数理解析研究所講究録
982巻 1997年 30-40

Tunneling time and Nelson's Quantum Mechanics

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(February 28, 1996)

I. INTRODUCTION

A number of recent experimental developments together with recent theoretical interest in quantum cosmology have led to many approaches to tunneling time, that is, a time associated with the passage of a particle under a tunneling barrier [1–13](see also Refs. [14–16] and references therein). Though the problem of tunneling time is of pure quantum mechanics but seems very simple, it turns out to be very deceptive and has continued to be controversial. Unfortunately, there is no clear consensus about the existence of a simple expression for tunneling time and the exact nature of the expression. **The lack of a time operator** T has created a variety of proposals for the possible definition of time spent by a particle within classically forbidden barrier [17,1–6,9]. It is commented that discussions of tunneling time are realistic and fruitful only when the tunneling effects should be treated in a detailed time-dependent and fully quantum mechanical way.

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As is well known, the quantum phenomena are inexplicable in classical terms. It is believed that a key feature of quantum effects distinct from classical effects is their apparent indeterminism, that is, individual microscopic events are unpredictable and uncontrollable. By means of the wave function and the Schrödinger equation, quantum mechanics can describe statistical properties of ensembles of dynamical systems prepared under the same conditions and the law of time evolution of these statistical ensembles, and its predictions are in good accord with the experiments. Though it predicts the outcomes of measurements performed on statistical ensembles of physical systems, it does not provide a description of the actual individual events of experiments. For the measurement of an observable one can only say that one of eigenvalues of a self-adjoint operator is obtained with a certain probability. We emphasize that tunneling time is not even an observable without having a corresponding operator, as stated above.

An attempt at providing a description of an individual sample path of a particle was proposed by Bohm and Vigier [18]. They supposed that the particle is constantly subjected to random fluctuation coming from some background source and it moves just as a kind of Brownian motion. Subsequently this idea was developed to **the stochastic interpretation of quantum mechanics by Nelson** . In this paper we pursue to formulate tunneling time based on this alternative approach to quantum mechanics by Nelson [19]. In the framework of Nelson's stochastic quantization, a quantized motion of a particle is subjected to a Markov process $x_i(t)$ described by a stochastic differential equation. A specified initial variable $x_i(t_0)$ will develop according to the Langevin equation and its solution gives **a sample path of experiments** in the space-time every event by event. In a scattering process, each sample path has a distinct feature and is divided into **the ensemble of transmitting paths** or **the ensemble of reflecting paths**. Actually the aggregates of sample paths reproduce the same predictions given by the ordinary quantum mechanics. Furthermore, in Nelson's approach each sample path has not only the its own passage but also its time-dependent history. Therefore one can calculate any 'observable' belonging to the specific ensemble, such as even the tunneling time which is only a parameter and has no corresponding operator in the quantum mechanics [20].

II. FORMULATION

We give a brief review of Nelson's approach to quantum mechanics. It is known that this approach, formulated as the real time stochastic process, gives the same expectation values of dynamical quantities as the conventional quantum mechanics does, described by the Schrödinger equation

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t}=(-\frac{\hbar^2}{2m}\Delta+V(\vec{x},t))\psi(\vec{x},t). \tag{1}$$

The diffusion process which corresponds to $|\psi(\vec{x},t)|^2$ is written down as the Fokker-Planck equation of a forward time evolution,

$$\frac{\partial P(\vec{x},t)}{\partial t}=-\vec{\nabla}\cdot\left\{\vec{b}(\vec{x},t)P(\vec{x},t)\right\}+\frac{\hbar}{2m}\Delta P(\vec{x},t), \tag{2}$$

where the drift term $\vec{b}(\vec{x},t)$ is given through the solution of (1),

$$\vec{b}(\vec{x},t)=\frac{\hbar}{m}\vec{\nabla}(\mathrm{Im}+\mathrm{Re})\ln\psi(\vec{x},t) \tag{3}$$

and the diffusion constant is $\frac{\hbar}{2m}$. Equation (2) lead us to the stochastic prosesses represented by the stocahstic differential equation,

$$dx_i(t)=b_i(\vec{x}(t),t)dt+dw_i(t), \tag{4}$$

$dw_i(t)$ is the Gaussian white noise (representing the quantum fluctuation) with the statistical properties of

$$<dw_i(t)>=0 \quad ,<dw_i(t)dw_j(t)>=\frac{\hbar}{m}\delta_{ij}dt. \tag{5}$$

Here $<\cdots>$ means a sample average.

