数理解析研究所講究録
983 巻 1997 年 121-132

A REMARK ON THE ASYMPTOTIC PROPERTIES OF EIGENVALUES OF ELLIPTIC OPERATORS ON $\mathbf{T}^n$

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1. Introduction

Let M be an n-dimensional compact Riemannian manifold without boundary and P a partial differential operator on M of order m. We assume that P is self-adjoint and elliptic, say the principal symbol $p_m(x,\xi) \in C^\infty(T^*M \setminus 0)$ of P is strictly positive. The famous Weyl formula says that the number $N(\lambda)$ of eigenvalues of P which is not greater than λ behaves like

$$N(\lambda) = c\lambda^{n/m} + O(\lambda^{(n-1)/m}); \qquad c = (2\pi)^{-n} \iint_{p_m(x,\xi)\leq 1} dxd\xi,$$

as $\lambda \to +\infty$. The order $(n-1)/m$ of the error term cannot be improved if we take the sphere S^n as M. The spectrum of the standard Laplacian $-\triangle$ on the sphere is well known and the case $P = (-\triangle)^{m/2}$ is the counterexample. Refer to Hörmander [4] for these matters.

On the other hand, we can expect better result $o(\lambda^{(n-1)/m})$ for the error term if M and P satisfy extra conditions. For example, this is true if the closed orbits of Hamilton flow H_{p_m} generated by the principal symbol form a set of measure 0 in $T^*M\setminus 0$ (Duistermaat-Guillemin [2]). If P is the Laplace-Beltrami operator, then H_{p_m} is the geodesic flow, and the torus $M = \mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ satisfies the condition above if $n \geq 2$ while $M = S^n$ not. We remark $\mathbf{T}^1 = S^1$. Then our next question is what the exact order of the error term is for each Riemannian manifold M which satisfies this

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global condition. In other word, our target is the best number $d > 0$ for $O(\lambda^{(n-1)/m-d})$ to be true for the error term.

We can answer this question to some extent for $M = \mathbf{T}^n$ with $n \geq 2$ and P with constant coefficients, which imply the global condition above. The general answer is $d \geq d_{m,n} = (m^2n - m)^{-1}$ (Theorem 1). We can improve it if P has a kind of convexity (Theorem 2). In the special case when P is homogeneous, that is, P has only principal part and no lower terms, the answers can be translated into those for the problem to know the asymptotic distribution of the number of lattice points inside the region $R\Omega = \{R\xi; \xi \in \Omega\}$ as $R \to +\infty$. Here $\Omega \subset \mathbf{R}^n$ is a compact set which contains the origin. In fact, we can take $R = (2\pi)^{-1}\lambda^{1/m}$ and $\Omega = \{\xi; p_m(\xi) \leq 1\}$, where $p_m(\xi)$ is the principal symbol of $P = P(D)$, since the number λ is an eigenvalue of P if and only if the Diophantus equation $p_m(2\pi\xi) = \lambda$ has a solution $\xi \in \mathbf{Z}^n$ (Lemma 1). Especially, in the case when P is the standard Laplacian on the 2-dimensional torus $\mathbf{T}^2$, this is known as Gauss's circle problem, and better results than our answer $d \geq 1/6$ have been shown from the number theoretical aspects (Remark 3). But we would like to emphasize here that we can treat more general n and P and the order $d_{m,n} = (m^2n - m)^{-1}$ can be determined only by the dimension of the manifold and the order of the operator.

