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Scaling limit of a model of quantum electrodynamics with N-nonrelativistic particles

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1 INTRODUCTION

The main problem presented in this paper is to consider a scaling limit of a model in quantum electrodynamics which describes an interaction of N-nonrelativistic charged particles and a quantized radiation field in the Coulomb gauge with the dipole approximation. The model we consider is called "the Pauli-Fierz model". Authors in [5,6] have studied a scaling limit of the Pauli-Fierz model with one-nonrelativistic charged particle. We may well extend the scaling limit of one-particle system to N-particles system.

The Pauli-Fierz Hamiltonians $H_{\vec{\rho}}$ with N-nonrelativistic charged particles in the Coulomb gauge with the dipole approximation are defined as operators acting in the Hilbert space $\underbrace{L^2(\mathbb{R}^d)\otimes\ldots\otimes L^2(\mathbb{R}^d)}_{N}\otimes\mathcal{F}(\mathcal{W})\cong L^2(\mathbb{R}^{dN})\otimes\mathcal{F}(\mathcal{W})$ by

$$H_{\vec{\rho}} = \frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(-i\hbar D_\mu^j\otimes I - eI\otimes A_\mu(\rho_j)\right)^2 + I\otimes H_b,$$

where D_μ^j is the differential operator with respect to the j-th variable in the μ-th direction, $A_\mu(\rho_j)$ the quantized radiation field in the μ-th direction with an ultraviolet cut-off function ρ_j in the Coulomb gauge, H_b the free Hamiltonian in $\mathcal{F}(\mathcal{W})$, and $m, e, \hbar$ the mass of the particles, the charge of the particles, the Planck constant divided 2π, respectively.

Note that A_μ is depend on the speed of light c. We introduce the following scaling.

$$c(\kappa) = c\kappa, e(\kappa) = e\kappa^{-\frac{1}{2}}, m(\kappa) = m\kappa^{-2}. \tag{1. 1}$$

Then the scaled Hamiltonian $H_{\vec{\rho}}(\kappa)$ amounts to

$$-\frac{\hbar^2\kappa^2}{2m}\Delta\otimes I+\kappa I\otimes H_b+\frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(\kappa 2e\hbar i D_\mu^j\otimes A_\mu(\rho_j)+e^2 I\otimes A_\mu^2(\rho_j)\right).$$

Defining a pseudo differential operator $E^{REN}(D,\kappa)$ in $L^2(\mathbb{R}^{dN})$ with a symbol $E^{REN}(p,\kappa)$ such that $E^{REN}(p,\kappa)\to\infty$ as $\kappa\to\infty$, we define a Hamiltonian $H_{\vec{\rho}}^{REN}(\kappa)$ by

$$-E^{REN}(D,\kappa)\otimes I+\kappa I\otimes H_b+\frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(\kappa 2e\hbar i D_\mu^j\otimes A_\mu(\rho_j)+e^2 I\otimes A_\mu^2(\rho_j)\right).$$

Consequently, we shall show the following for some $\vec{\rho}=(\rho_1,...,\rho_N)$ and scalar potentials V with some conditions (Theorem 3.7):

$$s-\lim_{\kappa\to\infty}(H_{\vec{\rho}}^{REN}(\kappa)+V\otimes I-z)^{-1}=\mathcal{U}(\infty)\left\{(E^\infty(D)+V_{eff}-z)^{-1}\otimes P_0\right\}\mathcal{U}^{-1}(\infty),$$

where $E^\infty(D)$ is a pseudo differential operator in $L^2(\mathbb{R}^{dN})$, V_{eff} a multiplication operator, which is called "effective potential", and P_0 a projection on $\mathcal{F}(\mathcal{W})$. Despite the fact that in the case of one-particle system the effective potential V_{eff} is the Gaussian transformation of a given scalar potential V, we shall show that in N-particles system, it is not necessary to be the Gaussian transformation. Actually it is determined by a matrix $\tilde{\Delta}^\infty=(\tilde{\Delta}_{ij}^\infty)_{1\le ij\le N}$ which is defined by the ultraviolet cut-off functions ρ_j;

$$\tilde{\Delta}_{ij}^\infty=\frac{1}{2}\frac{d-1}{d}\left(\frac{\hbar}{mc}\right)\frac{e^2}{\hbar c}\int_{\mathbb{R}^d}dk\frac{\hat{\rho}_i(k)\hat{\rho}_j(k)}{\omega(k)^3}.$$

2 THE PAULI-FIERZ MODEL

To begin with, let us introduce some preliminary notations. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. We denote the inner product and the associated norm by $<*,\cdot>_{\mathcal{H}}$ and $||\cdot||_{\mathcal{H}}$ respectively. The inner product is linear in $\cdot$ and antilinear in $*$. The domain of an operator A in $\mathcal{H}$ is denoted by $D(A)$. A notation $\hat{f}$ (resp.$\check{f}$) denotes the Fourier transformation (resp.the inverse Fourier transformation) of f and $\bar{f}$ the complex conjugate of f. Let

$$\mathcal{W}\equiv\underbrace{L^2(\mathbb{R}^d)\oplus...\oplus L^2(\mathbb{R}^d)}_{d-1}.$$