Using this Nelson approach, we analyze tunneling phenomena dependent on a real-time. Here, we restrict our discussion to a one-dimensional system with a static square well potential

$$V(x) = V_0\theta(x)\theta(d-x) \tag{6}$$

for simplicity. We assume a wave packet

$$\psi(x,t) = \int_{-\infty}^{\infty} A(k)\varphi_k(x)e^{-i\frac{E}{\hbar}t}dk \tag{7}$$

is incident from the left, where $\varphi_k(x)$ is a plane wave (stationary) solution of this potential problem and

$$A(k) = A_{k_0}(k) = C\exp\left\{-\frac{(k_0-k)^2}{4\sigma^2}\right\}, \tag{8}$$

with a normalization constant C. k_0 is a centeral wave number of the wave packet.

III. CLASSIFICATION OF SAMPLE PATHS

Suppose a simulation of tunneling phenomena based on (4), starting at $t = -\infty$ and ending at $t = \infty$. As we treat a wave packet in a scattering problem, the wave packet is located in the region I($x < 0$) initially and turns finally into two spatially separated wave packets which are in the regions I and III($x > d$). In this situation, we introduce transmission ensemble (ET) as a set of sample paths with random variables $x(t \to \infty) \in$ III. Likewise reflection ensemble (ER) is introduced as a set of sample paths with random variables $x(t \to \infty) \in$ I. Once each sample path is classified into either ET or ER, we consider the averages over ET only and ER only as

$$< \cdots >_{ET} = < \cdots >_{\{x(t)|x(t\to\infty)\in \mathrm{III}\}}, \tag{9}$$

$$< \cdots >_{ER} = < \cdots >_{\{x(t)|x(t\to\infty)\in \mathrm{I}\}}\ . \tag{10}$$

This enables us to obtain, for example, the mean transmission path $X_T(t) =< x(t) >_{ET}$ and the mean reflection one $X_R(t) =< x(t) >_{ER}$. This $X_T(t)$ informs us how long it takes for a particle to pass a potential barrier on the average (*average passing time*), and how long it takes for a particle to interact with the potential on the average (*average interacting time*).

Figure 1 shows two typical sample paths.