2. Main results

In the rest of this paper we always assume that $n \geq 2$ and $P = P(D)$ is an elliptic self-adjoint partial differential operators on $\mathbf{T}^n$ of order m with constant coefficients. Then the symbol of P can be expressed as

$$p(\xi) = p_m(\xi) + p_{m-1}(\xi) + \cdots + p_0(\xi),$$

where $p_j(\xi)$ is a real polynomial of $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ of order j ($j = 0, 1, \ldots m$). We remark that m must be even and $p_m(\xi)$ identically positive or negative for $\xi \neq 0$. We assume here the positivity, otherwise take $-P$ as P. Now, we set

$$\Omega = \{\xi \in \mathbf{R}^n; p_m(\xi) \leq 1\}.$$

We shall call its boundary $\partial\Omega$ the *cosphere* of P, which is a real analytic compact hypersurface in $\mathbf{R}^n$, that is, submanifolds of codimension 1. Before stating our main results, we shall introduce indices for hypersurfaces which were defined in Sugimoto [10] and [11] for another purpose.

Definition 1. Let Σ be a hypersurface in $\mathbf{R}^n$. Then, for a point $p \in \Sigma$ and for a plane H (of dimension 2) which contains the normal line of Σ at p, we define the index $\gamma(\Sigma; p, H)$ to be the order of contact of the curve $\Sigma \cap H$ to the line $T \cap H$ at p. Here T denotes the tangent hyperplane of Σ at p. Furthermore, we define the indices $\gamma(\Sigma)$ and $\gamma_0(\Sigma)$ by

$$\gamma(\Sigma) = \sup_p \sup_H \gamma(\Sigma; p, H), \quad \gamma_0(\Sigma) = \sup_p \inf_H \gamma(\Sigma; p, H).$$

Remark 1. We have $2 \leq \gamma_0(\Sigma) \leq \gamma(\Sigma)$ by definition. Equality $\gamma_0(\Sigma) = \gamma(\Sigma)$ holds when $n = 2$.

Hereafter Σ always denotes the cosphere of P, that is,

$$\Sigma = \{\xi \in \mathbf{R}^n; p_m(\xi) = 1\}.$$

Then we have the following inequality:

Proposition 1 ([11;Proposition 2]). $2 \leq \gamma_0(\Sigma) \leq \gamma(\Sigma) \leq m$.

Remark 2. In the case $m = 2$, the case of Laplacian $P = -\triangle$ for instance, we have $\gamma_0(\Sigma) = \gamma(\Sigma) = 2$. Even in the higher order case $m \geq 3$, this is true when the Gaussian curvature of Σ never vanishes.

We shall state our main theorems. In the following, $N(\lambda)$ denotes the number of eigenvalues of P which is not greater than λ (counted with respect to multiplicity), and $|\Omega|$ the Lebesgue measure of Ω.

Theorem 1. *We have the asymptotic distribution*

$$N(\lambda) = (2\pi)^{-n}|\Omega|\lambda^{\frac{n}{m}} + O(\lambda^{\frac{n-1}{m} - \frac{\alpha}{m(n-\alpha)}}) \tag{1}$$

with $\alpha = 1/\gamma_0(\Sigma)$, *hence with* $\alpha = 1/m$.

Remark 3. In the case when $n = m = 2$, Theorem 1 is an answer to Gauss's circle problem. In fact, λ is an eigenvalue of $-\triangle$ if and only if the Diophantus equation $|2\pi\xi|^2 = \lambda$ has a solution $\xi \in \mathbf{Z}^n$ (see Lemma 1 in Section 3). Hence we can know that the number of lattice points inside the disk $\{\xi; |\xi| \leq R\}$ behaves like $\pi R^2 + O(R^{2/3})$ by Theorem 1. This corresponds to classical results of Sierpinski [8]. There has been a series of improvements to this result, replacing $O(R^{2/3})$ by $O(R^\kappa)$, where $\kappa < 2/3$. For example, Chen [1] proved $\kappa > 24/37$. It has also been shown that $\kappa = 1/2$ is not possible (Landau [6]). The final result $\kappa > 1/2$ is conjectured but remained unsolved.

