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Formal power series solutions of non-linear partial differential equations of the first order

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In 1903, Maillet [1] proved that if an algebraic ordinary differential equation has a formal power series solution then it is in some formal Gevrey class. Later, this result was extended to general analytic ordinary differential equations by Gérard [2] and Malgrange [3].

In this note, I will try to generalize this result to partial differential equations.

§1. Maillet's theorem

First, let us recall the general form of Maillet's theorem.

Denote by $\boldsymbol{C}[[t]]$ the ring of formal power series in t, and by $\boldsymbol{C}\{t\}$ the subring of $\boldsymbol{C}[[t]]$ of convergent power series in t.

Definition 1. If a formal power series $\sum_{i\geq 0} a_i t^i \in \boldsymbol{C}[[t]]$ satisfies

$$\sum_{i\geq 0} \frac{a_i}{i!^{s-1}} t^i \in \boldsymbol{C}\{t\}$$

for some $s \geq 1$, we say that u is in the formal Gevrey class $\boldsymbol{C}\{t\}_s$.

Let $n \in \boldsymbol{N}(=\{0,1,2\ldots\})$, let $G(t,X_0,X_1,\ldots,X_n)$ be a holomorphic function defined in a neighborhood of the origin of $\boldsymbol{C}^{n+2}$ satisfying

$A_1)$ $\quad G(t,X_0,X_1,\ldots,X_n) \not\equiv 0,$

$A_2)$ $\quad G(0,0,\ldots,0)=0$

and let us consider the following ordinary differential equation

$$(e_1) \qquad G\left(t, u, \theta u, \ldots, \theta^n u\right) = 0,$$

where $\theta = t\frac{d}{dt}$ and $u = u(t)$ is an unknown function. Under A_1) and A_2) we have

Theorem A ([2], [3]). If (e_1) has a formal power series solution $u(t) \in \boldsymbol{C}[[t]]$ satisfying $u(0) = 0$ then $u(t)$ belongs to the formal Gevrey class $\boldsymbol{C}\{t\}_s$ for some $s \geq 1$.

The general form of Maillet's theorem is formulated as follows:
Let $F(t, X_0, X_1, \ldots, X_n)$ be a holomorphic function defined in a neighborhood of $(0, a_0, a_1, \ldots, a_n) \in \boldsymbol{C}^{n+2}$ satisfying

B_1) $\quad F(t, X_0, X_1, \ldots, X_n) \not\equiv 0,$
B_2) $\quad F(0, a_0, a_1, \ldots, a_n) = 0$

and let us consider the following ordinary differential equation

$$(e_2) \qquad F\left(t, u, u', u^{(2)}, \ldots, u^{(n)}\right) = 0$$

with an unknown function $u = u(t)$.

Maillet's Theorem. If (e_2) has a formal power series solution $u(t) \in \boldsymbol{C}[[t]]$ satisfying $u^{(p)}(0) = a_p$ for $p = 0, 1, \ldots, n$ then $u(t)$ belongs to the formal Gevrey class $\boldsymbol{C}\{t\}_s$ for some $s \geq 1$.

From Theorem A to Maillet's Theorem. Let us explain here how to reduce (e_2) to (e_1). Let $u(t) = \sum_{i\geq 0} u_i t^i \in \boldsymbol{C}[[t]]$ be a formal solution of (e_2). Put

$$u = \varphi + t^n w, \quad \text{where } \varphi = \sum_{i\geq 0} u_i t^i, \ w = \sum_{i \geq 1} w_{n+i} t^i.$$

Then the equation (e_2) with respect to u is rewritten into an equation with respect to w:

$$(*) \qquad F\left(t, \varphi + t^n w, \varphi' + t^{n-1}\Theta_1 w, \ldots, \varphi^{(n)} + \Theta_n w\right) = 0$$

where

$$\begin{aligned}\Theta_1 &= (\theta+n)\\ \Theta_2 &= (\theta+(n-1))(\theta+n)\\ &\cdots\cdots\\ &\cdots\cdots\\ \Theta_n &= (\theta+1)\cdots(\theta+(n-1))(\theta+n).\end{aligned}$$

By applying Theorem A to $(*)$ we can get $w \in \boldsymbol{C}\{t\}_s$ for some $s \geq 1$; this implies $u \in \boldsymbol{C}\{t\}_s$.

