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The Functorial Construction of the Sheaf of Small 2-microfunctions

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1 Introduction

The theory of second microlocalization along regular involutive submanifolds was begun by M. Kashiwara and J. M. Bony. M. Kashiwara has constructed the sheaf $\mathcal{C}_V^2$ of 2-microfunctions microlocalizing the sheaf $\mathcal{O}_X$ of germs of holomorphic functions two times. (Refer to Kashiwara–Laurent [3].) Since this sheaf is too large to decompose second microlocal singularities of microfunctions, Kataoka–Tose [7] and Kataoka–Okada–Tose [6] introduced a new sheaf what is called the sheaf of *small 2-microfunctions.* Schapira–Takeuchi [9] also constructed functorially the same sheaf defining a bimicrolocalization functor. Here we will also give another functorial construction, that is the idea of K. Kataoka, for the purpose of studying microfunction solutions for some degenerate elliptic operators.

We give this construction of the sheaf $\hat{\mathcal{C}}_V^2$ of small 2-microfunctions in a simple way in chapter 2.

In chapter 3 we give a support theorem, that is, when a regular involutive submanifold V is defined by

$$V = \left\{(x; \sqrt{-1}\xi \cdot dx) \in \sqrt{-1}\mathring{T}^*\mathbf{R}^n; \xi_1 = \cdots = \xi_{n-1} = 0\right\}, \tag{1.1}$$

we study a simple sufficient condition on which solution complexes in $\hat{\mathcal{C}}_V^2$ vanish locally in the derived category.

As application we obtain results of solvability in the sheaf $\mathcal{C}_M$ of microfunctions for linear differential operators of the form

$$P(x, D_x) = D_{x_1}^2 + D_{x_2}^2 + \cdots + D_{x_{n-1}}^2 + x_n^{2k} D_{x_n}^2 + (lower) \tag{1.2}$$

with $D_{x_j} = \dfrac{\partial}{\partial x_j}$ $(1 \leq j \leq n)$, $k \in \mathbf{N}$.

Though these operators $P(x, D_x)$ are not partially elliptic along V on $\{x_n = 0\}$, we will prove that the operators P induce isomorphisms

$$P : \hat{\mathcal{C}}^2_V \xrightarrow{\sim} \hat{\mathcal{C}}^2_V. \tag{1.3}$$

As a general rule, furthermore, the same isomorphisms hold for linear differential operators defined by

$$P(x, D_x) = Q(x, D_{x'}) + x_n^{2k} D_{x_n}^2 + (lower), \tag{1.4}$$

where $x' = (x_1, \ldots, x_{n-1})$, $k \in \mathbf{N}$, and Q is a second order differential operator satisfying the property

$$\operatorname{Re} \sigma(Q)(x; \xi') > 0 \quad (\forall x \in \mathbf{R}^n,\ \forall \xi' \in \mathbf{R}^{n-1} \setminus \{0\}). \tag{1.5}$$

See chapter 4.

2 The sheaf $\hat{\mathcal{C}}^2_V$

2.1 2-microlocal analysis

Let M be an open subset of $\mathbf{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$ and X a complex neighborhood of M in $\mathbf{C}^n$ with coordinates $z = (z_1, \ldots, z_n)$. Let (z, ζ) be the associated coordinates on T^*X and $z = x + \sqrt{-1}y$, $\zeta = \xi + \sqrt{-1}\eta$. Then $(x; \sqrt{-1}\xi \cdot dx)$ denotes a point of $T^*_M X (\simeq \sqrt{-1}T^*M)$ with $\xi \in \mathbf{R}^n$. Let V be the following regular involutive submanifold of $\overset{\circ}{T}{}^*_M X (= T^*_M X \setminus M)$:

$$V = \left\{(x; \sqrt{-1}\xi \cdot dx) \in \overset{\circ}{T}{}^*_M X; \xi_1 = \cdots = \xi_d = 0\right\} \qquad (1 \leq d < n). \tag{2.1}$$