Example 1. Suppose $n \geq 3$, $N \in \mathbf{N}$. Let

$$p(\xi) = p_{4N}(\xi) = (\xi_1^2 + \cdots + \xi_{n-1}^2 - \xi_n^2)^{2N} + \xi_n^{4N}.$$

Then $\gamma(\Sigma) = 4N$, $\gamma_0(\Sigma) = 2N$, hence we have the asymptotic distribution (1) with $\alpha = (2N)^{-1}$. For the proof of it, refer to [11;Example 1].

The cosphere Σ in this example is not convex. But, Theorem 1 can be improved if P has some convexity property. We set

$$\Sigma_\varepsilon = \{\xi \in \mathbf{R}^n; p_m(\xi) + \varepsilon_1 p_{m-1}(\xi) + \varepsilon_2 p_{m-2}(\xi) + \cdots + \varepsilon_m p_0(\xi) = 1\},$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$. We always assume that $|\varepsilon|$ is sufficiently small so that Σ_ε is a real analytic compact hypersurface.

Definition 2. The cosphere Σ of P is called stably convex if there is $a > 0$ such that Σ_ε is convex for $|\varepsilon| \leq a$.

Remark 4. The stable convexity implies the convexity. If P is homogeneous, that is $p_{m-1}(\xi) = \cdots = p_0(\xi) = 0$, the convexity is equivalent to the stable convexity. If the Gaussian curvature of the cosphere never vanishes, then it is stably convex. In fact, the curvature condition implies the convexity (Kobayashi-Nomizu [5;Chap.7]) and this condition is stable under the lower term perturbation. Accordingly the cosphere is always stably convex if $m = 2$.

Theorem 2. *If the cosphere of P is stably convex, we have the asymptotic distribution (1) with $\alpha = (n-1)/\gamma(\Sigma)$, hence with $\alpha = (n-1)/m$.*

Example 2. Suppose $n \geq 3$, $N \in \mathbf{N}$. Let

$$p(\xi) = p_{2N}(\xi) = \xi_1^{2N} + \xi_2^{2N} + \cdots + \xi_n^{2N}.$$

Then $\gamma(\Sigma) = \gamma_0(\Sigma) = 2N$, hence we have the asymptotic distribution (1) with $\alpha = (n-1)/2N$.

Remark 5. In the case when the Gaussian curvature of the cospher Σ never vanishes, Theorem 2 was essentially proved by Hlawka [3].

3. Proofs

We shall show Theorems 1 and 2 by proving the following sequence of lemmata. We remark that the capital "C" (with some suffices) in estimates always denotes a positive constant (depending on the suffices) which may be different in each occasion.

We first notice that all $f \in L^2(\mathbf{T}^n)$ has the Fourier series expansion $f(x) = \sum_{k \in \mathbf{Z}^n} c_k e^{2\pi i k \cdot x}$, therefore $Pf(x) = \sum_{k \in \mathbf{Z}^n} c_k p(2\pi k) e^{2\pi i k \cdot x}$. Hence we have

Lemma 1. *The number λ is an eigenvalue of $P = P(D)$ if and only if the Diophantus equation $p(2\pi\xi) = \lambda$ has a solution $\xi \in \mathbf{Z}^n$.*

Now, for $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$, we shall denote by Ω_ε the closed region surrounded by Σ_ε, that is,

$$\Omega_\varepsilon = \{\xi \in \mathbf{R}^n; p_m(\xi) + \varepsilon_1 p_{m-1}(\xi) + \varepsilon_2 p_{m-2}(\xi) + \cdots + \varepsilon_m p_0(\xi) \le 1\}.$$

Especially we have

$$\Omega_{\varepsilon(\lambda)} = \{\xi \in \mathbf{R}^n; p\,(R(\lambda)\xi) \le \lambda\},$$

where

$$R(\lambda) = \lambda^{\frac{1}{m}}, \qquad \varepsilon(\lambda) = (R(\lambda)^{-1}, R(\lambda)^{-2}, \dots, R(\lambda)^{-m}).$$

We may assume that λ is sufficiently large so that $|\varepsilon(\lambda)|$ is sufficiently small. Let χ_ε be the characteristic function of Ω_ε. From Lemma 1, we easily obtain

Lemma 2. $N(\lambda) = \sum_{k \in \mathbf{Z}^n} \chi_{\varepsilon(\lambda)}(2\pi R(\lambda)^{-1} k)$.