§2. Formulation in PDE

Following (e_1), let us consider the non-linear partial differential equation:

$$(E) \qquad G\left(t, x, u, t\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = 0,$$

where $(t,x) \in \boldsymbol{C}^2$, $u = u(t,x)$ is an unknown function and $G(t,x,X_1,X_2,X_3)$ is a holomorphic function defined in a neighborhood of the origin of $\boldsymbol{C}^5$ satisfying

$C_1)$ $\quad G(t,x,X_1,X_2,X_3) \not\equiv 0,$

$C_2)$ $\quad G(0,x,0,0,0) \equiv 0$ (near $x=0$).

Denote by $\boldsymbol{C}[[t,x]]$ the ring of formal power series in (t,x), and by $\boldsymbol{C}\{t,x\}$ the subring of $\boldsymbol{C}[[t,x]]$ of convergent power series in (t,x).

Definition 2. If a formal power series $\sum a_{i,j}t^i x^j \in \boldsymbol{C}[[t,x]]$ satisfies

$$\sum \frac{a_{i,j}}{i!^{s-1}j!^{\sigma-1}} t^i x^j \in \boldsymbol{C}\{t,x\}$$

for some $s \geq 1$ and $\sigma \geq 1$, we say that u is in the formal Gevrey class $\boldsymbol{C}\{t,x\}_{(s,\sigma)}$.

The following is a direct analogy of the case in §1 to PDE:

Problem 1. Under what condition is the following assertion (M) valid ?

(M) If (E) has a formal power series solution then it is in some formal Gevrey class.

Example 1. Let us consider

$$\left(t\frac{\partial u}{\partial t}\right)^2 - u^2 = a(x)t^3 + \left(\frac{\partial u}{\partial x}\right)^3, \tag{2.1}$$

where $a(x) \in \boldsymbol{C}\{x\}$. It is easy to see:

1) For any $\phi(x) \in \boldsymbol{C}[[x]]$ satisfying $\phi(0) \neq 0$ the equation (2.1) has a unique formal power series solution $u(t,x) \in \boldsymbol{C}[[t,x]]$ of the form

$$u(t,x) = \phi(x)t + \sum_{k\geq 2} \phi_k(x)t^k.$$

(We denote this by $U(\phi) \in \boldsymbol{C}[[t,x]]$.)

2) If we choose $\phi(x)$ being out of formal Gevrey classes, then $U(\phi)$ does not belong to any formal Gevrey class.

3) This implies that (2.1) does not satisfy (M).

Usually, as is seen above, in the case of PDE some data (for example, initial data or boundary data, ect) can be given freely and therefore by choosing this free data as a formal power series out of formal Gevrey classes we can easily conclude that (M) is not valid.

In order to include such cases as Example 1, instead of (M) let us consider the following problem:

Problem 2. Under what condition is the following assertion (AM) valid ?

(AM) If (E) has a formal power series solution then (E) has a formal power series solution in some formal Gevrey class.

Of course, (AM) is valid for the equation (2.1).

§3. Some definitions

A formal power series $f(t,x) \in \boldsymbol{C}[[t,x]]$ is expressed in the form

$$f(t,x) = \sum_{i\geq 0} f_i(x)t^i, \quad f_i(x) \in \boldsymbol{C}[[x]].$$

Definition 3. The valuation $v_t(f)$ of $f(t,x)$ in t is defined by the following:

(1) if $f \equiv 0$, $v_t(f) = \infty$;

(2) if $f \not\equiv 0$, $v_t(f) = \inf\{\, i\,;\, f_i(x) \not\equiv 0\}$.