We put

$$x = (x', x''), \quad x' = (x_1, \ldots, x_d), \quad x'' = (x_{d+1}, \ldots, x_n), \tag{2.2}$$

etc. We set, moreover,

$$N = \{z \in X; \operatorname{Im} z'' = 0\}, \tag{2.3}$$

$$\widetilde{V} = T^*_N X. \tag{2.4}$$

This space $\widetilde{V}$ is called a partial complexification of V. It is equipped with the sheaf

$$\mathcal{C}_{\widetilde{V}} = \mu_N(\mathcal{O}_X)[n - d] \tag{2.5}$$

of microfunctions with holomorphic parameters z', where μ_N denotes the functor of Sato's microlocalization along N. Refer to Kashiwara–Schapira [4]. M. Kashiwara constructed the sheaf $\mathcal{C}_V^2$ of 2-microfunctions along V on $T_V^*\widetilde{V}$ from $\mathcal{C}_{\widetilde{V}}$ by

$$\mathcal{C}_V^2 = \mu_V(\mathcal{C}_{\widetilde{V}})[d]. \tag{2.6}$$

We also define

$$\mathcal{A}_V^2 = \mathcal{C}_{\widetilde{V}}|_V, \tag{2.7}$$

$$\mathcal{B}_V^2 = R\Gamma_V(\mathcal{C}_{\widetilde{V}})[d] = \mathcal{C}_V^2|_V. \tag{2.8}$$

We call $\mathcal{B}_V^2$ the sheaf of 2-hyperfunctions along V. Note that these complexes $\mathcal{C}_{\widetilde{V}}$, $\mathcal{C}_V^2$ and $\mathcal{B}_V^2$ are concentrated in degree 0.

There are fundamental exact sequences concerning $\mathcal{C}_V^2$. On V,

$$0 \longrightarrow \mathcal{A}_V^2 \longrightarrow \mathcal{B}_V^2 \longrightarrow \mathring{\pi}_{V*}\left(\mathcal{C}_V^2|_{\mathring{T}_V^*\widetilde{V}}\right) \longrightarrow 0, \tag{2.9}$$

$$0 \longrightarrow \mathcal{C}_M|_V \longrightarrow \mathcal{B}_V^2, \tag{2.10}$$

where $\mathring{\pi}_V$ is the restriction of the projection $\pi_V : T_V^*\widetilde{V} \to V$ to $\mathring{T}_V^*\widetilde{V}$, and $\mathcal{C}_M(=\mu_M(\mathcal{O}_X)[n])$ is the sheaf of Sato microfunctions on M.

Moreover there exists a canonical spectrum map

$$\mathrm{Sp}_V^2 : \pi_V^{-1}\mathcal{B}_V^2 \longrightarrow \mathcal{C}_V^2 \tag{2.11}$$

on $T_V^*\widetilde{V}$. By using Sp_V^2 we define

$$\mathrm{SS}_V^2(u) = \mathrm{supp}\left(\mathrm{Sp}_V^2(u)\right) \tag{2.12}$$

for $u \in \mathcal{B}_V^2$. This subset $\mathrm{SS}_V^2(u)$ is called the second singular spectrum of u along V.

Refer to Kashiwara–Laurent [3] for more details.

From the exact sequence (2.9) $\mathcal{C}_V^2$ is the sheaf which means second microlocal analytic singularities of elements of $\mathcal{B}_V^2$. This sheaf $\mathcal{C}_V^2$ is too large to study microfunction solutions for some differential equations because of (2.10).

For this reason Kataoka–Tose [7] constructed the subsheaf $\mathring{\mathcal{C}}_V^2$ of $\mathcal{C}_V^2|_{\mathring{T}_V^*\widetilde{V}}$ with the comonoidal transform to get a exact sequence

$$0 \longrightarrow \mathcal{A}_V^2 \longrightarrow \mathcal{C}_M|_V \longrightarrow \mathring{\pi}_{V*}\left(\mathring{\mathcal{C}}_V^2\right) \longrightarrow 0. \tag{2.13}$$

On the other hand Kataoka-Okada–Tose [6] constructed the same sheaf $\tilde{\mathcal{C}}_V^2$ as the image sheaf of morphisms

$$\mathring{\pi}_V^{-1}(\mathcal{C}_M|_V) \longrightarrow \mathring{\pi}_V^{-1}(\mathcal{B}_V^2) \longrightarrow \mathcal{C}_V^2|_{\mathring{T}_V^*\widetilde{V}}. \tag{2.14}$$

Schapira–Takeuchi [9] also constructed the same sheaf

$$\mathcal{C}_{MN} = \mu_{MN}(\mathcal{O}_X)[n] \tag{2.15}$$

with the functor of Schapira–Takeuchi's bimicrolocalization.