Let us fix a smooth positive function $\psi(x)$ which is supported in a sufficiently small ball $\{x; |x| \le a\}$ and satisfies $\int \psi(x)\, dx = 1$. We define, for $R, \tau > 0$,

$$N_\varepsilon(R, \tau) = \sum_{k \in \mathbf{Z}^n} \left[\chi_\varepsilon(2\pi R^{-1} \cdot) * \tau^{-n} \psi(\tau^{-1} \cdot)\right](k).$$

By the argument of Friedrichs' mollifier, we have easily

Lemma 3. *For* $\tau > 0$, $N_{\varepsilon(\lambda)}(R(\lambda) - \tau, \tau) \leq N(\lambda) \leq N_{\varepsilon(\lambda)}(R(\lambda) + \tau, \tau)$.

On the other hand, direct application of Poisson's summation formula to $N_\varepsilon(R,\tau)$ yields

Lemma 4. $N_\varepsilon(R,\tau) = (2\pi)^{-n}|\Omega_\varepsilon|R^n + \sum_{\substack{k\in\mathbf{Z} \\ k\neq 0}}(2\pi)^{-n}R^n\widehat{\chi_\varepsilon}(Rk)\hat{\psi}(2\pi\tau k)$.

In order to use Lemma 4 with $\varepsilon = \varepsilon(\lambda)$, we shall estimate the difference between $|\Omega_{\varepsilon(\lambda)}|$ and $|\Omega|$. By using Heaviside function $Y(t)$, we express them in the form of oscillatory integrals as

$$\begin{aligned}|\Omega_{\varepsilon(\lambda)}| &= \int \chi_{\varepsilon(\lambda)}(\xi)\, d\xi \\ &= \int Y\left(1 - \lambda^{-1}p\left(R(\lambda)\xi\right)\right) d\xi \\ &= \frac{1}{2\pi}\iint e^{it(1-p_m(\xi)-r_\lambda(\xi))}\hat{Y}(t)\, dtd\xi,\end{aligned}$$

where

$$r_\lambda(\xi) = R(\lambda)^{-1}p_{m-1}(\xi) + R(\lambda)^{-2}p_{m-2}(\xi) + \cdots + R(\lambda)^{-m}p_0(\xi).$$

Similarly we have

$$|\Omega| = \frac{1}{2\pi}\iint e^{it(1-p_m(\xi))}\hat{Y}(t)\, dtd\xi.$$

Then, by Taylor's formula,

$$\begin{aligned}&|\Omega_{\varepsilon(\lambda)}| - |\Omega| \\ &= \frac{1}{2\pi}\iint e^{it(1-p_m(\xi))} \\ &\quad \times \left(-itr_\lambda(\xi) + (itr_\lambda(\xi))^2\int_0^1 (1-\theta)e^{-it\theta r_\lambda(\xi)}\, d\theta\right)\hat{Y}(t)\, dtd\xi \\ &= -\int \delta(1-p_m(\xi))r_\lambda(\xi)\, d\xi \\ &\quad + \int_0^1 (1-\theta)\left\{\int \delta'(1-p_m(\xi)-\theta r_\lambda(\xi))\left(r_\lambda(\xi)\right)^2 d\xi\right\} d\theta,\end{aligned}$$

where $\delta(t)$ is a formal expression of Dirac's delta function. The first term of the line above is $O\left(R(\lambda)^{-2}\right)$ since

$$\begin{aligned}&\int \delta(1-p_m(\xi))r_\lambda(\xi)\,d\xi = \int_\Sigma r_\lambda(\xi)\,d\Sigma\\ =&R(\lambda)^{-2}\int_\Sigma \left(p_{m-2}(\xi)+R(\lambda)^{-1}p_{m-3}(\xi)+\cdots+R(\lambda)^{-(m-2)}p_0(\xi)\right)\,d\Sigma.\end{aligned}$$