We define the Boson Fock space over $\mathcal{W}$ by

$$\mathcal{F}(\mathcal{W}) \equiv \bigoplus_{n=0}^{\infty} \otimes_s^n \mathcal{W} \equiv \bigoplus \mathcal{F}_n(\mathcal{W}),$$

where $\otimes_s^0 \mathcal{W} \equiv \mathbb{C}$ and $\otimes_s^n \mathcal{W}$ $(n \geq 1)$ denotes the n-fold symmetric tensor product. Put

$$\mathcal{F}^\infty(\mathcal{W}) \equiv \bigcup_{N=0}^{\infty} \bigoplus_{n=0}^{N} \mathcal{F}_n(\mathcal{W}) \bigoplus_{n \geq N+1} \{0\}.$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ $(f \in \mathcal{W})$ act on $\mathcal{F}^\infty(\mathcal{W})$ and leave it invariant with the canonical commutation relations (CCR): for $f, g \in \mathcal{W}$

$$\begin{aligned} [a(f), a^\dagger(g)] &= \left\langle \bar{f}, g \right\rangle_{\mathcal{W}}, \\ [a^\sharp(f), a^\sharp(g)] &= 0, \end{aligned}$$

where $[A, B] = AB - BA$, $a^\sharp$ denotes either a or $a^\dagger$. Furthermore,

$$\left\langle a^\dagger(f)\Phi, \Psi \right\rangle_{\mathcal{F}(\mathcal{W})} = \left\langle \Phi, a(\bar{f})\Psi \right\rangle_{\mathcal{F}(\mathcal{W})}, \quad \Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W}).$$

We define polarization vectors $e^r (r = 1, ..., d-1)$ as measurable functions $e^r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that

$$e^r(k)e^s(k) = \delta_{rs}, \quad e^r(k)k = 0, \quad a.e. k \in \mathbb{R}^d.$$

The μ-th direction time-zero smeared radiation field in the Coulomb gauge with the dipole approximation is defined as operators acting in $\mathcal{F}(\mathcal{W})$ by

$$A_\mu(f) = \frac{1}{\sqrt{2}} \left\{ a^\dagger \left(\oplus_{r=1}^{d-1} \frac{\sqrt{\hbar} e_\mu^r \hat{f}}{\sqrt{c\omega}} \right) + a \left(\oplus_{r=1}^{d-1} \frac{\sqrt{\hbar} e_\mu^r \tilde{\hat{f}}}{\sqrt{c\omega}} \right) \right\}, \tag{2. 1}$$

where $\omega(k) = |k|$ and $\tilde{g}(k) = g(-k)$. Let $\Omega = (1, 0, 0, ...) \in \mathcal{F}(\mathcal{W})$. For a nonnegative self-adjoint operator $h : \mathcal{W} \to \mathcal{W}$, we denote "the second quantization of h"by $d\Gamma(h)$. Put $\tilde{\omega} = \underbrace{\omega \oplus ... \oplus \omega}_{d-1}$. The free Hamiltonian H_b in $\mathcal{F}(\mathcal{W})$ is defined by

$$H_b \equiv \hbar c d\Gamma(\tilde{\omega}).$$

The Pauli-Fierz Hamiltonians with N-nonrelativistic charged particles interacting with the quantized radiation field with the dipole approximation in the Coulomb gauge read as follows:

$$H_{\vec{\rho}} \equiv H_{\rho_1,\ldots,\rho_N} \equiv \frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(-i\hbar D_\mu^j \otimes I - eI\otimes A_\mu(\rho_j)\right)^2 + I\otimes H_b,$$

acting in

$$\underbrace{L^2(\mathbb{R}^d)\otimes\ldots\otimes L^2(\mathbb{R}^d)}_{N}\bigotimes\mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{dN})\otimes\mathcal{F}(\mathcal{W}) \cong \int_{\mathbb{R}^{dN}}^{\oplus}\mathcal{F}(\mathcal{W})dx.$$

We introduce the scaling (1.1). For objects A containing of the parameters c, e, m, we denote the scaled object by $A(\kappa)$ throughout this paper. We define classes P and $\widetilde{P}$ of sets of functions as follows:

Definition 2.1 $\vec{\rho} = (\rho_1, \ldots, \rho_N)$ *is in* P *if and only if*

(1) $\hat{\rho}_j, j = 1, \ldots, N$ *are rotation invariant,* $\hat{\rho}_j(k) = \hat{\rho}_j(|k|)$*, and real-valued,*

(2) $\hat{\rho}_j/\omega, \hat{\rho}_j/\sqrt{\omega}, \hat{\rho}_j, \sqrt{\omega}\hat{\rho}_j \in L^2(\mathbb{R}^d)$.

Moreover $\vec{\rho}$ *is in* $\widetilde{P}$ *if and only if in addition to* **(1)** *and* **(2)** *above*

(3) $\hat{\rho}_j/\omega\sqrt{\omega} \in L^2(\mathbb{R}^d)$ *and there exist* $0 < \alpha < 1$ *and* $1 \le \epsilon$ *such that* $\hat{\rho}_i(\sqrt{\cdot})\hat{\rho}_j(\sqrt{\cdot})(\sqrt{\cdot})^{d-2} \in Lip(\alpha)\cap L^\epsilon([0,\infty))$*, where* $Lip(\alpha)$ *is the set of the Lipschitz continuous functions on* $[0,\infty)$ *with the degree* α*,*

(4) $\sup_k |\hat{\rho}_j(k)\omega^{\frac{d}{2}-\frac{3}{2}}(k)| < \infty, \sup_k |\hat{\rho}_j(k)\omega^{\frac{d}{2}-\frac{1}{2}}(k)| < \infty, j = 1, \ldots, N.$.

Put

$$H_0 = -\frac{1}{2m}\hbar^2\Delta\otimes I + I\otimes H_b,$$

where Δ is the Laplacian in $\mathbb{R}^{dN}$. It is well known that H_0 is a nonnegative self-adjoint operator on $D(H_0) = D\left(-\frac{1}{2m}\hbar^2\Delta\otimes I\right)\cap D(I\otimes H_b)$.

