator-valued distributions but continuous operators for themselves. Moreover, both $t \mapsto a_t \in \mathcal{L}((E),(E))$ and $t \mapsto a_t^* \in \mathcal{L}((E)^*,(E)^*)$ are smooth flows.

For each $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ there exists a unique operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\langle \Xi_{l,m}(\kappa)\phi,\, \psi\rangle\rangle = \langle \kappa,\, \eta_{\phi,\psi}\rangle\,, \qquad \phi,\psi \in (E),$$

where

$$\eta_{\phi,\psi}(s_1,\cdots,s_l,t_1,\cdots,t_m) = \left\langle\!\left\langle a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}\phi,\, \psi \right\rangle\!\right\rangle.$$

We use a formal (but descriptive) integral expression:

$$\Xi_{l,m}(\kappa) = \int_{\mathbb{R}^{l+m}} \kappa(s_1,\cdots,s_l,t_1,\cdots,t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m. \tag{1.3}$$

We call $\Xi_{l,m}(\kappa)$ an *integral kernel operator* with *kernel distribution* κ, see [10]. It is known that κ is uniquely determined whenever it is taken from the subspace

$$(E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)} = \left\{\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*\,;\ s_{l,m}(\kappa) = \kappa\right\},$$

where $s_{l,m}$ is the symmetrizing operator with respect to the first l and the last m variables independently. It is known [10, Chapter 4] that every operator in $\mathcal{L}((E),(E)^*)$, hence every bounded operator on the Boson Fock space $\Gamma(H_{\mathbb{C}}) \cong L^2(E^*,\mu)$ as well admits a strongly convergent infinite series expansion in terms of integral kernel operators (Fock expansion).

2 Brownian Motion and White Noise Process

Consider two white noise functions:

$$B_t \sim (0, 1_{[0,t]}, 0, \cdots), \quad t \ge 0; \qquad W_t \sim (0, \delta_t, 0, \cdots), \qquad t \in \mathbb{R}.$$

It is easy to see that $\{B_t\} \subset L^2(E^*,\mu)$ forms a Gaussian family satisfying

$$B_0 = 0, \qquad \mathbf{E}(B_t) = 0, \qquad \mathbf{E}(B_sB_t) = \min\{s,t\}, \quad s,t \ge 0,$$

namely, $\{B_t\}_{t\ge 0}$ is a Brownian motion. If necessary we put $B_{-t} \sim (0, -1_{[-t,0]}, 0, \cdots)$, $t \ge 0$. It is easily verified that $t \mapsto B_t \in L^2(E^*,\mu)$ is continuous but not differentiable; however, it becomes differentiable if regarded as a flow in $(E)^*$. This is due to the fact that $t \mapsto 1_{[0,t]} \in E_{\mathbb{C}}^*$ is a C^∞-flow with derivative δ_t. To summarize,

Proposition 2.1 *The map $t \mapsto B_t \in (E)^*$ is a C^∞-flow and it holds that*

$$\frac{d}{dt} B_t = W_t, \qquad t \in \mathbb{R},$$

with respect to the strong dual topology of $(E)^*$. Hence $t \mapsto W_t \in (E)^*$ is also a C^∞-flow.

The C^∞-flow $\{W_t\} \subset (E)^*$ is called the *white noise process*. In other words, the white noise is formulated as the time derivative of Brownian motion.

Remark There is another justification of white noise in terms of a generalized stochastic process. In general, let (Ω, P) be a probability space. A map $X : \Omega \times \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ is called a *generalized stochastic process* if

(i) for a fixed $\xi \in \mathcal{S}(\mathbb{R})$, a map $\omega \to X(\omega, \xi)$ is a random variable;

(ii) for a fixed $\omega \in \Omega$, a map $\xi \to X(\omega, \xi)$ is a continuous linear functional on $\mathcal{S}(\mathbb{R})$.

