数理解析研究所講究録
983巻 1997年 156-166

On Boundary Value Problems for Micro-hyperbolic Systems of Differential Equations

Motoo Uchida

Osaka University, Graduate School of
Science, Department of Mathematics

内田素夫（大阪大学大学院理学研究科）

In [KK], Kashiwara and Kawai formulate boundary value problems for elliptic systems of differential equations from a microlocal point of view, where they describe the obstruction of extension beyond the boundary in terms of a system of micro-differential equations induced on the boundary. In this short paper, we prove the same formula as established in [KK] for (semi-)micro-hyperbolic systems of differential equations. This enables us to understand boundary value problems for elliptic systems and for semi-hyperbolic systems in a unified manner.

The results proved in this paper[1] are more or less known to specialists, but are not found in the literature.

Notations. In this paper, we freely use the notations of [KS1] for sheaves and functors. For a complex manifold X, T^*X denotes the cotangent bundle of X. $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X, $\mathcal{D}_X$ the sheaf of rings of differential operators, and $\mathcal{E}_X$ the sheaf of rings of microdifferential operators. If M is a closed real submanifold of X, T^*_MX denotes the conormal bundle of M, $\pi_M : T^*_MX \to M$ the projection to the base space. We denote by H the Hamiltonian map $T^*T^*X \to TT^*X$. If M is a real submanifold of X, H induces an isomorphism $T^*T^*_MX \to T_{T^*_MX}T^*X$, which is also denoted simply by H.

[1]Its original version is in Research Reports in Mathematics 96-04, Osaka University (March 1996). The contents of this paper are not related to the author's seminar talk at RIMS; the author would like to thank the editor of this volume who has given the opportunity of reproducing here the preprint.

1. Main Theorems

Let M be a real analytic manifold of dimension $n \geq 1$, N a submanifold of M of codimension 1 defined by equation $f = 0$ for a real-valued analytic function f with $df|_N \neq 0$. Let Z_+ denote the closed subset $\{f \geq 0\}$ of M; then Z_+ is a real analytic submanifold of M with boundary. We set $N^+ = \{k \cdot df(x) \mid x \in N,\, k > 0\}$; then $N^+ \subset T^*_N M$. Let X be a complex neighborhood of M, Y a closed complex submanifold of X of codimension 1 such that $M \cap Y = N$. Denote by φ the closed embedding $Y \hookrightarrow X$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. $\mathrm{Ch}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$. We assume the following conditions :

(A.1) $\varphi : Y \to X$ is non characteristic for $\mathcal{M}$.

(A.2) At any point p of $(T^*_M X \cap T^*_N X \setminus N) \cap \mathrm{Ch}(\mathcal{M})$,

$$-H(\pi^* df) \notin C_p(\mathrm{Ch}(\mathcal{M}),\, Z_+ \times_M T^*_M X)/T_p T^*_M X, \tag{1.1}$$

where $\pi : T^*_M X \to M$ and $\pi^* : T^*_{\pi(p)} M \to T^*_p T^*_M X$.

In the right-hand side of (1.1), $C_p(\mathrm{Ch}(\mathcal{M}), Z_+ \times_M T^*_M X)$ denotes the normal cone at p (cf. [KS1, Def.4.1.1]), which is a closed cone in $T_p T^* X$, and $C_p(\cdot,\, \cdot)/T_p T^*_M X$ the image of the normal cone in $(T_{T^*_M X} T^* X)_p$ for short.

Let $(T^*_N X)^+$ be an open subset of $T^*_N X$ defined by $(T^*_N X)^+ = q^{-1}(N^+)$, with q being the canonical projection $T^*_N X \to T^*_N M$. Let ${}^t\varphi' : T^* X \times_X Y \to T^* Y$ the induced map of φ, $\rho : T^*_N X \to T^*_N Y$ the projection induced from ${}^t\varphi'$ on N.