Proposition 2.2 ([3,4]) *For $\vec{\rho} \in P$ and $\kappa > 0$, the operator $H_{\vec{\rho}}(\kappa)$ is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of H_0 and nonnegative.*

Let $\mathbf{F} = F \otimes I$, where F denotes the Fourier transform in $L^2(\mathbb{R}^{dN})$. It is clear that operators $\mathbf{F}H_{\vec{\rho}}\mathbf{F}^{-1}$ can be decomposable as follows:

$$\mathbf{F}H_{\vec{\rho}}(\kappa)\mathbf{F}^{-1} = \int_{\mathbb{R}^{dN}}^{\oplus} H_{\vec{\rho}}(p,\kappa)dp,$$

where

$$H_{\vec{\rho}}(p,\kappa) = \frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(\kappa\hbar p_{\mu}^{j} - eA_{\mu}(\rho_j)\right)^2 + \kappa H_b.$$

Proposition 2.3 ([3,4]) *For $\vec{\rho} \in P$ and $\kappa > 0$, the operator $H_{\vec{\rho}}(p,\kappa)$ is self-adjoint on $D(H_b)$ and essentially self-adjoint on any core of H_b and nonnegative.*

Set Hilbert spaces $M_d = \left\{f \middle| \int |f(k)|^2\omega(k)^d dk < \infty\right\}$ and put $\mathcal{W}_\alpha = \underbrace{M_\alpha \oplus \ldots \oplus M_\alpha}_{d-1}$, $\alpha \in \mathbb{R}$. The following lemma is the key lemma to investigating the scaling limits.

Lemma 2.4 ([9]) *Let $\vec{\rho} \in \widetilde{P}$ and $\kappa > 0$ be sufficiently large. Then there exist a Hilbert Schmidt operator $\mathbf{W}_-$, a bounded operator $\mathbf{W}_+$, and $\mathbf{L}_j = (\mathbf{L}_j^1, \ldots, \mathbf{L}_j^d), \mathbf{L}_j^\mu \in \mathcal{W}, j = 1, \ldots, N$, $\mu = 1, \ldots, d$ such that, if we put for $p^j \in \mathbb{R}^d, j = 1, \ldots, N$*

$$B(\mathbf{f},p) = a^\dagger(\mathbf{W}_-\mathbf{f}) + a(\mathbf{W}_+\mathbf{f}) + \sum_{j=1}^{N}\left\langle \mathbf{L}_j p^j, \mathbf{f}\right\rangle_{\mathcal{W}},$$

$$B^\dagger(\mathbf{f},p) = a^\dagger(\overline{\mathbf{W}}_+\mathbf{f}) + a(\overline{\mathbf{W}}_-\mathbf{f}) + \sum_{j=1}^{N}\left\langle \overline{\mathbf{L}}_j p^j, \mathbf{f}\right\rangle_{\mathcal{W}},$$

then

$$\begin{aligned}[B(\mathbf{f},p), B^\dagger(\mathbf{g},p)] &= \langle \mathbf{f}, \mathbf{g}\rangle_{\mathcal{W}}, \\ [B^\sharp(\mathbf{f},p), B^\sharp(\mathbf{g},p)] &= 0, \quad on\ \mathcal{F}^\infty(\mathcal{W}),\end{aligned}$$

and for $\Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W})$,

$$\left\langle B^\dagger(\mathbf{f},p)\Phi, \Psi\right\rangle_{\mathcal{F}(\mathcal{W})} = \left\langle \Phi, B(\bar{\mathbf{f}},p)\Psi\right\rangle_{\mathcal{F}(\mathcal{W})},$$

moreover

$$[\ H_{\vec{\rho}}(p), B^{\sharp}(\mathbf{f},p)\ ] = \pm B^{\sharp}(\hbar c\tilde{\omega}\mathbf{f},p), \ \ on \ \ \mathcal{F}^{\infty}(\mathcal{W}) \cap D(H_b^{\frac{3}{2}}),$$

where $\mathbf{f} \in \mathcal{W}_0 \cap \mathcal{W}_2$ *and* $+$ *(resp.-) corresponds to* $B^{\dagger}$ *(resp.B).*

By virtue of Lemma 2.4, we see the following.

Corollary 2.5 *Let* $\vec{\rho} \in \widetilde{P}$ *and* κ *be sufficiently large. Then for* $\Phi \in D(H_b)$*,*

$$\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right) B^{\sharp}(\mathbf{f},p) \exp\left(-i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right)\Phi = B^{\sharp}(e^{ic\tilde{\omega}t}\mathbf{f},p)\Phi$$

3 SCALING LIMITS

In this section, we construct a unitary operator which implements unitary equivalence of the Pauli-Fierz Hamiltonian and a decoupled Hamiltonian. Moreover we investigate a scaling limit of the Pauli-Fierz Hamiltonian. Unless otherwise stated in this section, we suppose that $\kappa > 0$ is sufficiently large. From Lemma 2.4 (1) it follows that there exist two unitary operators $U(\kappa)$ (p independent) and $S(p,\kappa)$ such that ([6,Section III])

$$U^{-1}(\kappa)S(p,\kappa)^{-1}B^{\sharp}(\mathbf{f},p,\kappa)S(p,\kappa)U(\kappa) = a^{\sharp}(\mathbf{f}), \quad \mathbf{f} \in \mathcal{W}. \tag{3. 1}$$

Concretely $S(p,\kappa)$ is given by

$$S(p,\kappa) = \exp\left(\sum_{i,j=1}^{N}\frac{e\hbar}{\kappa^2}p_{\mu}^{i}\left\{a\left(\oplus_{r=1}^{d-1}\frac{e_{\mu}^{r}M_{ij}(\kappa)\hat{\rho}_j}{\sqrt{2\hbar c^3\omega^3}}\right) - a^{\dagger}\left(\oplus_{r=1}^{d-1}\frac{e_{\mu}^{r}M_{ij}(\kappa)\hat{\rho}_j}{\sqrt{2\hbar c^3\omega^3}}\right)\right\}\right),$$

where $(M_{ij}(\kappa))_{1\leq ij\leq N}$ is a matrix such that

$$\lim_{\kappa\to\infty}\frac{M_{ij}(\kappa)}{\kappa^2} = \delta_{ij}\frac{1}{m}.$$