Here we have used the fact that m is even, hence $\int_\Sigma p_{m-1}(\xi)\,d\Sigma = 0$. Similarly, the second term is $O\left(R(\lambda)^{-2}\right)$ as well since the integrand with respect to θ is essentially an integral of derivatives of $r_\lambda(\xi)^2$ over the hypersurface $\Sigma_{\theta\varepsilon(\lambda)}$. Thus we have obtained

Lemma 5. $|\Omega_{\varepsilon(\lambda)}| = |\Omega| + O\left(R(\lambda)^{-2}\right)$.

In view of Lemma 4, an estimate for the Fourier transform of the characteristic function $\widehat{\chi_\varepsilon}(\xi)$ is needed for that for the error term. In fact, the following is true:

Lemma 6. *Let $\alpha \le n/2$. Suppose*

$$|\widehat{\chi_\varepsilon}(\xi)| \le C(1+|\xi|)^{-(1+\alpha)}, \tag{2}$$

where C is independent of ξ and small ε. Then we have (1).

To prove Lemma 6, we first note that ψ is rapidly decreasing. With this fact and the estimate (2), the summation part of the equality in Lemma 4 is estimated as

$$\begin{aligned}&\left|\sum_{k\in\mathbf{Z},k\neq 0}(2\pi)^{-n}R^n\widehat{\chi_\varepsilon}(Rk)\hat{\psi}(2\pi\tau k)\right|\\ \le&CR^n\sum_{k\in\mathbf{Z},k\neq 0}(1+|Rk|)^{-(1+\alpha)}(1+|\tau k|)^{-N}\\ \le&CR^n\int_{|\xi|\ge 1}(1+|R\xi|)^{-(1+\alpha)}(1+|\tau\xi|)^{-N}\,d\xi\\ \le&CR^n\int_{1\le|\xi|\le 1/\tau}|R\xi|^{-(1+\alpha)}\,d\xi + CR^n\int_{1/\tau\le|\xi|}|R\xi|^{-(1+\alpha)}|\tau\xi|^{-N}\,d\xi\\ \le&C(R/\tau)^{n-(1+\alpha)},\end{aligned}$$

where $N > n-(1+\alpha)$ and C is independent of ε, R, and τ. Here we have used $n \geq 1+\alpha$. Applying this estimate and Lemma 5 together to Lemma 4, we have

$$\begin{aligned}
&N_{\varepsilon(\lambda)}\left(R(\lambda) \pm \tau, \tau\right) \\
=&(2\pi)^{-n}\left(|\Omega| + O\left(R(\lambda)^{-2}\right)\right)\left(R(\lambda) \pm \tau\right)^n + O\left(\left((R(\lambda) \pm \tau)/\tau\right)^{n-(1+\alpha)}\right) \\
=&(2\pi)^{-n}|\Omega|R(\lambda)^n + O\left(\tau R(\lambda)^{n-1} + (R(\lambda)/\tau)^{n-(1+\alpha)}\right) + O\left(R(\lambda)^{n-2}\right) \\
=&(2\pi)^{-n}|\Omega|R(\lambda)^n + O\left(R(\lambda)^{n-1-\frac{\alpha}{n-\alpha}}\right)
\end{aligned}$$

if we take $\tau = R(\lambda)^{-\alpha/(n-\alpha)}$. Here we have used the binomial expansion $(R \pm \tau)^n = R^n + O\left(\tau R^{n-1}\right)$ and $n-1-\alpha/(n-\alpha) \geq n-2$. Then we have (1) by Lemma 3 and have completed the proof of Lemma 6.