2.2 The sheaf of small 2-microfunctions

Here we give another functorial construction in order to estimate the support of solution complexes in its sheaf. This construction is the idea of K. Kataoka. First of all we set

$$\begin{aligned} \widetilde{X} &= X \times (\mathbf{R}^d \setminus \{0\}), &(2.16)\\ H_c &= \left\{(z,\xi') \in \widetilde{X}; \langle \operatorname{Im} z', \xi'\rangle \le c|\operatorname{Im} z''|\right\}, &(2.17)\\ G &= \left\{(z,\xi') \in \widetilde{X}; \operatorname{Im} z'' = 0\right\} &(2.18)\end{aligned}$$

for $c > 0$. We identify

$$\left\{(z', x'', \xi'; \sqrt{-1}\xi'' \cdot dx'') \in \overset{\circ}{T}{}^*_G\widetilde{X}; \operatorname{Im} z' = 0\right\} \tag{2.19}$$

with $\overset{\circ}{T}{}^*_{\widetilde{V}}\widetilde{V}$ through the correspondence

$$(x, \xi'; \sqrt{-1}\xi'' \cdot dx'') \longleftrightarrow (x; \sqrt{-1}\xi'' \cdot dx''; \sqrt{-1}\xi' \cdot dx'). \tag{2.20}$$

Definition 2.1 (small 2-microfunction) *One sets*

$$\hat{\mathcal{C}}^2_V = \varinjlim_{c} H^n\left(\left(\mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right)\Big|_{\overset{\circ}{T}{}^*_V\widetilde{V}}\right) \tag{2.21}$$

on $\overset{\circ}{T}{}^*_V\widetilde{V}$, *where* $p : \widetilde{X} \to X$. *One calls* $\hat{\mathcal{C}}^2_V$ *the sheaf of small 2-microfunctions along* V.

We can find from the next theorem that this sheaf $\hat{\mathcal{C}}^2_V$ coincides with $\mathcal{C}_{MN}$ of Schapira–Takeuchi. Therefore $\hat{\mathcal{C}}^2_V$ is the sheaf which means second microlocal analytic singularities of microfunctions, that is to say, we have

$$0 \longrightarrow \mathcal{A}^2_V \longrightarrow \mathcal{C}_M|_V \longrightarrow \overset{\circ}{\pi}_{V*}(\hat{\mathcal{C}}^2_V) \longrightarrow 0. \tag{2.22}$$

Theorem 2.2 *Let* $q_\circ = (x_\circ; \sqrt{-1}\xi''_\circ \cdot dx''; \sqrt{-1}\xi'_\circ \cdot dx') \in \overset{\circ}{T}{}^*_V\widetilde{V}$. *Then we have*

$$\hat{\mathcal{C}}^2_{V,q_\circ} = \varinjlim_{Z} H^n_Z(\mathcal{O}_X)_{x_\circ}. \tag{2.23}$$

Here Z ranges through the family of closed subsets of X such that

$$Z = M + \sqrt{-1}(\Gamma + (\Gamma' \times \{\mathrm{o}\})) \tag{2.24}$$

and Γ (resp. Γ') is closed convex cone with the vertex 0 in $\mathbf{R}^n$ (resp. $\mathbf{R}^d$) satisfying the properties

$$\begin{aligned} \Gamma &\subset \{(y', y'') \in \mathbf{R}^n; \langle y'', \xi''_\circ\rangle < 0\} \cup \{0\}, & (2.25)\\ \Gamma' &\subset \{y' \in \mathbf{R}^d; \langle y', \xi'_\circ\rangle < 0\} \cup \{0\}. & (2.26)\end{aligned}$$

It suffices to prove the above theorem, since the same result holds as to the sheaf $\mathcal{C}_{MN}$. (Refer to Schapira–Takeuchi [9].) In the same manner in Kataoka [5], we get the following proposition.

Proposition 2.3 *In the preceding situation of Definition 2.1, let $p : \widetilde{X} \to X$ be a projection and V an open subset of $\widetilde{X}$ with convex fibers. Assume that there exists a compact subset K of $\mathbf{R}^d \setminus \{0\}$ such that $V \subset X \times K$. Then for any sheaf F on X and any $q \in \mathbf{Z}$*

$$H^q(V, p^{-1}F) \simeq H^q(p(V), F). \tag{2.27}$$

Proof of Theorem 2.2. We assume that $q_\circ = (x_\circ; \sqrt{-1}dx_n; \sqrt{-1}dx_1)$ for the sake of simplicity. The stalk of $\hat{\mathcal{C}}^2_V$ at $q_\circ$ is described as

$$\begin{aligned} \hat{\mathcal{C}}^2_{V, q_\circ} &\simeq \varinjlim_{c} H^n\left(\mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right)_{q_\circ} \\ &= \varinjlim_{c, \widetilde{Z}, \pi(q_\circ)\in\widetilde{U}} H^n_{\widetilde{Z}\cap H_c\cap\widetilde{U}}\left(\widetilde{U}, p^{-1}\mathcal{O}_X\right). \end{aligned} \tag{2.28}$$