Theorem 3.1 *Suppose* $\vec{\rho} \in \widetilde{P}$*. Then putting* $S(p,\kappa)U(\kappa) = \mathcal{U}(p,\kappa)$*, we see that* $\mathcal{U}(p,\kappa)$ *maps* $D(H_b)$ *onto itself with*

$$\mathcal{U}(p,\kappa)H_{\vec{\rho}}(p,\kappa)\mathcal{U}^{-1}(p,\kappa) = \kappa H_b + E(p,\kappa), \tag{3. 2}$$

where

$$
\begin{aligned}
E(p,\kappa) &= \frac{\hbar^2}{2m}\sum_{i=1}^{N}\sum_{\mu=1}^{d}\left(\kappa p_\mu^i + \kappa\sum_{j=1}^{N} p_\nu^j \Delta_{\nu\mu}^{ji}(\kappa)\right)^2 + \Box(\kappa),\\
\Delta_{\nu\mu}^{ji}(\kappa) &= \frac{1}{\kappa^3}\frac{e^2}{2c^2}\sum_{k=1}^{N}\sum_{r,s=1}^{d-1}\left\langle \frac{e_\nu^r M_{ij}(\kappa)\hat{\rho}_k}{\sqrt{\omega^3}}, \left(I + \mathbf{W}_-(\kappa)\mathbf{W}_+^{-1}(\kappa)\right)^{(r,s)} \frac{e_\mu^s \hat{\rho}_i}{\sqrt{\omega}}\right\rangle_{L^2(\mathbb{R}^d)},\\
\Box(\kappa) &= \frac{e^2\hbar}{4mc}\sum_{i=1}^{N}\sum_{r,s=1}^{d-1}\left\langle \frac{e_\mu^r \hat{\rho}_i}{\sqrt{\omega}}, \left(I - \mathbf{W}_-(\kappa)\mathbf{W}_+^{-1}(\kappa)\right)^{(r,s)} \frac{e_\mu^s \hat{\rho}_i}{\sqrt{\omega}}\right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

Proof: For simplicity, we omit the symbol κ. Put $\mathcal{U}(p)\Omega \equiv \Omega(p)$. From [6,Proposition 2.4, Lemma 5.9] it follows that $\Omega(p) \in D(H_b)$. Then $\Omega(p) \in D(B(\mathbf{f},p))$. By virtue of Corollary 2.5 and (3.1), we can see that for all $\mathbf{f} \in \mathcal{W}$

$$B(\mathbf{f},p)\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right)\Omega(p) = 0. \tag{3. 3}$$

The equation (3.3) implies that there exists a positive constant $E(p)$ such that

$$\exp\left(i\frac{t}{\hbar}H_{\vec{\rho}}(p)\right)\Omega(p) = \exp\left(i\frac{t}{\hbar}E(p)\right)\Omega(p). \tag{3. 4}$$

Hence from Corollary 2.5, (3.1), (3.4) and the denseness of

$$\mathcal{L}\left\{B^\dagger(\mathbf{f}_1)....B^\dagger(\mathbf{f}_n)\Omega(p), \Omega(p) \,\middle|\, \mathbf{f}_j \in \mathcal{W}, j = 1,...,n, n \geq 1\right\},$$

one can get (3.2). The constant $E(p)$ is explicitly given by

$$E(p) = \frac{< H_{\vec{\rho}}(p)\Omega(p), \Omega >_{\mathcal{F}(\mathcal{W})}}{< \Omega(p), \Omega >_{\mathcal{F}(\mathcal{W})}}.$$

It completes the proof. □

The positive constant $E(p,\kappa)$ can be rewritten by:

$$E(p,\kappa) = \frac{\kappa^2\hbar^2}{2m}p^2 + E^{REN}(p,\kappa) + \widetilde{E}(p,\kappa),$$

where

$$\widetilde{E}(p,\kappa) = \frac{\kappa^2\hbar^2}{2m}\sum_{i,j=1}^{N}\sum_{\mu,\nu=1}^{d} p_\mu^i b_{\mu\nu}^{ij}(\kappa) p_\nu^j, \tag{3. 5}$$

$$
\begin{aligned}
b^{ij}_{\mu\nu}(\kappa) &= \sum_{k=1}^{N}\sum_{\alpha=1}^{d}\left(\frac{\Delta^{jk}_{\nu\alpha}(\kappa)+\overline{\Delta^{jk}_{\nu\alpha}(\kappa)}}{2}\right)\left(\frac{\Delta^{ik}_{\mu\alpha}(\kappa)+\overline{\Delta^{ik}_{\mu\alpha}(\kappa)}}{2}\right),\\
E^{REN}(p,\kappa) &= E(p,\kappa)-\frac{\kappa^2\hbar^2}{2m}p^2-\widetilde{E}(p,\kappa).
\end{aligned}
$$

Note that since $(b^{ij}_{\mu\nu}(\kappa))_{1\leq i,j\leq N, 1\leq \mu,\nu\leq d}$ is nonnegative and symmetric $dN\times dN$ matrix, we have $\widetilde{E}(p,\kappa)\geq 0$ for any $p\in\mathbb{R}^{dN}$. We define

$$
\begin{aligned}
H^{REN}_{\vec{\rho}}(\kappa) &= -E^{REN}(D,\kappa)\otimes I+\kappa I\otimes H_b\\
&\quad+\frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left(-2\kappa e\hbar i D^j_\mu\otimes A_\mu(\rho_j)+e^2 I\otimes A_\mu(\rho_j)^2\right),\\
\widetilde{H_{\vec{\rho}}}(\kappa) &= \widetilde{E}(D,\kappa)\otimes I+\kappa I\otimes H_b,
\end{aligned}
$$

where $E^{REN}(D,\kappa)$ and $\widetilde{E}(D,\kappa)$ are pseudo differential operators on $L^2(\mathbb{R}^{dN})$ with symbols $E^{REN}(p,\kappa)$ and $\widetilde{E}(p,\kappa)$ respectively.