This concept was introduced by Gelfand and by Itô around 1955. Then $W : E^* \times \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ defined by $W(x, \xi) = \langle x, \xi \rangle$ satisfies the above properties, and hence is a generalized stochastic process. This W is also referred to as the white noise. Formal writing

$$W(x, \xi) = \langle x, \xi \rangle = \int_{\mathbb{R}} \xi(t) x(t) dt = \int_{\mathbb{R}} \xi(t) W_t(x) dt$$

suggests the situation very well. The white noise $\{W_t\}$ has no meaning (as a usual random variable) at each time point t but is considered as a generalized stochastic process, i.e., a distribution in t. In other words, the white noise receives a rigorous meaning only after smearing the time parameter and hence it is no longer a flow (time-parametrized stochastic process). In contrast, once the space $(E)^*$ of white noise distributions is introduced, the white noise $W_t(x)$ is a distribution in x and forms a C^∞-flow in $(E)^*$.

3 Quantum Stochastic Processes

Definition ([12]) A family of operators $\{\Xi_t\} \subset \mathcal{L}((E), (E)^*)$ is called a *quantum stochastic process (on Fock space)* if $t \mapsto \Xi_t$ is continuous, where t runs over an interval of $\mathbb{R}$. A continuous linear map $\Xi : E_{\mathbb{C}} \to \mathcal{L}((E), (E)^*)$ is called a *generalized quantum stochastic process.* A generalized quantum stochastic process Ξ is called *regular* if it admits a continuous extension from $E_{\mathbb{C}}^*$ into $\mathcal{L}((E), (E)^*)$. The extension will be denoted by the same symbol.

The continuity condition is not very restrictive as is seen below; also recall that the white noise process is a C^∞-flow within our formulation. If a generalized quantum stochastic process Ξ is regular, $\{\Xi_t = \Xi(\delta_t)\}_{t \in \mathbb{R}}$ is a quantum stochastic process. However, not every quantum stochastic process is of this form. From now on a regular generalized quantum stochastic process is also called a regular quantum stochastic process.

It is known that both $\{a_t\}$ and $\{a_t^*\}$ are regular quantum stochastic processes. In fact, $t \mapsto a_t \in \mathcal{L}((E), (E))$ and $t \mapsto a_t^* \in \mathcal{L}((E)^*, (E)^*)$, both of whish are C^∞-flows. The integral kernel operators:

$$A_t = \Xi_{0,1}(1_{[0,t]}), \qquad A_t^* = \Xi_{1,0}(1_{[0,t]}), \tag{3.1}$$

form quantum stochastic processes. These are called the *annihilation* and *creation processes*, respectively.

Using the natural inclusion: $(E)^* \hookrightarrow \mathcal{L}((E), (E)^*)$ by multiplication, we regard a continuous flow $t \mapsto \Phi_t \in (E)^*$ as a quantum stochastic process. In that sense the white noise process $\{W_t\}$ and the Brownian motion $\{B_t\}$ give rise to the *quantum white noise* and the *quantum Brownian motion*, respectively. It is known that

$$W_t = a_t + a_t^*, \quad t \in \mathbb{R}; \qquad B_t = A_t + A_t^*, \quad t \geq 0.$$

Obviously, the quantum white noise is regular.

There are many examples arising from quantum stochastic differential equations, see e.g., [5], [9], [16]. A systemtic study of such examples from our viewpoint would be interesting, see also [13].

4 Integral Kernel Expansion of a Quantum Stochastic Process

Recall first the canonical isomorphism:

$$\mathcal{L}((E),(E)^*) \cong ((E)\otimes(E))^* \cong \operatorname*{ind\,lim}_{p\to\infty}((E)\otimes(E))_{-p},$$

where $((E)\otimes(E))_{-p}$ is the Hilbert space obtained by completing $((E)\otimes(E))$ with respect to the norm $\|\cdot\|_{-p}$. Then, for $p\geq 0$ let $\mathcal{L}_p((E),(E)^*)$ denote the subspace of all $\Xi\in\mathcal{L}((E),(E)^*)$ which correspond to elements in $((E)\otimes(E))_{-p}$. In particular, $\mathcal{L}_0((E),(E)^*)$ is identified with the space of all Hilbert-Schmidt operators on $L^2(E^*,\mu)$. The topology of $\mathcal{L}_p((E),(E)^*)$ is naturally induced from the norm of $((E)\otimes(E))_{-p}$ which is denoted also by $\|\cdot\|_{-p}$. With this notation,