Here π denotes the projection $\pi : T^*_G\widetilde{X} \to G$ and $\widetilde{Z}$ ranges through the family of closed subsets of $\widetilde{X}$ such that

$$C_G\left(\widetilde{Z}\right)_{\pi(q_\circ)} \subset \left\{v \in T_G\widetilde{X}; \langle v, q_\circ\rangle > 0\right\} \cup \{0\}, \tag{2.29}$$

and $\widetilde{U}$ through the family of open neighborhoods of $\pi(q_\circ) = (x_\circ, 1, 0, \ldots, 0)$ in $\widetilde{X}$. Refer to Kashiwara–Schapira [4] for the notion of normal cones.

Note that there exists the following exact sequence,

$$0 \longrightarrow \varinjlim H^{j-1}\left(\widetilde{U} \setminus \left(\widetilde{Z} \cap H_c\right), p^{-1}\mathcal{O}_X\right) \longrightarrow \varinjlim H^j_{\widetilde{Z}\cap H_c\cap\widetilde{U}}\left(\widetilde{U}, p^{-1}\mathcal{O}_X\right) \longrightarrow 0, \tag{2.30}$$

for $j \geq 2$.

One easily checks that the right-hand side of (2.28) is equal to that of (2.23) by using Proposition 2.3. $\square$

Remark 2.4 Assume $d+1=n$. In this case, we find that

$$H^j\left(\left(\mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right)\Big|_{\overset{\circ}{T^*_V}\tilde{V}}\right)=0 \qquad (j\neq n) \tag{2.31}$$

from the theorem of Edge of the Wedge.

3 A vanishing theorem of solution complexes in the sheaf $\hat{\mathcal{C}}^2_V$

3.1 Microlocal study of sheaves

In this section, we recall some notation on microlocal study of sheaves. (Refer to Kashiwara–Schapira [4].) Let X be a real manifold and A a unitary ring. We denote by $\mathbf{D}(X)$ the derived category of the abelian category of sheaves of A-modules on X and $\mathbf{D}^+(X)$ denotes the full subcategory of $\mathbf{D}(X)$ consisting of complexes with cohomology bounded from below. Let $F\in \mathrm{Ob}(\mathbf{D}^+(X))$. Then $\mathrm{SS}(F)$ denotes the micro-support of F. We quote some important formulae on the micro-support under several operations on sheaves.

Let Z be a locally closed subset of X, and $G\in\mathrm{Ob}(\mathbf{D}^+(X))$. Then

$$\mathrm{SS}(R\Gamma_Z(G))\subset \mathrm{SS}(G)\hat{+}\mathrm{SS}(\mathbf{Z}_Z)^a. \tag{3.1}$$

Here $\mathrm{SS}(\mathbf{Z}_Z)^a$ is the image of $\mathrm{SS}(\mathbf{Z}_Z)$ by the antipodal map $a: T^*X\to T^*X$, $(x;\xi)\mapsto(x;-\xi)$, and $\mathbf{Z}_Z$ is the zero sheaf on $X\setminus Z$ and the constant sheaf with the stalk $\mathbf{Z}$ on Z.

Now we describe the set of the right-hand side of (3.1) with a system of local coordinates. For two conic subsets A, B of T^*X , the subset $A\hat{+}B$ is defined. Let $(x_\circ;\xi_\circ)\in T^*X$, $\xi_\circ\neq 0$. Then $(x_\circ;\xi_\circ)$ does not belong to $A\hat{+}B$ if and only if there exists a positive number δ such that

$$\Big\{(x+\varepsilon y;\frac{x^*}{\varepsilon}+y^*)\in T^*X;(x;x^*)\in B^a,$$
$$|x-x_\circ|+|x^*|+|y|+|y^*-\xi_\circ|<\delta,\quad 0<\varepsilon<\delta\Big\}\cap A=\emptyset. \tag{3.2}$$

Next let Y and X be manifolds, and assume that $f:Y\to X$ is smooth. For $F\in\mathrm{Ob}(\mathbf{D}^+(X))$ one has

$$\mathrm{SS}(f^{-1}F)=\rho(\varpi^{-1}(\mathrm{SS}(F))), \tag{3.3}$$

where ρ, ϖ are the natural maps associated to f, from $Y \underset{X}{\times} T^*X$ to T^*Y and T^*X respectively:

$$T^*X \xleftarrow{\varpi} Y \underset{X}{\times} T^*X \xrightarrow{\rho} T^*Y. \tag{3.4}$$

Moreover, let Z be a closed subset of Y. Then:

$$\mathrm{SS}(\mathbf{Z}_Z) \subset N^*(Z). \tag{3.5}$$

Here $N^*(Z)$ is the conormal cone to Z.