Theorem 3.2 *Suppose $\vec{\rho}\in\widetilde{P}$. Then $H^{REN}_{\vec{\rho}}(\kappa)$ and $\widetilde{H_{\vec{\rho}}}(\kappa)$ are essentially self-adjoint on any core of H_0 and bounded from below.*

Remark 3.3 *Write*

$$
E(p,\kappa)=\frac{\hbar^2\kappa^2}{2m}p^2+\sum_{\mu=1}^{d}\sum_{i=1}^{N}\frac{\hbar^2\kappa^2}{m}p^i_\mu\widetilde{p^i_\mu}(\kappa)+\sum_{\mu=1}^{d}\sum_{i=1}^{N}\frac{\hbar^2\kappa^2}{2m}\widetilde{p^i_\mu}(\kappa)^2+\square(\kappa). \tag{3. 6}
$$

Then the first and second terms on the right hand side of (3.6) diverge as $\kappa\to\infty$ for $p\neq 0$, but the rest terms not. Actually we see that

$$
\begin{aligned}
\lim_{\kappa\to\infty}\frac{\hbar^2\kappa^2}{2m}\sum_{\mu=1}^{d}\sum_{i=1}^{N}\widetilde{p^i_\mu}(\kappa)^2 &= \frac{1}{2m}\left(\frac{e^2}{2mc^2}\right)\left(\frac{d-1}{d}\right)^2\sum_{\alpha=1}^{d}\sum_{k=1}^{N}\left(\sum_{j=1}^{N}\hbar p^j_\alpha\left\langle\frac{\hat{\rho}_j}{\sqrt{\omega^3}},\frac{\hat{\rho}_k}{\sqrt{\omega}}\right\rangle_{L^2(\mathbb{R}^d)}\right)^2,\\
&\equiv E^\infty(p).
\end{aligned}
$$

Then, by (3.2), concerning an asymptotic behavior of $H_{\vec{\rho}}(\kappa)$ as $\kappa\to\infty$, we should subtract the first and second terms in the right hand side of (3.6) from the original Hamiltonian $H_{\vec{\rho}}(\kappa)$. However one can not say that $\widetilde{p^i_\mu}(\kappa)^2$ is real and nonnegative for any $p\in\mathbb{R}^{dN}$. To *guarantee the nonnegative self-adjointness of the Hamiltonian $H_{\vec{\rho}}^{REN}(\kappa)$ with the divergence terms subtracted, we should define $\tilde{E}(p,\kappa)$ such as (3.5). In this sense, we may say that the operator $H_{\vec{\rho}}^{REN}(\kappa)$ has an interpretation of the Hamiltonian $H_{\vec{\rho}}(\kappa)$ with the infinite self-energy of the nonrelativistic particles subtracted.*

We define

$$\mathcal{U}(\kappa) = \mathbf{F}^{-1}\left(\int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{U}(\kappa,p)dp\right)\mathbf{F}.$$

Then we have the following theorem.

Theorem 3.4 ([6]) *Suppose that $\vec{\rho}\in\tilde{P}$. Then*

$$\begin{aligned} s-\lim_{\kappa\to\infty}\mathcal{U}(\kappa) &= \exp\left(\sum_{j=1}^{N}\frac{e\hbar}{m}D_\mu^j\otimes\left\{a\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r\hat{\rho}_j}{\sqrt{2\hbar c^3\omega^3}}\right)-a^\dagger\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r\hat{\rho}_j}{\sqrt{2\hbar c^3\omega^3}}\right)\right\}\right), \\ &\equiv \mathcal{U}(\infty). \end{aligned}$$

We take scalar potentials V to be real-valued measurable functions on $\mathbb{R}^{dN}$ and put

$$C_\kappa(V)=\mathcal{U}^{-1}(\kappa)(V\otimes I)\mathcal{U}(\kappa),\quad C(V)=\mathcal{U}^{-1}(\infty)(V\otimes I)\mathcal{U}(\infty).$$

We introduce conditions $(\mathbf{V-1})$ and $(\mathbf{V-2})$ as follows.

(V-1) For sufficiently large $\kappa>0$, $D(\tilde{E}(D,\kappa))\subset D(V)$ and for $\lambda>0$, $V(\tilde{E}(D,\kappa)+\lambda)^{-1}$ is bounded with

$$\lim_{\lambda\to\infty}||V(\tilde{E}(D,\kappa)+\lambda)^{-1}||=0, \tag{3. 7}$$

where the convergence is uniform in sufficiently large $\kappa>0$.

(V-2) For $\lambda>0$, $V(\tilde{E}(D,\kappa)+\lambda)^{-1}$ is strongly continuous in κ and

$$s-\lim_{\kappa\to\infty}V(\tilde{E}(D,\kappa)+\lambda)^{-1}=V(E^\infty(D)+\lambda)^{-1}.$$

The condition (3.7) yields that, by the Kato-Rellich theorem and commutativity of $\mathcal{U}(\kappa)$ and $(\widetilde{E}(D,\kappa)+\lambda)^{-1}$, operators $\widetilde{E}(D,\kappa)\otimes I + C_\kappa(V)$ are essentially self-adjoint on any core of $D(\widetilde{E}(D,\kappa)\otimes I)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. Moreover since $I \otimes H_b$ is nonnegative and commute with $\widetilde{E}(D,\kappa)\otimes I$, one can see that

$$\widetilde{H_{\vec{\rho}}}(V,\kappa) \equiv \widetilde{E}(D,\kappa)\otimes I + C_\kappa(V) + \kappa I \otimes H_b$$

is essentially self-adjoint on any core of $D(\widetilde{E}(D,\kappa)\otimes I + \kappa I \otimes H_b)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. In particular, $D(H_0)$ is a core of $\widetilde{H_{\vec{\rho}}}(V,\kappa)$. Put

$$H_{\vec{\rho}}^{REN}(V,\kappa) \equiv H_{\vec{\rho}}^{REN}(\kappa) + V \otimes I.$$