$$\|\Xi\phi\|_{-p}\leq\|\Xi\|_{-p}\|\phi\|_p, \qquad \phi\in(E). \tag{4.1}$$

It is noted that

$$\mathcal{L}((E),(E)^*) \cong \operatorname*{ind\,lim}_{p\to\infty}\mathcal{L}_p((E),(E)^*) \left(=\bigcup_{p\geq 0}\mathcal{L}_p((E),(E)^*) \quad \text{as vector spaces}\right).$$

On the other hand, it is also known that $\mathcal{L}((E),(E)^*)$ contains all bounded operator on $L^2(E^*,\mu)$.

Lemma 4.1 *Let $\Xi=\sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m})$ be the expansion of $\Xi\in\mathcal{L}_p((E),(E)^*)$, $p\geq 0$, in terms of integral kernel operators. Then*

$$|\kappa_{l,m}|_{-(p+1)}\leq G_{l,m;p}\|\Xi\|_{-p}, \tag{4.2}$$

where

$$G_{l,m;p}=\left(l^l m^m\right)^{-1/2}\left(e^3\delta^2(1+\rho^{2p})\right)^{(l+m)/2}. \tag{4.3}$$

PROOF. Since by definition

$$\widehat{\Xi}(\xi,\eta)=\langle\!\langle\Xi\phi_\xi,\,\phi_\eta\rangle\!\rangle=\langle\!\langle\Xi,\,\phi_\xi\otimes\phi_\eta\rangle\!\rangle, \qquad \xi,\eta\in E_{\mathbb{C}},$$

we have

$$|\widehat{\Xi}(\xi,\eta)|\leq\|\Xi\|_{-p}\|\phi_\xi\otimes\phi_\eta\|_p\leq\|\Xi\|_{-p}\exp\frac{1}{2}\left(|\xi|_p^2+|\eta|_p^2\right).$$

The assertion then follows by applying the result in [10, Theorem 4.4.6]. qed

We now prepare a general lemma. Let $\mathfrak{X}$ be a countable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Then there exists a sequence of Hilbert spaces $\cdots\subset H_2\subset H_1\subset H_0\subset H_{-1}\subset H_{-2}\subset\cdots$ such that

$$\mathfrak{X}\cong\operatorname*{proj\,lim}_{p\to\infty}H_p, \qquad \mathfrak{X}^*\cong\operatorname*{ind\,lim}_{p\to\infty}H_{-p}.$$

If $\mathfrak{X}$ is a nuclear space, we may assume without loss of generality that the natural injection $H_{p+1}\to H_p$ is of Hilbert–Schmidt type for any $p\geq 0$. We denote by $|\cdot|_p$ the norm of H_p.

Lemma 4.2 *Let $\mathfrak{X}$ be a countable Hilbert nuclear space and $\{H_p\}$ the same as above. Let Ω be a locally compact space. Then for a map $f : \Omega \to \mathfrak{X}^*$ the following two conditions are equivalent:*

(i) *f is continuous;*

(ii) *for each $\omega_0 \in \Omega$ there exists $p \geq 0$ such that $f(\omega_0) \in H_{-p}$ and*

$$\lim_{\omega \to \omega_0} | f(\omega) - f(\omega_0) |_{-p} = 0.$$

In that case for any compact subset $\Omega_0 \subset \Omega$ there exists $p \geq 0$ such that $f : \Omega_0 \to H_{-p}$ is continuous.