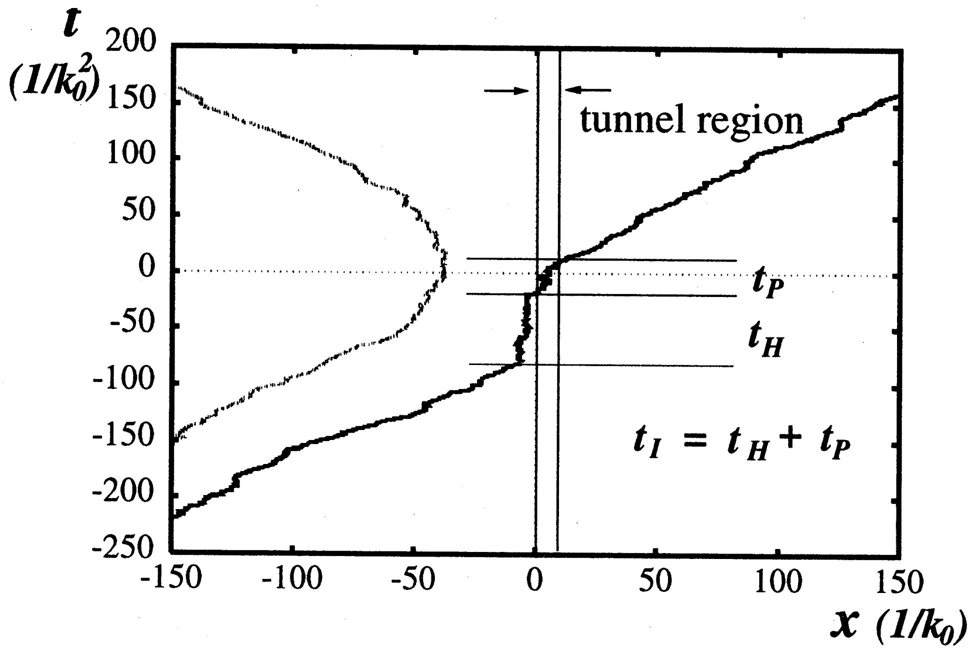

FIG. 1. The two typical sample paths; One is a transmission sample path and another is a reflection one. The potential width d is $\frac{10}{k_0}$, and the potential height $V = 2E_0$.

One is a transmission sample path and another is a reflection one. For some interval from the start, both paths seem to be like a free particle path. Then they are distorted in front of the potential. At last, the transmission one passes the tunnel and seems to take a path like a free particle again in region of III. On the other hand, the other is reflected and seems to be like a free particle which go to $x \to -\infty$ in region of I. We can see from these exemplar paths the following three characteristic time intervals related to tunneling phenomena. First the *passage time* t_P is defined as the time interval in which the random variables $x(t)$ is in the region of II$(0 \leq x \leq d)$. Secondly one may introduce the concept of the *hesitation time* t_H in which the random variable stays in front of the potential for some period. For this time interval, a strong interference between the incident wave and the reflecting one exists. Finally the *interaction time* t_I, meaning the effective interaction time interval, is introduced as $t_I = t_P + t_H$.

The above method, however, has a practical disadvantage in some numerical simulations. When we simulate tunneling phenomena with a large and/or wide potential, it occurs that for a finite number of whole samples one does not find sufficient number of transmission samples for meaningful analyses. We suggest the use of backward formulation of Nelson's approach.

We start our simulation on the stochastic differential equation of backward time evolution, with an "initial" distribution $|\psi(x, t \to \infty)|^2$. Transmission samples "begin" in the transmission region of III, while reflection samples "begin" in the reflection region of I. Focusing on transmission samples, we can collect sufficient number of the sample paths and therefore sufficient information for our purpose.

IV. TUNNELING TIMES IN A SIMPLE MODEL

Here we show the average value t_P, t_H and t_I obtained from a simulation in a narrow ($k_0 d = 1$) and a wide ($k_0 d = 10$) potential cases. We set the "initial ($t = \infty$)" distribution concentrated at the peak of the transmission wave packet, and collected 80 samples per each case.

Generally, the average $< t_P >$ decreases as the potential height $\frac{V_0}{E}$ becomes larger(Fig. 2).