Theorem 3.5 *Let $\vec{\rho} \in \widetilde{P}$. Suppose that V satisfies* $(\mathbf{V-1})$ *and* $(\mathbf{V-2})$*. Then, for sufficiently large $\kappa > 0$, the operator $H_{\vec{\rho}}^{REN}(V,\kappa)$ is essentially self-adjoint on $D(H_0)$ and bounded from below uniformly in sufficiently large $\kappa > 0$. Moreover the unitary operator $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself and for $z \in \mathbb{C}\setminus\mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,*

$$\left(H_{\vec{\rho}}^{REN}(V,\kappa) - z\right)^{-1} = \mathcal{U}(\kappa)\left(\widetilde{H_{\vec{\rho}}}(V,\kappa) - z\right)^{-1}\mathcal{U}^{-1}(\kappa). \tag{3. 8}$$

Proof: Since $\mathcal{U}(\kappa)$ maps $D(I\otimes H_b)$ onto itself (see Theorem 3.1) and $-\Delta\otimes I$ commutes with $\mathcal{U}(\kappa)$ on $D(-\Delta\otimes I)$, $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself. Put

$$S_0^\infty(\mathbb{R}^{dN}) = \left\{ f \in L^2(\mathbb{R}^{dN}) | \hat{f} \in C_0^\infty(\mathbb{R}^{dN}) \right\}.$$

At first , by Theorem 3.1, we see that for $\Phi \in S_0^\infty(\mathbb{R}^{dN})\hat{\otimes} D(H_b)$,

$$H_{\vec{\rho}}^{REN}(V,\kappa)\Phi = \mathcal{U}(\kappa)\widetilde{H_{\vec{\rho}}}(V,\kappa)\mathcal{U}^{-1}(\kappa)\Phi. \tag{3. 9}$$

By a limiting argument we can extend (3.9) to $\Phi \in D(H_0)$. Since $D(H_0)$ is a core of $\widetilde{H_{\vec{\rho}}}(V,\kappa)$ and $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself, the right hand side of (3.9) is essentially self-adjoint on $D(H_0)$. So is the left hand side of (3.9). (3.8) can be easily shown. □

We want to consider a scaling limit of $H_{\vec{p}}^{REN}(V,\kappa)$ as $\kappa \to \infty$. Let V satisfy $(\mathbf{V-1})$. Then since $D(C(V)) \supset D(-\Delta)\hat{\otimes}D(H_b)$, one can define, for $\Phi \in \mathcal{F}(\mathcal{W})$ and $\Psi \in D(H_b)$, a symmetric operator $E_{\Phi,\Psi}(C(V))$ with $D(E_{\Phi,\Psi}(C(V)) = D(-\Delta)$ by

$$\langle f, E_{\Phi,\Psi}(C(V))g\rangle_{L^2(\mathbb{R}^{dN})} = \langle f\otimes\Phi, C(V)(g\otimes\Psi)\rangle_{\mathcal{F}}, \quad f \in L^2(\mathbb{R}^{dN}), g \in D(-\Delta).$$

In particular, we call $E_{\Omega,\Omega}(C(V)) \equiv E_\Omega(C(V))$ "the partial expectation of $C(V)$ with respect to Ω".

Theorem 3.6 *Let $\vec{p} \in \tilde{P}$. Suppose that V satisfies the conditions $(\mathbf{V-1})$ and $(\mathbf{V-2})$. Then for $z \in \mathbb{C}\setminus\mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,*

$$s-\lim_{\kappa\to\infty}(H_{\vec{p}}^{REN}(V,\kappa)-z)^{-1} = \mathcal{U}(\infty)\left\{(E^\infty(D)+E_\Omega(C(V))-z)^{-1}\otimes P_0\right\}\mathcal{U}^{-1}(\infty), \tag{3. 10}$$

where P_0 is the projection from $\mathcal{F}(\mathcal{W})$ to the one dimensional subspace $\{\alpha\Omega|\alpha\in\mathbb{C}\}$.

Proof: By $(\mathbf{V-1})$ and $(\mathbf{V-2})$, we see that

(V-1)' For sufficiently large $\kappa > 0$, $D(\tilde{E}(D,\kappa)) \subset D(C_\kappa(V))$ and for $\lambda > 0$, $C_\kappa(V)(\tilde{E}(D,\kappa)+\lambda)^{-1}$ is bounded with

$$\lim_{\lambda\to\infty}||C_k(V)(\tilde{E}(D,\kappa)+\lambda)^{-1}|| = 0,$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V-2)' For $\lambda > 0$, $C_\kappa(V)(\tilde{E}(D,\kappa)+\lambda)^{-1}$ is strongly continuous in κ and

$$s-\lim_{\kappa\to\infty} C_k(V)(\tilde{E}(D,\kappa)+\lambda)^{-1} = C(V)(E^\infty(D)+\lambda)^{-1}.$$

From $(\mathbf{V-1})'$, $(\mathbf{V-2})'$ and iterating the second resolvent formula with respect to the pair $(\widetilde{H_{\vec{p}}}(\kappa), \widetilde{H_{\vec{p}}}(V,\kappa))$, it follows that

$$s-\lim_{\kappa\to\infty}\left(\widetilde{H_{\vec{p}}}(V,\kappa)-z\right)^{-1} = (E^\infty(D)\otimes I + (I\otimes P_0)C(V)(I\otimes P_0)-z)^{-1}I\otimes P_0.$$