PROOF. (i) $\Longrightarrow$ (ii) Let $V \subset \Omega$ be an open neighborhood of ω_0 with compact closure. Since f is cotinuous, $f(\overline{V}) \subset \mathfrak{X}^*$ is compact and hence bounded. Then $f(\overline{V}) \subset H_{-p}$ is bounded for some p. In other words, there exists $M \geq 0$ such that

$$| f(\omega) |_{-p} \leq M, \qquad \omega \in V.$$

Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of H_{p+1}. Then by definition,

$$| f(\omega) - f(\omega_0) |_{-(p+1)}^2 = \sum_{j=1}^{\infty} \langle f(\omega) - f(\omega_0), e_j \rangle^2 .$$

We note that

$$\langle f(\omega) - f(\omega_0), e_j \rangle^2 \leq | f(\omega) - f(\omega_0) |_{-p}^2 | e_j |_p^2 \leq 4M^2 | e_j |_p^2 , \qquad \omega \in V.$$

Given $\epsilon > 0$ we choose N such that

$$4M^2 \sum_{j>N} | e_j |_p^2 < \frac{\epsilon}{2},$$

which is possible since $H_{p+1} \to H_p$ is of Hilbert–Schmidt type and hence $\sum_{j=1}^{\infty} | e_j |_p^2 < \infty$. On the other hand, $\omega \mapsto \langle f(\omega), e_j \rangle$ is continuous by assumption. Then for each $j = 1, \cdots, N$ one may find an open neighborhood $U_j \subset \Omega$ of ω_0 such that

$$| \langle f(\omega), e_j \rangle - \langle f(\omega_0), e_j \rangle | < \sqrt{\frac{\epsilon}{2N}}, \qquad \omega \in U_j.$$

Put $U = V \cap U_1 \cap \cdots \cap U_N$. Then

$$\begin{aligned} | f(\omega) - f(\omega_0) |_{-(p+1)}^2 &= \sum_{j=1}^{N} \langle f(\omega) - f(\omega_0), e_j \rangle^2 + \sum_{j>N} \langle f(\omega) - f(\omega_0), e_j \rangle^2 \\ &\leq \sum_{j=1}^{N} \frac{\epsilon}{2N} + 4M^2 \sum_{j>N} | e_j |_p^2 \\ &< N \times \frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon, \qquad \omega \in U. \end{aligned}$$

This is the assertion of (ii).

(ii) $\Longrightarrow$ (i) The topology of $\mathfrak{X}^*$ is defined by the seminorms

$$\| f \|_B = \sup_{\omega \in B} | \langle f, \omega \rangle |, \qquad f \in \mathfrak{X}^*,$$

where B runs over the bounded subsets of $\mathfrak{X}$. Then for any B we have

$$\begin{aligned} \| f(\omega) - f(\omega_0) \|_B &\leq \sup_{\omega \in B} | f(\omega) - f(\omega_0) |_{-p} | \omega |_p \\ &= | B |_p | f(\omega) - f(\omega_0) |_{-p} \longrightarrow 0, \qquad \omega \longrightarrow \omega_0, \end{aligned}$$

by assumption, which shows that f is continuous at ω_0.

The rest of the statement is already clear. qed

Theorem 4.3 *Let $\{\Xi_t\}$ be a quantum stochastic process wit the integral kernel expansion: $\Xi_t = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}(t))$. Then $t \mapsto \kappa_{l,m}(t) \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ is continuous.*

PROOF. We shall prove the continuity at a fixed t. Since $s \mapsto \Xi_s \in \mathcal{L}((E), (E)^*) \cong ((E) \otimes (E))^*$ is continuous, there exists $p \geq 0$ such that $\| \Xi_t \|_{-p} < \infty$ and

$$\lim_{s \to t} \| \Xi_s - \Xi_t \|_{-p} = 0$$

by Lemma 4.2. On the other hand, by Lemma 4.1 we have

$$| \kappa_{l,m}(s) - \kappa_{l,m}(t) |_{-(p+1)} \leq G_{l,m;p} \| \Xi_s - \Xi_t \|_{-p},$$

and therefore

$$\lim_{s \to t} | \kappa_{l,m}(s) - \kappa_{l,m}(t) |_{-(p+1)} = 0.$$

Again from Lemma 4.2 we see that $t \mapsto \kappa_{l,m}(t)$ in continuous. qed

Thus a question about a quantum stochastic process can be discussed in terms of continuous flows in the space of kernel distributions. A similar result as above was proved in [12] under a superfluous assumption which is now eliminated.