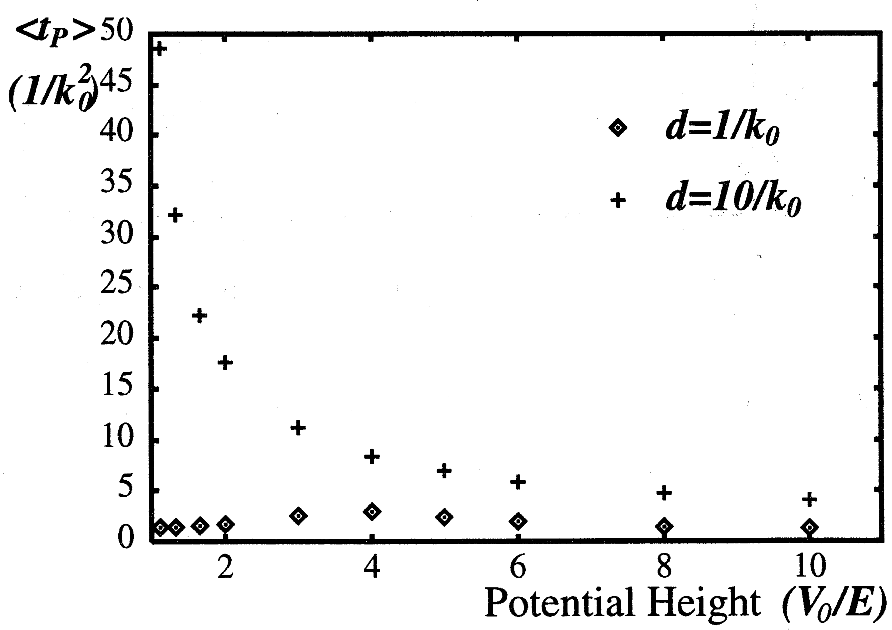

FIG. 2. The mean values of "*passage time*" versus potential height.

It is noted that the real part of the exponential of the wave function, or

$$-\frac{\partial}{\partial x}\mathrm{Re}\log\psi(x,t). \tag{11}$$

is dominant in the drift in the high potential tunneling region of II and mainly controls the quantitative behavior of t_P. Let us estimate the *passage time* in the extreme case ($\kappa_0 d \gg 1, \kappa = \sqrt{2m(V_0 - E)/\hbar}$). Here, we approximate the drift term $b(x)$ near the tunnel region as $b(x) \sim \frac{\hbar\kappa_0}{m}$. There are two characteristic time intervals ; the diffusion time $t_d \sim \frac{md^2}{\hbar}$ and the current time $t_c \sim \frac{md}{\hbar\kappa_0}$. The condition $\kappa_0 d \gg 1$ lead to the relation of $t_d \gg t_c$. Thus the time interval t_c becomes the *passage time*. This agrees with the absolute value of the imaginary time from the Euclidean classical equation, and with Büttiker-Landauer time [1].

The average $< t_H >$ increases as the potential height becomes larger (Fig. 3).

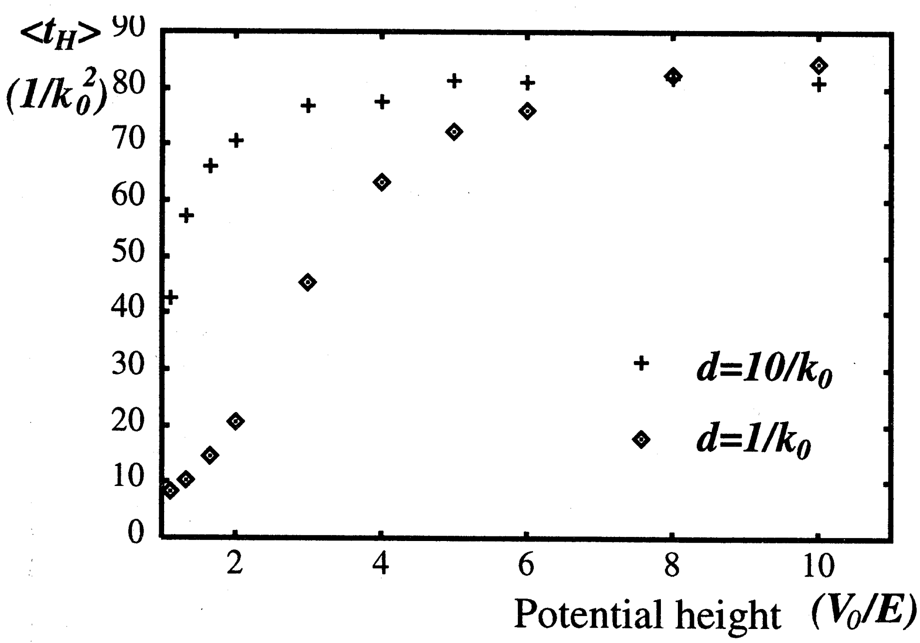

FIG. 3. The mean values of "*hesitation time*" versus potential height.