Let $G(t, x, X_1, X_2, X_3)$ be a holomorphic function defined in a neighborhood of the origin of $\boldsymbol{C}^5$ satisfying C_1) and C_2), and let us consider

$$(E) \qquad G\left(t, x, u, t\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = 0.$$

For simplicity we write

$$\begin{aligned} Du &= \left(u, t\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) \\ Du(t,0) &= \left(u, t\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right)\Big|_{x=0}. \end{aligned}$$

For $u(t,x) \in \boldsymbol{C}[[t,x]]$ with $v_t(u) \geq 1$ we put

$$\begin{aligned} q_i(u) &= v_t\left(\frac{\partial G}{\partial X_i}(t, x, Du)\right), \quad i = 1,2,3; \\ q_i(u;0) &= v_t\left(\frac{\partial G}{\partial X_i}(t, 0, Du(t,0))\right), \quad i = 1,2,3; \\ \rho(u;t) &= -\frac{\dfrac{\partial G}{\partial X_1}(t, 0, Du(t,0))}{\dfrac{\partial G}{\partial X_2}(t, 0, Du(t,0))}. \end{aligned}$$

Introduce new variables $\xi_1, \xi_2, \ldots, \eta_1, \eta_2, \ldots, \zeta_1, \zeta_2, \ldots$, and put

$$\mathcal{X} = \sum_{i \geq 1} \xi_i t^i, \quad \mathcal{Y} = \sum_{i \geq 1} \eta_i t^i, \quad \mathcal{Z} = \sum_{i \geq 1} \zeta_i t^i.$$

We define $d_1(G)$, $d_2(G)$, $d_3(G)$ by the following:

$$d_1(G) = v_t\left(G(t, x, \mathcal{X}, \mathcal{Y}, \mathcal{Z}) - G(t, x, 0, \mathcal{Y}, \mathcal{Z})\right),$$

$$d_2(G) = v_t\left(G(t, x, \mathcal{X}, \mathcal{Y}, \mathcal{Z}) - G(t, x, \mathcal{X}, 0, \mathcal{Z})\right),$$

$$d_3(G) = v_t\left(G(t,x,\mathcal{X},\mathcal{Y},\mathcal{Z}) - G(t,x,\mathcal{X},\mathcal{Y},0)\right).$$

By the definition it is easy to see:

Lemma 1. (1) $d_i(G) \geq 1$ $(i = 1, 2, 3)$.
(2) $q_i(u) = 0$ is equivalent to $(\partial G/\partial X_i)(0, x, 0, 0, 0) \not\equiv 0$.
(3) $q_i(u) \geq 1$ implies $d_i(G) \geq 2$.

§4. Results.

Let $u(t,x) \in \boldsymbol{C}[[t,x]]$ be a formal power series solution of (E) satisfying $u(0,x) \equiv 0$. Put

$$q(u) = \min\{q_1(u), q_2(u), q_3(u)\},$$
$$l(u) = \max\{\, i\,;\, q_i(u) = q(u)\}.$$

It is clear that $q_{l(u)}(u) = q(u)$ holds. Assume:

a-1) $\quad q(u) < \infty$,
a-2) $\quad q_{l(u)}(u;0) = q(u)$,
a-3) $\quad \rho(u;0) - q(u) \notin \{1, 2 \dots\}$, if $l(u) = 2$.

Note that in case $l(u) = 2$ we have $q_2(u;0) = q_2(u) \leq q_1(u) \leq q_1(u;0)$ and therefore $\rho(u;0) = \rho(u;t)|_{t=0}$ is well-defined.

Theorem 1. If (E) has a formal power series solution $u(t,x) \in \boldsymbol{C}[[t,x]]$ satisfying $u(0,x) \equiv 0$ and if the conditions a-1), a-2), a-3) and

$$q(u) \leq \frac{d_3(G) - 1}{2} \tag{4.1}$$

are valid, then (E) has a formal power series solution $w(t,x)$ such that

$$w(t,x) \in \begin{cases} \boldsymbol{C}\{t,x\}_{(2,1)}, & \text{when } l(u) = 1, \\ \boldsymbol{C}\{t,x\}_{(1,1)}, & \text{when } l(u) = 2 \text{ or } 3. \end{cases}$$

Remark 1. (1) In case $l(u) = 2$ or 3, the conclusion of Theorem 1 asserts the existence of a holomorphic solution.