Thus all we have to show is the estimate (2). Let Γ be a conic neighborhood of $x = (0, \ldots, 0, 1)$ and $\varphi(x)$ a smooth function which is positve, homogeneous of order 0 for large $|\xi|$, and is supported in Γ. It suffices to show the same estimate for $\widehat{\chi_\varepsilon \varphi}(\xi)$ instead of $\widehat{\chi_\varepsilon}(\xi)$ and we may take Γ sufficiently small in need. We shall express

$$\Sigma_\varepsilon \cap \Gamma = \{(y, h_\varepsilon(y)); y = (y_1, y_2, \cdots, y_{n-1}) \in U\},$$

where $U \subset \mathbf{R}^{n-1}$ is a neighborhood of the origin and $h_\varepsilon(y) \in C^\omega(U)$. We remark that h_ε is real analytic with respect to ε as well. Then we have, by the change of variables $x = (ty, th_\varepsilon(y))$ and integration by parts,

$$\begin{aligned}
\widehat{\chi_\varepsilon \varphi}(\xi) &= \int_{\Omega_\varepsilon} e^{-ix\cdot\xi}\varphi(x)\,dx \\
&= \int_0^1 \int_U e^{-it\omega_n|\xi|(y\cdot\eta + h_\varepsilon(y))} g_\varepsilon(t,y)\,dtdy \\
&= \frac{i}{\omega_n|\xi|}\int_U e^{-i\omega_n|\xi|(y\cdot\eta + h_\varepsilon(y))}\frac{g_\varepsilon(1,y)}{y\cdot\eta + h_\varepsilon(y)}\,dy \\
&\quad - \frac{i}{\omega_n|\xi|}\int_0^1\left\{\int_U e^{-i\omega_n t|\xi|(y\cdot\eta + h_\varepsilon(y))}\frac{(\partial g_\varepsilon/\partial t)(t,y)}{y\cdot\eta + h_\varepsilon(y)}\,dy\right\}dt
\end{aligned}$$

where $g_\varepsilon(t,y) = \varphi(ty, th_\varepsilon(y))t^{n-1}|h_\varepsilon(y) - y\cdot h'_\varepsilon(y)|$, $\omega = \xi/|\xi| = (\omega', \omega_n)$, $\omega' = (\omega_1, \omega_2, \cdots, \omega_{n-1})$, and $\eta = \omega'/\omega_n$. We remark that $g_\varepsilon(t,y)$ vanishes identically for small t. We may assume here that ω_n is away from 0 since integration by parts argument yields the better estimate than we need in the direction $\omega = (\omega', 0)$. By all of this, the estimate (2) is reduced to that for the oscillatory integral of the type

$$I_\varepsilon(t;\eta) = \int_U e^{it(y\cdot\eta + h_\varepsilon(y))} g(y)\, dy; \qquad g \in C_0^\infty(U).$$

That is, we have

Lemma 7. *Suppose, for some N,*

$$|I_\varepsilon(t;\eta)| \le C_g |t|^{-\alpha}, \tag{3}$$

where $C_g = C\sum_{|\alpha|\le N} \|\partial^\alpha g/\partial y^\alpha\|_{L^\infty(U)}$ and $C > 0$ is independent of t, η, and small ε. Then we have the estimate (2), hence the asymptotic distribution (1).

In order to obtain (3), we shall use the following *scaling principle* for oscillatory integrals:

Lemma 8. *Let $f(t) \in C^\infty(\mathbf{R})$ be real-valued and let $\zeta(t) \in C_0^\infty(\mathbf{R})$. Suppose, for $\nu \ge 2$ and $\mu > 0$,*

$$|f^{(\nu)}(t)| \ge \mu \quad \textit{on} \quad \operatorname{supp}\zeta.$$

Then we have

$$\left| \int e^{itf(t)} \zeta(t)\, dt \right| \le C\,(\|\zeta\|_{L^\infty} + \|\zeta'\|_{L^1})\, |t|^{-1/\nu},$$

where the constant $C > 0$ depends only on ν and μ.