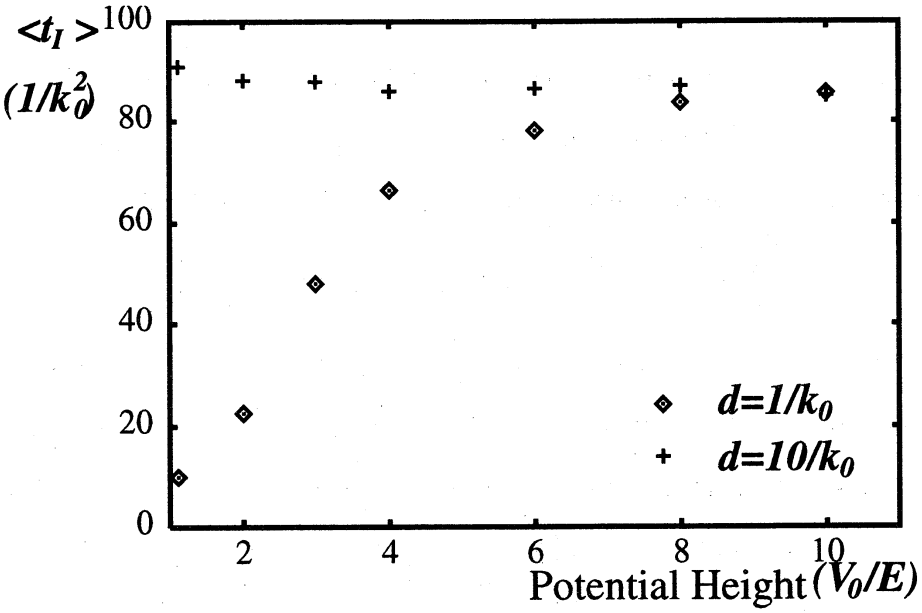

FIG. 4. The mean values of "*interaction time*" versus potential height.

This trend is intuitively natural since the particle "hesitates" for longer period for higher potential. In terms of the wave function, $\psi(x,t)$ varies more rapidly in the neighborhood of the potential as its height becomes larger, and so does the change of the drift term in the Langevin equation, which keeps the particle in the small region in front of the potential barrier. The random variable $x(t)$ stays for some time interval in not only pre-tunnel region but also in tunnel region. In the comparatively low potential case ($\frac{V_0}{E} \sim 1$), the interval in pre-tunnel region and one in tunnel region are almost equal. But in high potential case ($\frac{V_0}{E} \gg 1$), the former is dominant over the latter. In terms of forward time evolution, the random variable $x(t)$ for a transmission sample path through a higher potential starts from the front part of a wave packet arrives in the pre-tunneling region and "*hesitates*" there until the wave function begin to be split into transmission and reflection parts. At last, driven by the drift, the particle goes through the potential barrier.

V. EXTENSION TO MORE GENARAL CASES

In order to take the effect of absorption or transition to other cannels, we generalize Nelson's stochastic quantization to the case including optical potential and/or multi-channel coupling. Schrödinger equation with optical potential iU is written down as

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} = (-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x},t) + iU(\vec{x},t))\psi(\vec{x},t). \tag{12}$$

The corresponding drift term and the stochastic equation are the same as the previous ones, while the Fokker-Planck equation and the conservation law of probability current have additional terms which represent the absorption effects as

$$\frac{\partial P(\vec{x},t)}{\partial t} = -\vec{\nabla}\cdot\left\{\vec{b} - \frac{\hbar}{2m}\vec{\nabla}\right\}P(\vec{x},t) + \frac{2U}{\hbar}P, \tag{13}$$

$$\frac{\partial P}{\partial t} + \vec{\nabla}\cdot(\vec{v}P) = \frac{2U}{\hbar}P. \tag{14}$$

Fig. 5 shows the three typical sample paths in this simulation. There is a path which is absorbed in the passage through the tunneling region as well as transmission and reflection ones.