5 Generalized Integral Kernel Operators and Quantum Stochastic Integrals

In [13] we introduced an operator of which formal expression is given as

$$\int_{\mathbb{R}^{l+m}} a_{s_1}^* \cdots a_{s_l}^* L(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m, \tag{5.1}$$

where L is an $\mathcal{L}((E), (E)^*)$-valued distribution on $\mathbb{R}^{l+m}$, more precisely, L is taken from the space

$$\mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E), (E)^*)) \cong (E_{\mathbb{C}}^{\otimes(l+m)})^* \otimes \mathcal{L}((E), (E)^*).$$

Such an operator is called a *(generalized) integral kernel operator.* In fact, by the characterization theorem of operator symbols [10, Chapter 4.4] an operator $\Xi \in \mathcal{L}((E), (E)^*)$ is uniquely determined by

$$\langle\langle \Xi \phi_\xi, \phi_\eta \rangle\rangle = \left\langle\left\langle L(\eta^{\otimes l} \otimes \xi^{\otimes m}) \phi_\xi, \phi_\eta \right\rangle\right\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}. \tag{5.2}$$

Then (5.1) is a formal integral expression for Ξ. An integral kernel operator $\Xi_{l,m}(\kappa)$ as in (1.3) is obtained from (5.1) by taking L to be a scalar-operator-valued distribution.

Let $\{L_t\}_{t\in J}$ be a quantum stochastic process, J being an interval, and fix $a \in J$ as a time origin. Then for any $t \in J$ there exists a unique operator $\Xi_t \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\!\langle \Xi_t\phi,\, \psi\rangle\!\rangle = \int_a^t \langle\!\langle L_s\phi,\, \psi\rangle\!\rangle\, ds, \qquad \phi, \psi \in (E). \tag{5.3}$$

We write

$$\Xi_t = \int_a^t L_s\, ds. \tag{5.4}$$

It is proved by a standard argument that $\{\Xi_t\}_{t\in J}$ is a quantum stochastic process.

Theorem 5.1 *If two quantum stochastic processes $\{L_t\}$ and $\{\Xi_t\}$ are related as in (5.4), then $t \mapsto \Xi_t$ is differentiable with respect to the topology of $\mathcal{L}((E),(E)^*)$ and it holds that*

$$\frac{d}{dt}\,\Xi_t = L_t.$$

Corollary 5.2 *For the annihilation process $\{A_t\}$ and the creation process $\{A_t^*\}$ defined in (3.1) it holds that*

$$A_t = \int_0^t a_s\, ds, \qquad \frac{d}{dt}\,A_t = a_t, \qquad \text{and} \qquad A_t^* = \int_0^t a_s^*\, ds, \qquad \frac{d}{dt}\,A_t^* = a_t^*.$$

If $\{L_t\}$ is a quantum stochastic process, so are $\{L_t a_t\}$ and $\{a_t^* L_t\}$. The proof being given in [12], we only note that the bilinear map $\mathcal{L}((E),(E)^*) \times \mathcal{L}((E),(E)) \to \mathcal{L}((E),(E)^*)$ defined by composition is separately continuous but is *not* (jointly) continuous. We are now in a position to introduce quantum stochastic integrals in terms of white noise distribution theory.

Definition Let $\{L_t\}$ be a quantum stochastic process. Then the quantum stochastic processes defined as

$$\int_a^t L_s a_s\, ds, \qquad \int_a^t a_s^* L_s\, ds \tag{5.5}$$

are called the *quantum stochastic integral of L_t against the annihilation and creation processes*, respectively. The latter is also called a *quantum Hitsuda–Skorokhod integral.*

Obviously the quantum stochastic integrals in (5.5) are dual to each other. The number process $\{\Lambda_t\}$ is defined as

$$\Lambda_t = \int_0^t a_s^* a_s\, ds, \qquad \frac{d}{dt}\,\Lambda_t = a_t^* a_t. \tag{5.6}$$

This is the quantum stochastic integral of $\{a_t^*\}$ against the annihilation process as well as the quantum stochastic integral of $\{a_t\}$ against the creation process.