(2) In case $q(u) = 0$ and $l(u) = 3$, the above result is nothing but the Cauchy-Kowalewski theorem.

(3) This result was extended to general non-linear higher order partial differential equations in [4],[5].

The following theorem fills the case where the condition (4.1) is not satisfied.

Theorem 2. If (E) has a formal power series solution $u(t,x) \in \boldsymbol{C}[[t,x]]$ satisfying $u(0,x) \equiv 0$ and if the conditions a-1), a-2), a-3) and

$$q(u) \leq \max\{d_1(G), d_2(G)\} \tag{4.2}$$

are valid, then (E) has a formal power series solution $w(t,x)$ such that

$$w(t,x) \in \begin{cases} \boldsymbol{C}\{t,x\}_{(3,2)}, & \text{when } l(u) = 1, \\ \boldsymbol{C}\{t,x\}_{(2,2)}, & \text{when } l(u) = 2, \\ \boldsymbol{C}\{t,x\}_{(1,2)}, & \text{when } l(u) = 3. \end{cases}$$

§5. Examples.

Example 1. Let us consider

$$\left(t\frac{\partial u}{\partial t}\right)^2 - u^2 = a(x)t^3 + \left(\frac{\partial u}{\partial x}\right)^3 \tag{5.1}$$

where $a(x) \in \boldsymbol{C}\{x\}$.

1) For any $\phi(x) \in \boldsymbol{C}[[x]]$ satisfying $\phi(0) \neq 0$ the equation (5.1) has a unique formal power series solution $u(t,x) \in \boldsymbol{C}[[t,x]]$ of the form

$$u(t,x) = \phi(x)t + \sum_{k\geq 2} \phi_k(x)t^k.$$

2) In this case we have $q_1(u) = 1$, $q_2(u) = 1$, $q_3(u) \geq 2$, $q_1(u;0) = 1$, $q_2(u;0) = 1$, $q_3(u;0) \geq 2$, $d_1(G) = 2$, $d_2(G) = 2$, $d_3(G) = 3$. Therefore,

$q(u) = 1$, $l(u) = 2$ and $\rho(u; 0) = 1$. Since the condition (4.1) is valid, we can apply Theorem 1 to the equation (5.1).

3) Conclusion. The equation (5.1) has a holomorphic solution $w(t, x)$ satisfying $w(0, x) \equiv 0$.

Example 2. Let us consider

$$\left(t\frac{\partial u}{\partial t}\right)^2 - u^2 = \left(x^2\frac{\partial u}{\partial x} - u + (1+x)t\right)^2 + a(x)t^3 \qquad (5.2)$$

where $a(x) \in \boldsymbol{C}\{x\}$.

1) By a calculation we see that the equation (5.2) has a unique formal solution $u(t, x)$ of the form

$$u(t,x) = \left(1 + x + \sum_{p\geq 2}(p-1)!x^p\right)t + \sum_{k\geq 2}\phi_k(x)t^k.$$

2) In this case we have $q_1(u) = 1$, $q_2(u) = 1$, $q_3(u) \geq 2$, $q_1(u; 0) = 1$, $q_2(u; 0) = 1$, $q_3(u; 0) = \infty$, $d_1(G) = 2$, $d_2(G) = 2$, $d_3(G) = 2$. Therefore, $q(u) = 1$, $l(u) = 2$ and $\rho(u; 0) = 1$. Since the condition (4.2) is valid, we can apply Theorem 2 to the equation (5.2).

3) Conclusion. The equation (5.2) has a unique formal power series solution $u(t, x)$ satisfying $u(0, x) \equiv 0$ and it belongs to the formal Gevrey class $\boldsymbol{C}\{t, x\}_{(2,2)}$.

4) Note that (5.2) does not satisfy the condition (4.1) and we can not apply Theorem 1 to this case.

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