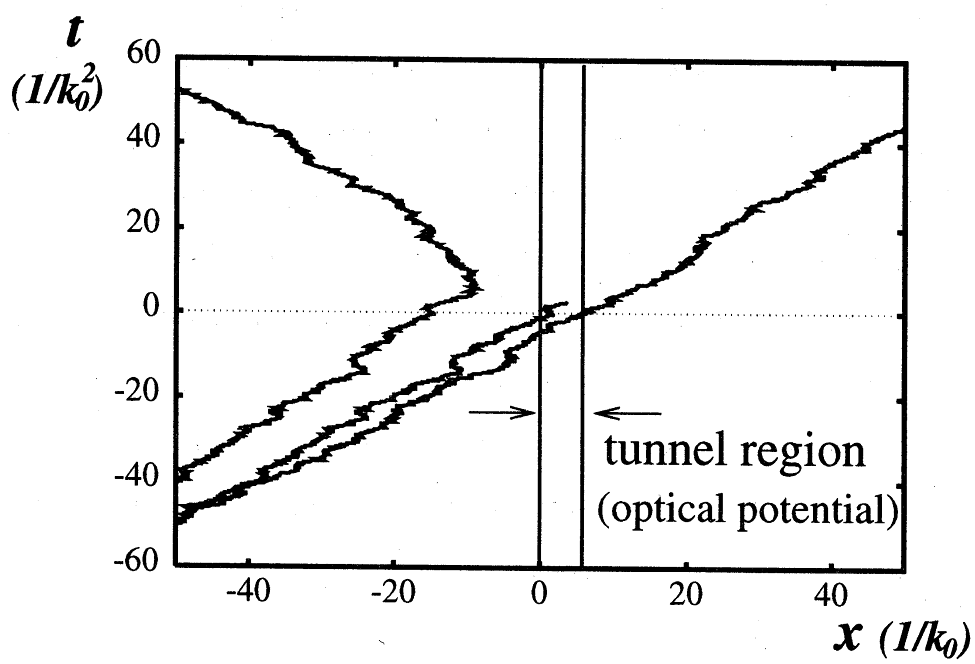

FIG. 5. The three typical sample paths in the case of optical potential. The potential width d is $\frac{5}{k_0}$, and the potential height $V - iU = (1.11 - 2i)E_0$.

Similarly starting from the N-channels Schrödinger equations

$$i\hbar\frac{\partial\psi_i}{\partial t} = (-\frac{\hbar^2}{2m_i}\nabla^2 + V_{ii})\psi_i + \sum_{j\neq i}^{N} V_{ij}\psi_j, \tag{15}$$

$$V_{ij} = V_{ji}^*, \tag{16}$$

we have the Fokker-Planck equations

$$\frac{\partial P_i}{\partial t} = -\vec{\nabla}\cdot\left\{\vec{b}^{(i)} - \frac{\hbar}{2m_i}\vec{\nabla}\right\}P_i + \sum_{j\neq i}^{N} P_{(j\to i)}, \tag{17}$$

$$P_{(j\to i)} = \frac{2}{\hbar}\mathrm{Im}(\psi_j^* V_{ij}\psi_i), \tag{18}$$

the stochastic differential equatins

$$d\vec{x}^{(i)}(t) = \vec{b}^{(i)}(\vec{x},t) + d\vec{W}^{(i)}(t), \tag{19}$$

with

$$< dW_k^{(i)}(t) dW_l^{(j)}(t) >= 2\frac{\hbar}{2m_i}\delta_{ij}\delta_{kl}dt. \tag{20}$$

and the conservation law of probability current

$$\frac{\partial P_i}{\partial t} + \vec{\nabla}\cdot(\vec{v}^{(i)}\,P_i) = \sum_{j\neq i}^{N} P_{(j\to i)}. \tag{21}$$

Here the drift terms are given by

$$\vec{b}^{(i)}(\vec{x},t) = \frac{\hbar}{m}\vec{\nabla}(\mathrm{Im}+\mathrm{Re})\ln\psi_i(\vec{x},t). \tag{22}$$

VI. CONCLUDING REMARKS

In this paper we have analyzed time intervals associated with tunneling phenomena, using Nelson's stochastic approach to quantum mechanics. This approach allows us to study the phenomena an event by event in their sample paths, in particular, to have information on the time evolution of each sample path. This is the reason for getting the unambiguous definitions of tunneling time in our approach.

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