There are slightly different definitions of a quantum Hitsuda–Skorokhod integral, see Lindsay [8] where the gradient operator plays a role; see also [13] where the integral is defined only for a regular quantum stochastic process $\{L_t\}$.

6 Stochastic Integral Representation of a Quantum Stochastic Process

Given $\Xi \in \mathcal{L}((E),(E)^*)$ we devide its integral kernel expansion into three partial sums:

$$\Xi = \sum_{l\geq 0, m\geq 1} \Xi_{l,m}(\kappa_{l,m}) + \sum_{l\geq 1} \Xi_{l,0}(\kappa_{l,0}) + \Xi_{0,0}(\kappa_{0,0}). \tag{6.1}$$

The third term is a scalar operator: $\Xi_{0,0}(\kappa_{0,0}) = \langle\langle \Xi\phi_0, \phi_0\rangle\rangle I$. Note that each integral kernel operator in the first term of (6.1) contains at least one annihilation operator, and that each in the second term consists only creation operators. Then one may expect an expression of the form:

$$\sum_{l\geq 0, m\geq 1} \Xi_{l,m}(\kappa_{l,m}) = \int_{\mathbb{R}} L(t)a_t dt, \qquad \sum_{l\geq 1} \Xi_{l,0}(\kappa_{l,0}) = \int_{\mathbb{R}} a_s^* M^*(s) ds. \tag{6.2}$$

In fact, by iterated integration (guaranteed by the Fubini type theorem, see [13]) and by a routine examination of convergence by means of norm estimates we obtain the following

Theorem 6.1 *Let $\Xi \in \mathcal{L}_p((E),(E)^*)$, $p \geq 0$. Then it admits an expression of the form:*

$$\Xi = \int_{\mathbb{R}} L(t)a_t\, dt + \int_{\mathbb{R}} a_s^* M^*(s)\, ds + cI, \tag{6.3}$$

where (i) $L \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*))$ *satisfies*

$$\| L(\xi)\phi \|_{-(p+q+1)} \leq C_1 \| \Xi \|_{-p} |\xi|_{p+q+1} \| \phi \|_{p+q+1}, \qquad \xi \in E_{\mathbb{C}}, \quad \phi \in (E), \tag{6.4}$$

for $q > q_1 = q_1(p) \geq 0$ and $C_1 = C_1(p,q) \geq 0$;
(ii) *$M \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)))$ satisfies $[M(\xi), a_t] = 0$ for all $\xi \in E_{\mathbb{C}}$ and $t \in \mathbb{R}$ and*

$$\| M(\xi)\phi \|_{p} \leq C_2 \| \Xi \|_{-p} |\xi|_{p+q+1} \| \phi \|_{p+q+1}, \qquad \xi \in E_{\mathbb{C}}, \quad \phi \in (E), \tag{6.5}$$

for $q > q_2 = q_2(p) \geq 0$ and $C_2 = C_2(p,q) \geq 0$;
(iii) *$c \in \mathbb{C}$ is given by $c = \langle\langle \Xi\phi_0, \phi_0\rangle\rangle$.*

With the help of Theorem 6.1 one can derive stochastic integral representation (in a broad sense) of a quantum stochastic process.

Theorem 6.2 *Let $\{\Xi_t\}$ be a quantum stochastic process. Then there exist continuous maps $t \mapsto L_t \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*))$, $t \mapsto M_t \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)))$ and $t \mapsto c_t \in \mathbb{C}$ such that*

$$\Xi_t = \int_{\mathbb{R}} L_t(s)a_s\, ds + \int_{\mathbb{R}} a_s^* M_t^*(s)\, ds + c_t I.$$

PROOF. The topology of $\mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E),(E)^*))$ is defined by the seminorms

$$\| L \|_{K,B_1,B_2} = \sup\{ | \langle\langle L(\xi)\phi, \psi\rangle\rangle | \,;\, \xi \in K, \phi \in B_1, \psi \in B_2\},$$

where $K \subset E_{\mathbb{C}}$, $B_1, B_2 \subset (E)$ are bounded subsets. Then by (6.4) we obtain

$$\| L_s - L_t \|_{K,B_1,B_2} \leq C_1 \| \Xi_s - \Xi_t \|_{-p} |K|_{p+q+1} \| B_1 \|_{p+q+1} \| B_2 \|_{p+q+1}. \tag{6.6}$$