3.2 A vanishing theorem of solution complexes in the sheaf $\hat{\mathcal{C}}^2_V$

In this, and all forthcoming sections unless otherwise specified, we assume that $d+1 = n$, that is to say, a regular involutive submanifold V is defined by $\xi_1 = \cdots = \xi_{n-1} = 0$.

In order to study microfunction solutions for linear differential equations, we shall send them to the sheaf $\hat{\mathcal{C}}^2_V$ through the morphism (2.22) and reduce the results to that in the sheaf $\mathcal{C}_M$. Here using the construction of the sheaf $\hat{\mathcal{C}}^2_V$ that we have given in chapter 2, we have an estimate of the support of solution complexes in $\hat{\mathcal{C}}^2_V$. We prove this theorem by means of the micro-support.

We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order holomorphic differential operators on X. Let $\mathcal{M}$ be an arbitrary coherent $\mathcal{D}_X$-module, and we denote by $\mathrm{char}(\mathcal{M})$ the characteristic variety of $\mathcal{M}$.

Theorem 3.1 *Let* $q_\circ = (x_\circ; \pm\sqrt{-1}dx_n; \sqrt{-1}\eta'_\circ \cdot dx') \in \overset{\circ}{T}{}^*_V\tilde{V}$. *Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \hat{\mathcal{C}}^2_V)_{q_\circ} = 0, \tag{3.6}$$

if there exists a positive number δ *such that*

$$\Big\{(z; (\xi' + \sqrt{-1}\varepsilon\eta') \cdot dz' \pm (\xi_n + \sqrt{-1}) \cdot dz_n) \in T^*X; \\ |z - x_\circ| + |\eta' - \eta'_\circ| < \delta, \quad |\mathrm{Im}\, z_n| + |\xi| < \varepsilon\delta\Big\} \cap \mathrm{char}(\mathcal{M}) = \emptyset \tag{3.7}$$

for any ε *with* $0 < \varepsilon < \delta$.

Remark 3.2 In the situation of Theorem 3.1, one gets not only (3.6) but also

$$q_\circ \notin \mathrm{supp}\left(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \hat{\mathcal{C}}^2_V)\right) \tag{3.8}$$

if the same condition (3.7) holds.

Proof of Theorem 3.1. We may assume that $q_\circ = (x_\circ; \sqrt{-1}dx_n; \sqrt{-1}\eta'_\circ \cdot dx') \in \overset{\circ}{T}{}^*_{\widetilde{V}}\widetilde{V}$ by a coordinate transformation. Using Remark 2.4, one has:

$$\begin{aligned}
\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \hat{\mathcal{C}}^2_{\widetilde{V}})_{q_\circ} &= \varinjlim_c H^j R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \left(\mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right)\Big|_{\overset{\circ}{T}{}^*_{\widetilde{V}}\widetilde{V}}[n]\right)_{q_\circ} \\
&= \varinjlim_c H^{j+n}\mu_G R\Gamma_{H_c}(p^{-1}F)_{q_\circ} \\
&\simeq \varinjlim_c H^{j+n} R\Gamma_{\widetilde{Z}\cap H_c}(p^{-1}F)_{(x_\circ,\eta'_\circ)},
\end{aligned} \tag{3.9}$$

similarly as we did in Proof of Theorem 2.2. Here we set $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ and $\widetilde{Z} = \left\{(z,\xi') \in \widetilde{X}; y_n \leq 0\right\}$.

Hence the j-th cohomology group (3.9) on $\mathcal{M}$ vanishes at $q_\circ$ for all $j \in \mathbf{Z}$ provided that there exists a positive number $c_0 > 1$ such that

$$R\Gamma_{\widetilde{Z}\cap H_c}(p^{-1}F)_{(x_\circ,\eta'_\circ)} \simeq 0 \tag{3.10}$$

for any $c > c_0$. Therefore it suffices to study a sufficient condition in order that we have (3.10).

On the other hand, we define a real analytic function f_c on $\widetilde{X} = X \times (\mathbf{R}^d \setminus \{0\})$ by

$$f_c(z,\xi') = -c \cdot y_n - \langle y', \xi' \rangle. \tag{3.11}$$

Assume that

$$(x_\circ, \eta'_\circ; df_c(x_\circ, \eta'_\circ)) \notin \mathrm{SS}\left(R\Gamma_{\widetilde{Z}}(p^{-1}F)\right), \tag{3.12}$$

and we find that (3.10) holds by the definition of the micro-support and the fact that $f_c(x_\circ, \eta'_\circ) = 0$.