Since

$$(I \otimes P_0)C(V)(I \otimes P_0) = E_\Omega(C(V)),$$

we see that

$$s - \lim_{\kappa\to\infty} \left(\widetilde{H_{\vec{\rho}}}(V,\kappa) - z\right)^{-1} = (E^\infty(D) + E_\Omega(C(V)) - z)^{-1} \otimes P_0.$$

Thus by Theorems 3.4 and 3.5, we get (3.10). □

We want to see $E_\Omega(C(V))$ more explicitly. For $\vec{\rho} \in \widetilde{P}$, let $\widetilde{\Delta}^\infty = (\widetilde{\Delta}^\infty_{ij})_{1\le i,j\le d}$, where

$$\widetilde{\Delta}^\infty_{ij} = \frac{1}{2}\frac{d-1}{d}\left(\frac{\hbar}{mc}\right)^2 \frac{e^2}{\hbar c}\int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k)\hat{\rho}_j(k)}{\omega(k)^3}.$$

Let $\mathbf{I}_{d\times d}$ denote $d \times d$-identity matrix. Since $\Delta^\infty \equiv \widetilde{\Delta}^\infty \otimes \mathbf{I}_{d\times d}$ is a nonnegative symmetric matrix, there exist unitary matrices $\mathbf{T}$ so that

$$\mathbf{T}\Delta^\infty\mathbf{T}^{-1} = \begin{pmatrix} \lambda_1\mathbf{I}_{d\times d} & & & \\ & \lambda_2\mathbf{I}_{d\times d} & & \\ & & \ddots & \\ & & & \lambda_N\mathbf{I}_{d\times d} \end{pmatrix}, \tag{3. 11}$$

where $\lambda_1 \ge \lambda_2 ... \ge \lambda_N \ge 0$.

Theorem 3.7 *Suppose $\lambda_1 \ge \lambda_2 ... \ge \lambda_K > 0$, $\lambda_{K+1} = ... = \lambda_N = 0$ and fix a unitary operator $\mathbf{T}$ in (3.11). Let $x = (x_1, ..., x_N)$, $x_j \in \mathbb{R}^d$, $j = 1, ..., N$ and V satisfy*

$$\int_{\mathbb{R}^{dK}} dy_1...dy_K |V| \circ \mathbf{T}^{-1}(y_1, .., y_K, (\mathbf{T}x)_{K+1}, .., (\mathbf{T}x)_N) \exp\left(-\frac{\sum_{j=1}^K |(\mathbf{T}x)_j - y_j|^2}{2\lambda_1...\lambda_K}\right) < \infty. \tag{3. 12}$$

Moreover we suppose that the left hand side of (3.12) is locally bounded. Then the partial expectation $E_\Omega(C(V))$ is given by a multiplication operator V_{eff};

$$\begin{aligned} V_{eff}(x) &= (2\pi\lambda_1...\lambda_K)^{-\frac{d}{2}} \int_{\mathbb{R}^{dK}} dy_1...dy_K V \circ \mathbf{T}^{-1}(y_1, ..., y_K, (\mathbf{T}x)_{K+1}, ..., (\mathbf{T}x)_N) \\ &\quad \times \exp\left(-\frac{\sum_{j=1}^K |(\mathbf{T}x)_j - y_j|^2}{2\lambda_1...\lambda_K}\right). \end{aligned}$$

In particular, in the case where $\tilde{\Delta}^\infty$ is non-degenerate, V_{eff} is given by

$$V_{eff}(x) = (2\pi \det \tilde{\Delta}^\infty)^{-\frac{d}{2}} \int_{\mathbb{R}^{dN}} V(y) \exp\left(-\frac{|x-y|^2}{2\det \tilde{\Delta}^\infty}\right) dy.$$

Proof: Suppose $V \in \mathcal{S}(\mathbb{R}^{dN})$, which is the set of the rapidly decreasing infinitely continuously differentiable functions on $\mathbb{R}^{dN}$. Then the direct calculation shows that for $f, g \in L^2(\mathbb{R}^{dN})$

$$\langle f, E_\Omega(C(V))g\rangle_{L^2(\mathbb{R}^{dN})} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dk \bar{f}(x) g(x) e^{ikx} \hat{V}(k) e^{-\frac{1}{2}\sum_{\mu=1}^{d}\sum_{ij=1}^{N} \Delta_{ij}^\infty k_\mu^i k_\mu^j}.$$

Hence we have

$$\langle f, E_\Omega(C(V))g\rangle_{L^2(\mathbb{R}^{dN})} = \langle f, V_{eff} g\rangle_{L^2(\mathbb{R}^{dN})}. \tag{3. 13}$$

We next consider the case where V is bounded. In this case we can approximate V by a sequence $\{V_n\}_{n=1}^\infty$, $V_n \in \mathcal{S}(\mathbb{R}^{dN})$, such that

$$||V - V_n||_\infty \to 0 \ (n \to \infty),$$

where $||\cdot||_\infty$ denotes the sup norm. Then we have

$$E_\Omega(C(V_n)) \to E_\Omega(C(V)) \ (n \to \infty),$$

strongly. Moreover $(V_n)_{eff}(x) \to V_{eff}(x)$ for all $x \in \mathbb{R}^{dN}$. Thus for $f, g \in L^2(\mathbb{R}^{dN})$, (3.13) follows for such V. Finally, let V satisfy (3.12). Define

$$V_n = \begin{cases} V(x) & |V(x)| \le n, \\ n & |V(x)| > n. \end{cases}$$

Hence for $f \in L^2(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$, we have

$$\langle f, E_\Omega(C(V_n))g\rangle_{L^2(\mathbb{R}^{dN})} \to \langle f, E_\Omega(C(V))g\rangle_{L^2(\mathbb{R}^{dN})} \ (n \to \infty).$$

On the other hand, since the left hand side of (3.12) is locally bounded, we can see that for $f \in C_0^\infty(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$,

$$\langle f, (V_n)_{eff} g\rangle_{L^2(\mathbb{R}^{dN})} \to \langle f, V_{eff} g\rangle_{L^2(\mathbb{R}^{dN})} \ (n \to \infty),$$

which completes the proof. □

Remark 3.8 *In Theorem 3.7, in the case where* $\widetilde{\Delta}^\infty$ *is non-degenerate, since the left hand side of (3.12) is continuous in* $x \in \mathbb{R}^{dN}$, *it is necessarily locally bounded.*

We call V_{eff} "the effective potential with respect to V". We give a typical example of scalar potentials V and ultraviolet cut-off functions $\vec{\rho}$.