For a fixed t there exists $p \geq 0$ such that $\| \Xi_t \|_{-p} < \infty$ and

$$\lim_{s \to t} \| \Xi_s - \Xi_t \|_{-p} = 0,$$

see Lemma 4.2. Then from (6.6) we see that $s \mapsto L_s$ is continuous at t. That $t \mapsto M_t$ is continuous is proved similarly. The continuity of $t \mapsto c_t = \langle\langle \Xi_t \phi_0, \phi_0 \rangle\rangle$ is obvious. qed

A similar result was proved in [12] under a superfluous condition which is now eliminated. There are similar results for a generalized quantum stochastic process and a regular quantum stochastic process, see [12].

7 Admissible White Noise Distributions and Quantum Itô Formula

The quantum Itô formula is related to the rule of composition of two quantum stochastic integrals. The original discussion due to Hudson–Parthasarathy [5] was restricted to adapted processes and the recent work of Belavkin [1] discusses without assuming adaptedness. A white noise approach was discussed by Huang [4] at somehow formal level and needs more careful study. As the first step we here discuss the simplest but an instructive case of $\{A_t\}$ and $\{A_t^*\}$. Before going into the discussion we note first the following

Proposition 7.1 *Let $\{L_t\}$ and $\{M_t\}$ be two quantum stochastic processes and put*

$$\Xi_t = \int_a^t L_s \, ds, \qquad \Omega_t = \int_a^t M_s \, ds.$$

If $t \mapsto M_t \in \mathcal{L}((E),(E))$ is continuous, or if $t \mapsto L_t \in \mathcal{L}((E)^*,(E)^*)$ is continuous, then $\{\Xi_t \Omega_t\} \subset \mathcal{L}((E),(E)^*)$ is a quantum stochastic process and it holds that

$$d(\Xi_t \Omega_t) = d\Xi_t \cdot \Omega_t + \Xi_t \cdot d\Omega_t,$$

or equivalently

$$\Xi_t \Omega_t = \int_a^t L_s \Omega_s \, ds + \int_a^t \Xi_s M_s \, ds.$$

The proof is obvious. Modelled after the discussion in [12, §3.3] we can prove that

$$\int_a^t \Xi_s M_s \, ds = \lim_{\Delta} \sum \Xi_{s_i} (\Omega_{s_{i+1}} - \Omega_{s_i}) \equiv \int_a^t \Xi_s d\Omega_s, \tag{7.1}$$

$$\int_a^t L_s \Omega_s \, ds = \lim_{\Delta} \sum (\Xi_{s_{i+1}} - \Xi_{s_i}) \Omega_{s_i} \equiv \int_a^t d\Xi_s \Omega_s. \tag{7.2}$$

The Itô type Riemmanian approximations in (7.1) and (7.2) are not essential just for the convergence; but play an important role when we consider adapted processes. Namely, if $[\Xi_{t+h} - \Xi_t, \Omega_t] = 0$ whenever $h > 0$, then (7.2) becomes

$$\int_a^t L_s \Omega_s \, ds = \lim_{\Delta} \sum \Omega_{s_i} (\Xi_{s_{i+1}} - \Xi_{s_i}) \equiv \int_a^t \Omega_s d\Xi_s.$$

As a direct consequence of Proposition 7.1 we have

$$\begin{aligned} d(A_tA_t) &= dA_t \cdot A_t + A_t \cdot dA_t = 2A_t a_t dt, & (7.3)\\ d(A_t^*A_t) &= dA_t^* \cdot A_t + A_t^* \cdot dA_t = a_t^* A_t dt + A_t^* a_t dt, & (7.4)\\ d(A_t^*A_t^*) &= dA_t^* \cdot A_t^* + A_t^* \cdot dA_t^* = 2A_t^* a_t^* dt. & (7.5) \end{aligned}$$

On the other hand, $A_tA_t^*$ has no meaning in $\mathcal{L}((E),(E)^*)$ and, of course, is not differentiable. To study further we need the idea of admissible white noise distributions introduced in [11], [14], where conditional expectations and quantum martingales are discussed.