For the proof of this lemma, consult Stein [9, Chapter VIII, 1.2]. By an appropriate change of coordinates and taking U sufficiently small in need, we may assume $|(\partial^\nu h_\varepsilon/\partial y_1^\nu)(y)| \ge \mu > 0$ for $y \in U$ and small ε, where $\nu = \gamma_0(\Sigma)$. Hence, from Lemma 8, we obtain

$$\left| \int e^{it(y\cdot\eta + h_\varepsilon(y))} g(y)\, dy_1 \right| \le C \left(\sum_{|\alpha|\le 1} \left\| \frac{\partial^\alpha g}{\partial y^\alpha} \right\|_{L^\infty(U)} \right) |t|^{-1/\gamma_0(\Sigma)}$$

for all $y' = (y_2, \dots, y_{n-1})$, therefore

$$\begin{aligned} |I_\varepsilon(t;\eta)| &\le \int \left| \int e^{it(y\cdot\eta + h_\varepsilon(y))} g(y)\, dy_1 \right| dy' \\ &\le C_g |t|^{-1/\gamma_0(\Sigma)}. \end{aligned}$$

We have thus proved the following lemma which implies Theorem 1 by Lemma 7 and Proposition 1.

Lemma 9. *The estimate (3) is true for $\alpha = 1/\gamma_0(\Sigma)$.*

On the other hand, when Σ is stably convex, the map $h'_\varepsilon : U \to -h'_\varepsilon(U) \subset \mathbf{R}^{n-1}$ is homeomorphism because of the compactness and the real analyticity of Σ_ε. Then we can define $z_\varepsilon = z_\varepsilon(\eta)$ by the relation $\eta + h'_\varepsilon(z_\varepsilon) = 0$, otherwise I_ε has better estimate than we need by integration by parts argument again. We set

$$\tilde{I}_\varepsilon(t;z) = \int_U e^{itE_\varepsilon(y;z)} g(y)\, dy; \quad E(y;z) = h_\varepsilon(y) - h_\varepsilon(z) - h'_\varepsilon(z)\cdot(y-z).$$

Hereafter we shall estimate $\tilde{I}_\varepsilon$ instead of I_ε since $|I_\varepsilon(t;\eta)| = |\tilde{I}_\varepsilon(t;z_\varepsilon)|$. For this purpose, we rewrite it with the polar coordinates as

$$\tilde{I}_\varepsilon(t;z) = \int_{S^{n-2}} G_\varepsilon(t;z,\omega)\, d\omega; \quad G_\varepsilon(t;z,\omega) = \int_0^\infty e^{itF_\varepsilon(\rho;z,w)} \beta(\rho;z,\omega)\, d\rho,$$

where

$$F_\varepsilon(\rho;z,\omega) = h_\varepsilon(\rho\omega + z) - h_\varepsilon(z) - \rho h'_\varepsilon(z)\cdot\omega, \quad \beta(\rho;z,\omega) = g(\rho\omega + z)\rho^{n-2}.$$

For the sake of simplicity, we shall often abbreviate parameters z and ω. We split the function $G_\varepsilon(t)$ into the following two parts:

$$\begin{aligned} G^1_\varepsilon(t) &= \int_0^\infty e^{itF_\varepsilon(\rho)} \beta_1(\rho,t)\, d\rho : \qquad \beta_1(\rho,t) = \beta(\rho)\Psi\left(|t|^{\frac{1}{\gamma(\Sigma)}}\rho\right), \\ G^2_\varepsilon(t) &= \int_0^\infty e^{itF_\varepsilon(\rho)} \beta_2(\rho,t)\, d\rho; \qquad \beta_2(\rho,t) = \beta(\rho)(1-\Psi)\left(|t|^{\frac{1}{\gamma(\Sigma)}}\rho\right), \end{aligned}$$