In this way we have been able to reduce the condition on the vanishing of the cohomology groups to that on the micro-support (3.12). It suffices to estimate the micro-support $\mathrm{SS}\left(R\Gamma_{\widetilde{Z}}(p^{-1}F)\right)$.

Applying the estimates on the micro-support in section 3.1, we can easily obtain the needed expression (3.7) by the formula:

$$\mathrm{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = \mathrm{char}(\mathcal{M}). \tag{3.13}$$

□

Remark 3.3 K. Takeuchi also got the same result as Theorem 3.1 at the same time.

4 Application

In this chapter, applying the support theorem in section 3.2 to the linear differential operators in the introduction, we argue the structure of solutions. In particular, we study its solvability in the sheaf of microfunctions.

In a general way, we take V and $\widetilde{V}$ as in section 2.1, and the following regular involutive submanifold of $\overset{\circ}{T}{}^*X$.

$$V^{\mathbf{C}} = \left\{(z; \zeta \cdot dz) \in \overset{\circ}{T}{}^*X; \zeta' = 0\right\}. \tag{4.1}$$

This space $V^{\mathbf{C}}$ is a complexification of V. We identify X with the diagonal set $\Delta_X = \{(z, w) \in X \times X; z = w\}$ of $X \times X$. Then there exist natural injections:

$$T^*X \simeq T^*_{\Delta_X}(X \times X) \hookrightarrow T^*(X \times X), \tag{4.2}$$

$$V^{\mathbf{C}} \hookrightarrow V^{\mathbf{C}} \times V^{\mathbf{C}}. \tag{4.3}$$

The space $\widetilde{V}^{\mathbf{C}}$ denotes the union of bicharacteristic leaves of $V^{\mathbf{C}} \times V^{\mathbf{C}}$ which pass through $V^{\mathbf{C}}$, that is,

$$\widetilde{V}^{\mathbf{C}} = \left\{(z, w; \zeta \cdot dz + \theta \cdot dw) \in \overset{\circ}{T}{}^*(X \times X); z'' = w'', \zeta' = \theta' = 0, \zeta'' + \theta'' = 0\right\}. \tag{4.4}$$

We remark that $T^*_{V^{\mathbf{C}}}\widetilde{V}^{\mathbf{C}}$ is a complexification of $T^*_V\widetilde{V}$.

We denote by $\mathcal{E}^2_{V^{\mathbf{C}}}$ the sheaf of rings of 2-microdifferential operators along $V^{\mathbf{C}}$ and $\sigma_{V^{\mathbf{C}}}(P)$ the principal symbol of a 2-microdifferential operator P. Let U be an open subset of $T^*_{V^{\mathbf{C}}}\widetilde{V}^{\mathbf{C}}$. Then, for a 2-microdifferential operator $P \in \mathcal{E}^2_{V^{\mathbf{C}}}(U)$, P is invertible on U if and only if $\sigma_{V^{\mathbf{C}}}(P) \neq 0$ on U.

We denote, moreover, by $\mathcal{E}_X$ the sheaf of rings of microdifferential operators on T^*X. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined on a neighborhood of a point of V. One says that $\mathcal{M}$ is *partially elliptic* along V if:

$$\mathrm{Ch}^2_{V^{\mathbf{C}}}(\mathcal{M}) \cap \overset{\circ}{T}{}^*_V\widetilde{V} = \emptyset. \tag{4.5}$$

Here the subset $\mathrm{Ch}^2_{V^{\mathbf{C}}}(\mathcal{M})$ of $T^*_{V^{\mathbf{C}}}\widetilde{V}^{\mathbf{C}}$ is the microcharacteritic variety of $\mathcal{M}$ along $V^{\mathbf{C}}$.

Let $P(z, D_z)$ be a microdifferential operator defined on a neighborhood of a point of V and partially elliptic along V. Since this operator P induces an isomorphism $P : \mathcal{C}^2_V \xrightarrow{\sim} \mathcal{C}^2_V$, any microfunction (2-hyperfunction) solution for the equation $Pu = 0$ always belongs to $\mathcal{A}^2_V$.

Refer to Laurent [8] and Bony–Schapira [1] for more details.