For a $\mathbb{C}$-valued measurable function f on $\mathbb{R}^n$ we put

$$\| f \|_r^2 = \int_{\mathbb{R}^n} |f(t_1,\cdots,t_n)|^2(1+t_1^2)^r\cdots(1+t_n^2)^r dt_1\cdots dt_n, \qquad r\in\mathbb{R}.$$

For $\phi \sim (f_n) \in (E)$ we put

$$\| \phi \|_{r,\beta}^2 = \sum_{n=0}^{\infty} n! e^{2\beta n} \| f_n \|_r^2, \qquad r,\beta\in\mathbb{R}. \tag{7.6}$$

Let $(\mathcal{A})_{r,\beta}$ be the completion of (E) with respect to the norm $\| \cdot \|_{r,\beta}$. In an obvious manner $\{(\mathcal{A})_{r,\beta}\}_{r,\beta\ge 0}$ forms a projective system of Hilbert spaces and $\{(\mathcal{A})_{-r,-\beta}\}_{r,\beta\ge 0}$ an inductive system of Hilbert spaces. Then we put

$$(\mathcal{A}) = \operatorname*{proj\,lim}_{r,\beta\to\infty}(\mathcal{A})_{r,\beta} = \bigcap_{r,\beta}(\mathcal{A})_{r,\beta}, \qquad (\mathcal{A})^* = \operatorname*{ind\,lim}_{r,\beta\to\infty}(\mathcal{A})_{-r,-\beta} = \bigcup_{r,\beta}(\mathcal{A})_{-r,-\beta}.$$

There holds an inclusion relation:

$$(E)\subset(\mathcal{A})\subset(\mathcal{A})_{0,0} = L^2(E^*\mu)\cong\Gamma(H_{\mathbb{C}})\subset(\mathcal{A})^*\subset(E)^*,$$

where the injections are all continuous. A white noise distribution belonging to $(\mathcal{A})^*$ is called *admissible.*

Proposition 7.2 *The annihilation process $\{A_t\}$ is a continuous flow in $\mathcal{L}((\mathcal{A}),(\mathcal{A}))$ as well as in $\mathcal{L}((\mathcal{A})^*,(\mathcal{A})^*)$. But it is not differentiable in either space.*

The proof is straightforward computation of norms and is omitted. Thus both $A_tA_t^* \in \mathcal{L}((\mathcal{A}),(\mathcal{A}))$ and $A_tA_t^* \in \mathcal{L}((\mathcal{A})^*,(\mathcal{A})^*)$ are well defined and, with the canonical commutation relation we obtain immediately

$$A_tA_t^* = A_t^*A_t + tI.$$

It is noteworthy that the right hand side belongs to $\mathcal{L}((E),(E)^*)$ and differentiable, see (7.4). Therefore

$$d(A_tA_t^*) = dA_t^*\cdot A_t + A_t^*\cdot dA_t + dt = A_t\cdot dA_t^* + A_t^*\cdot dA_t + dt. \tag{7.7}$$

On the other hand, as an operator on $(\mathcal{A})$, (7.4) becomes

$$d(A_t^*A_t) = dA_t\cdot A_t^* + A_t\cdot dA_t^* = A_t^*\cdot dA_t + A_t^*\cdot dA_t. \tag{7.8}$$

The relations (7.3), (7.8), (7.5) and (7.7) form the quantum Itô formula. They are sometimes summarized as follows:

$$dA_t \cdot dA_t = dA_t^* \cdot dA_t = dA_t^* \cdot dA_t^* = 0, \qquad dA_t \cdot dA_t^* = dt.$$

For the Brownian motion $B_t = A_t + A_t^*$ we have

$$d(B_t^2) = d((A_t + A_t^*)^2) = dB_t \cdot B_t + B_t \cdot dB_t + dt = 2B_t \cdot dB_t + dt.$$

This is an essence of the classical Itô formula: $(dB_t)^2 = dt$.

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