where the function $\Psi(\rho) \in C^\infty(\mathbf{R})$ equals to 1 for large ρ and vanishes near the origin. The estimate for the part $G^2_\varepsilon(t)$ is easy. In fact, we have

$$\begin{aligned}|G^2_\varepsilon(t)| &\le \int_0^\infty |\beta_2(\rho,t)|\,d\rho,\\ &\le C_g \int_0^\infty |\rho^{n-2}(1-\Psi)\left(|t|^{\frac{1}{\gamma(\Sigma)}}\rho\right)|\,d\rho\\ &\le C_g|t|^{-\frac{n-1}{\gamma(\Sigma)}}.\end{aligned}$$

On the other hand, integration by parts yields

$$G^1_\varepsilon(t) = \int_0^\infty e^{itF_\varepsilon(\rho)}\,(L^*)^l\,\beta_1(\rho,t)\,d\rho$$

for $l = 0, 1, 2, \cdots$. Here

$$L = \frac{1}{itF'_\varepsilon(\rho)}\frac{\partial}{\partial\rho}$$

and L^* is the transpose of L. By induction, we can easily have

$$(L^*)^l = \left(\frac{i}{t}\right)^l \sum C_{r,q,s_1,\cdots,s_q}\frac{F_\varepsilon^{(s_1)}\cdots F_\varepsilon^{(s_q)}}{(F'_\varepsilon)^{l+q}}(\rho)\frac{\partial^r}{\partial\rho^r},$$

where the summation $\sum$ is a finite sum of $r, q, s_1, \cdots, s_q \ge 0$ which satisfy $r+s_1+\cdots+s_q = l+q$. The derivatives of F_ε have the following estimate:

Lemma 10. *Suppose Σ is stable convex. Then there exist constants $C, C_\nu, a > 0$ such that the estimates*

$$\begin{aligned}|F'_\varepsilon(\rho)| &\ge C\rho^{\gamma(\Sigma)-1},\\ |F_\varepsilon^{(\nu)}(\rho)| &\le C_\nu\rho^{1-\nu}|F'_\varepsilon(\rho)|\end{aligned}$$

hold for $0 \le \rho, |z|, |\varepsilon| \le a$, $\omega \in S^{n-2}$, and $\nu = 0, 1, 2\cdots$.

If we use Lemma 10 and the estimate

$$\left|\frac{\partial^r\beta_1}{\partial\rho^r}(\rho,t)\right| \le C_g\rho^{n-2-r},$$

we have, for a large number l and a constant $b > 0$

$$\begin{aligned}
|G_\varepsilon^1(t)| &\le \frac{C}{|t|^l} \sum \int_0^\infty \left| \frac{F_\varepsilon^{(s_1)} \cdots F_\varepsilon^{(s_q)}}{(F_\varepsilon')^{l+q}}(\rho) \frac{\partial^r \beta_1}{\partial \rho^r}(\rho, t) \right| d\rho \\
&\le \frac{C_g}{|t|^l} \int_{b|t|^{-\frac{1}{\gamma(\Sigma)}}}^\infty \rho^{n-2-l\gamma(\Sigma)}\, d\rho \\
&\le C_g |t|^{-\frac{n-1}{\gamma(\Sigma)}}.
\end{aligned}$$

We have thus proved

Lemma 11. *If Σ is stably convex, then the estimate (3) is true for $\alpha = (n-1)/\gamma(\Sigma)$.*

From Lemma 7, Lemma 11, and Proposition 1, we obtain Theorem 2 if we can prove Lemma 10. Since the proof of it is carried out essentially by the same argument as used in Randol [7] and Sugimoto [10], we shall omit it. (See [10;Lemma 2].)

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