Example 3.9 *Let*

$$\widetilde{\Delta}_{ij}^\infty = \delta_{ij}\frac{1}{2}\frac{d-1}{d}\left(\frac{\hbar}{mc}\right)^2\frac{e^2}{\hbar c}\int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k)^2}{\omega(k)^3}.$$

Then there exist positive constants δ_1 *and* δ_2 *such that for sufficiently large* $\kappa > 0$

$$\delta_1 |p|^2 \leq \widetilde{E}(p,\kappa) \leq \delta_2 |p|^2. \tag{3. 14}$$

Let $d = 3$ *and* V *be the Coulomb potential;*

$$V(x_1, ..., x_N) = -\sum_{j=1}^N \frac{\alpha_j}{|x_j|} + \sum_{i \neq j} \frac{\beta_{ij}}{|x_i - x_j|}, \quad \alpha_j \geq 0, \beta_{ij} \geq 0.$$

Then V *is the Kato class potential ([10],Theorem X.16). Namely for any* $\epsilon > 0$, *there exists* $b \geq 0$ *such that* $D(V) \supset D(\ \Delta)$ *and*

$$||V\Phi||_{L^2(\mathbb{R}^{3N})} \leq \epsilon|| - \Delta\Phi||_{L^2(\mathbb{R}^{3N})} + b||\Phi||_{L^2(\mathbb{R}^{3N})}. \tag{3. 15}$$

Together with (3.14) and (3.15), one can see that V *satisfies* $(\mathbf{V-1})$, $(\mathbf{V-2})$ *and for any* $t > 0$

$$\int_{\mathbb{R}^{3d}} |V|(y) e^{-t|x-y|^2} dy < \infty.$$

Then the scaling limit of the Pauli-Fierz Hamiltonian with the Coulomb potential exists and has the effective potential given by

$$\begin{aligned} V_{eff}(x) &= (2\pi\gamma)^{-\frac{3}{2}} \int_{\mathbb{R}^{3N}} V(y) e^{-\frac{|x-y|^2}{2\gamma}} dy, \\ \gamma &= \left\{\frac{1}{3}\left(\frac{\hbar}{mc}\right)^2 \frac{e^2}{\hbar c}\right\}^N \Pi_{j=1}^N \left(\int_{\mathbb{R}^3} dk \frac{\hat{\rho}_j^2(k)}{\omega(k)^3}\right). \end{aligned}$$

4 CONCLUDING REMARK

As is seen in Theorem 3.7, the effective potential V_{eff} is characterized by the matrix-valued functional $\widetilde{\Delta}^{\infty} = \widetilde{\Delta}^{\infty}(\vec{\rho})$, which has the following mathematical meaning; putting

$$\mathcal{U}(\infty)(x_i \otimes I)\mathcal{U}^{-1}(\infty) - x_i \otimes I \equiv \Delta x_i, \quad i = 1, ..., N,$$

we see that the partial expectation of $\Delta x_i \Delta x_j$ with respect to Ω is as follows;

$$E_{\Omega}[(\Delta x_i \Delta x_j)] = \widetilde{\Delta}^{\infty}_{ij}(\vec{\rho})I.$$

In one-nonrelativistic particle case, the author in [5] show that the partial expectation $E_{\Omega}[(\Delta x)^2]$ with respect to Ω may be interpreted as the mean square fluctuation in position of one-nonrelativistic particle ([2]). In this sense, $\widetilde{\Delta}^{\infty}_{ij}(\vec{\rho})$ may also be interpreted as correlation of fluctuations in position of the i-th and the j-th nonrelativistic particles under the action of quantized radiation fields.

5 REFERENCES

[1]H.A.Bethe, The electromagnetic shift energy levels, Phys.Rev.72,(1947)339-342,

[2] T.A.Welton, Some observable effects of the quantum mechanical fluctuations of the electromagnetic field, Phys.Rev.74(1948)1157-1167,

[3] A.Arai, A note on scattering theory in non-relativistic quantum electrodynamics, J.Phys.A.Math.Gen.16,(1983)49-70,

[4] A.Arai, Rigorous theory of spectra and radiation for a model in a quantum electrodynamics, J.Math.Phys.24(1983)1896-1910,

[5] A.Arai, An asymptotic analysis and its applications to the nonrelativistic limit of the Pauli-Fierz and a spin-boson model, J.Math.Phys.31(1990)2653-2663,

[6] F.Hiroshima, Scaling limit of a model in quantum electrodynamics, J..Math.Phys.34(1993) 4478-4578,

[7] F.Hiroshima, Diamagnetic inequalities for a systems of nonrelativistic particles with a quantized radiation field. to appear Rev.Math.Phys. ,

[8] F.Hiroshima, Functional integral representation of a model in quantum electrodynamics, submitted to J.Funct.Anal.,

[9] F.Hiroshima, A scaling limit of a model in quantum electrodynamics with many nonrelativistic particles, preprint,

[10] M.Reed, B.Simon, Method of Modern Mathematical Physics II Fourier Analysis and Self-Adjoint operator, Academic Press(1975).