From now on, we assume that $d+1=n$. We consider the following linear differential equation on M.

$$Pu = \Big(Q(x, D_{x'}) + x_n^{2k} D_{x_n}^2 + (lower)\Big)\, u = 0. \tag{4.6}$$

Here we assume that ord Q = 2, k ∈ **N** and that

$$\mathrm{Re}\,\sigma(Q)(x;\xi') > 0 \tag{4.7}$$

for any $x \in M$ and any $\xi' \in \mathbf{R}^{n-1} \setminus \{0\}$. Outside the Lagrangian manifold of $\overset{\circ}{T}{}^*_M X$:

$$L := \left\{(x; \sqrt{-1}\xi \cdot dx) \in \overset{\circ}{T}{}^*_M X; x_n = 0,\ \xi' = 0\right\}, \tag{4.8}$$

(4.6) is uniquely solvable in the sheaf $\mathcal{C}_M$, because the principal symbol of P never vanishes there. One cannot apply the above theory to this operator P, since P is not partially elliptic along V on $\{x_n = 0\}$.

Hence we consider the equation (4.6) on the regular involutive submanifold $V = \{\xi' = 0\}$.

First, we will prove the following theorem by using the support theorem in section 3.2.

Theorem 4.1 *Let P be a differential operator of* (4.6) *on M and $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$. Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \hat{\mathcal{C}}_V^2) = 0. \tag{4.9}$$

Proof. Let $q_\circ = (x_\circ; \pm\sqrt{-1}dx_n; \sqrt{-1}\eta'_\circ \cdot dx') \in \overset{\circ}{T}{}^*_V \tilde{V}$. It suffices to show that

$$\sigma(P)(z; (\xi' + \sqrt{-1}\varepsilon\eta') \cdot dz' \pm (\xi_n + \sqrt{-1}) \cdot dz_n) \neq 0 \tag{4.10}$$

where

$$|z - x_\circ| + |\eta' - \eta'_\circ| < \delta, \quad |\mathrm{Im}\, z_n| + |\xi| < \varepsilon\delta, \quad 0 < \varepsilon < \delta \tag{4.11}$$

for a good small $\delta > 0$.

We have actually

$$\begin{aligned}
& \mathrm{Re}\,\sigma(P)(z; (\xi' + \sqrt{-1}\varepsilon\eta') \cdot dz' \pm (\xi_n + \sqrt{-1}) \cdot dz_n) \\
= \;& \mathrm{Re}\,\sigma(Q)(z; (\xi' + \sqrt{-1}\varepsilon\eta') \cdot dz') + \mathrm{Re}\,(z_n^{2k}(\xi_n + \sqrt{-1})^2) \\
= \;& \varepsilon^2 \mathrm{Re}\,\sigma(Q)\Big(z; (\frac{\xi'}{\varepsilon} + \sqrt{-1}\eta') \cdot dz'\Big) + (\xi_n^2 - 1)(x_n^{2k} + O(y_n^2)) - 2\xi_n O(y_n) \\
< \;& 0
\end{aligned} \tag{4.12}$$

for a good small $\delta > 0$ because of the inequality

$$\operatorname{Re} \sigma(Q)(x_\circ; \sqrt{-1}\eta'_\circ \cdot dx') < 0. \tag{4.13}$$

This completes the proof. □

In this case, using Theorem 4.1 and the fundamental exact sequence (2.22), we get the next isomorphism.

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_V^2) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M|_V). \tag{4.14}$$

This shows that the structure of P in $\mathcal{C}_M|_V$ has been reduced to that in $\mathcal{A}_V^2$.

Second, we will show the results of solvability in the sheaf of microfunctions.

Theorem 4.2 *Let P be a differential operator of the form*

$$P = D_{x_1}^2 + D_{x_2}^2 + \cdots + D_{x_{n-1}}^2 + x_n^{2k} D_{x_n}^2 + (lower) \tag{4.15}$$

on M with $k \in \mathbf{N}$. Then $P : \mathcal{C}_M \to \mathcal{C}_M$ is surjective, that is, for any $f \in \mathcal{C}_{M,x^*}$, the following equation

$$Pu = f, \quad u \in \mathcal{C}_{M,x^*} \tag{4.16}$$

is solvable at any point $x^* \in \overset{\circ}{T^*_M}X$.

Proof. In the situation of Theorem 4.1, we set $Q = D_{x_1}^2 + D_{x_2}^2 + \cdots + D_{x_{n-1}}^2$. We may assume that $x^* \in L = V \cap \{x_n = 0\}$. It suffices to prove the next lemma.

Lemma 4.3 *Let x^* be any point of L. Then for any $f \in \mathcal{A}_{V,x^*}^2$, there exists $u \in \mathcal{A}_{V,x^*}^2$ such that $Pu = f$.*

Proof of lemma 4.3. Recall first that $\mathcal{A}_V^2 = \mathcal{C}_{\tilde{V}}|_V$ and that $\mathcal{C}_{\tilde{V}}$ is the subsheaf of $\mathcal{C}_N$, that is to say, $f(x, y')(\in \mathcal{C}_{N,x^*})$ belongs to $\mathcal{C}_{\tilde{V},x^*}$ if and only if f satisfies the system of Cauchy–Riemann equations

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \sqrt{-1}\frac{\partial f}{\partial y_j}\right) = 0. \quad (1 \le j \le n-1) \tag{4.17}$$

Let's consider the following equations:

$$\begin{cases} P(z', x_n, D_{z'}, D_{x_n})u = f, & \\ \dfrac{\partial u}{\partial \bar{z}_j} = 0. & (1 \le j \le n-1) \end{cases} \tag{4.18}$$

We have to show the existence of $u \in \mathcal{C}_{N,x^*}$ which satisfies (4.18). One notes that P is of the form:

$$P = \sum_{j=1}^{n-1} D_{z_j}^2 + x_n^{2k} D_{x_n}^2 + \sum_{j=1}^{n-1} a_j(z', x_n) D_{z_j} + a_n(z', x_n) D_{x_n} + b(z', x_n) \qquad (4.19)$$

where a_j $(1 \leq j \leq n-1)$ and b are the restriction of holomorphic functions on X to N.

We define a differential operator P_0 on N by

$$P_0 = -D_{y_1}^2 + \sum_{j=2}^{n-1} D_{x_j}^2 + x_n^{2k} D_{x_n}^2 + \sum_{j=1}^{n} a_j(z', x_n) D_{x_j} + b(z', x_n). \qquad (4.20)$$

Then the equations (4.18) are equivalent to where one replace P with P_0 owing to the properties of the solution u.

Because P_0 is micro-hyperbolic in y_1-direction at x^*, we find easily that on a neighborhood of $x^* \in L$ there exists a unique microfunction solution of the following Cauchy problem:

$$\begin{cases} P_0 u = f, \\ u|_{y_1=0} = \dfrac{\partial u}{\partial y_1}\Big|_{y_1=0} = 0. \end{cases} \qquad (4.21)$$

Refer to Kashiwara–Kawai [2] for the notion of the micro-hyperbolicity.

Let u the solution of equations (4.21). We have $P_0(\dfrac{\partial u}{\partial \bar{z}_j}) = 0$ for all j with $1 \leq j \leq n-1$, since the operators P_0 and $\dfrac{\partial}{\partial \bar{z}_j}$ are commutative.

Therefore we get:

$$\frac{\partial u}{\partial \bar{z}_j}\Big|_{y_1=0} = \frac{\partial}{\partial y_1}(\frac{\partial u}{\partial \bar{z}_j})\Big|_{y_1=0} = 0, \quad (1 \leq j \leq n-1) \qquad (4.22)$$

and hence we have $\dfrac{\partial u}{\partial \bar{z}_j} = 0$ for all j with $1 \leq j \leq n-1$ from the uniqueness of the solution for the Cauchy problem:

$$\begin{cases} P_0 v = 0, \\ v|_{y_1=0} = \dfrac{\partial v}{\partial y_1}\Big|_{y_1=0} = 0. \end{cases} \qquad (4.23)$$

This completes the proof of Theorem 4.2. □

Remark 4.4 In the situation of Theorem 4.2, we can claim further that

$$\mathrm{Ker}(\mathcal{A}_{\mathrm{V}}^2 \underset{\mathrm{P}}{\longrightarrow} \mathcal{A}_{\mathrm{V}}^2) \simeq \mathrm{Ker}(\mathcal{C}_{\mathrm{M}}|_{\mathrm{V}} \underset{\mathrm{P}}{\longrightarrow} \mathcal{C}_{\mathrm{M}}|_{\mathrm{V}}) \qquad (4.24)$$

by the isomorphism (4.14). This fact is also familiar by means of an estimate of the support of solution complexes in the sheaf $\mathcal{C}^2_V$.

By this assertion and Theorem 4.2, we can get the following exact sequence.

$$0 \longrightarrow \mathcal{A}^{2\ P}_V \longrightarrow \mathcal{C}_M|_V \underset{P}{\longrightarrow} \mathcal{C}_M|_V \longrightarrow 0. \tag{4.25}$$

Here we set $\mathcal{A}^{2\ P}_V = \mathrm{Ker}(\mathcal{A}^2_V \underset{\mathrm{P}}{\longrightarrow} \mathcal{A}^2_V)$.

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