Let $\widetilde{\mathcal{M}} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$, with $\pi : T^* X \to X$. Denoting by $\varphi^* \widetilde{\mathcal{M}}$ the induced $\mathcal{E}_Y$-module of $\widetilde{\mathcal{M}}$ on Y, we have :

Lemma 1.1. *If we assume* (A.1) *and* (A.2), *there exists a coherent $\mathcal{E}_Y$-module $\mathcal{N}^+$ defined on $T^*_N Y \setminus N$ and an $\mathcal{E}_Y$-homomorphism $\mathcal{N}^+ \to \varphi^* \widetilde{\mathcal{M}}$ such that*

$$\mathcal{N}^+_q \cong \bigoplus_{p \in (T^*_N X)^+ \cap \mathrm{Ch}(\mathcal{M}) \cap \rho^{-1}(q)} (\mathcal{E}_{Y \to X} \otimes_{\mathcal{E}_X} \widetilde{\mathcal{M}})_p \tag{1.2}$$

for any $q \in T^*_N Y \setminus N$.

Let $\mathcal{B}_M$ be the sheaf of hyperfunctions on M, $\mathcal{C}_N$ the sheaf of microfunctions on N (cf. [SKK]). Let $\mathrm{or}_{N|M}$ be the relative orientation sheaf of N in M as $\mathbf{C}$-module.

Theorem 1.2. *Assume* (A.1) *and* (A.2). *There is an isomorphism*

$$\text{(1.3)}\quad \mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\,\mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M}[1] \cong \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+,\,\mathcal{C}_N),$$

where $\dot{\pi}_N : T_N^*Y \setminus N \to N$.

Remark 1. Theorem 1.2 is first proved for elliptic $\mathcal{D}_X$-modules by Kashiwara and Kawai [KK]. Note that (A.1) and (A.2) are automatically satisfied if $\mathcal{M}$ is elliptic. Let $(x_1, \ldots, x_n)$ be a system of local coordinates of M, $Z_+ = \{x_1 \geq 0\}$. A classical example of non-elliptic differential operators which satisfy condition (A.2) is $D_1^2 - x_1^k A(x,\, D')$, with $k \in \mathbf{Z}$, $k \geq 2$, where $D_1 = \partial/\partial x_1$ and $A(x,\, D')$ is a differential operator of order 2 such that $[x_1,\, A] = 0$ and its principal symbol $\sigma(A)$ is negative valued on $T_M^*X \cap T_N^*X \setminus \rho^{-1}(0_N)$, 0_N being the zero section of T_N^*Y (i.e. $\sigma(A)(x,\, i\eta') < 0$ if $\eta' \neq 0$).

Remark 2. Condition (1.1) is an analogue of micro-hyperbolicity [KS2] and naturally appears in microlocal study of boundary value problems (cf. [S2, SZ]). It is well known that, if we assume

$$+H(\pi^* df) \notin C_p(\mathrm{Ch}(\mathcal{M}),\, Z_+ \times_M T_M^*X)/T_pT_M^*X$$

at $p \in T_M^*X \cap T_N^*X$, this entails propagation of regularity up to the boundary point p from the positive side of N (see [Kt2, S1, S2, SZ]).

Let $\mathcal{A}_M$ be the sheaf of real analytic functions on M. In place of (A.1) and (A.2), consider the following slightly stronger assumption. ((B.1) is the same as (A.1).)

(B.1) $\varphi : Y \to X$ is non characteristic for $\mathcal{M}$.

(B.2) φ is micro-hyperbolic for $\mathcal{M}$ at all $p \in T_M^*X \cap T_N^*X \setminus N$ [KS2, Def.2.1.2]: For both $\pm$,

$$\pm H(\pi^* df) \notin C_p(\mathrm{Ch}(\mathcal{M}),\, T_M^*X)/T_pT_M^*X.$$

Theorem 1.3. *Assume* (B.1) *and* (B.2). *There is an isomorphism*

$$\text{(1.4)}\quad \mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\,\mathcal{A}_M)|_N \otimes \mathrm{or}_{N|M}[1] \cong \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+,\,\mathcal{C}_N)$$

as well as isomorphism (1.3), *where* $\mathcal{N}^+$ *is the coherent* $\mathcal{E}_Y$*-module on* $T_N^*Y \setminus N$ *given in Lemma 1.1.*

2. Proof of Theorem 1.2 and 1.3

As in [KK], the proof of Theorem 1.2 is divided into two steps. In the first step, we relate the left-hand side of (1.3) to a differential complex with coefficients in $\mathcal{C}_{N|X}$ induced from $\mathcal{M}$. In the second step, proving Lemma 1.1, we complete the proof of Theorem 1.2.

Let us recall the notion of the $\mathcal{E}_X$-module $\mathcal{C}_{Z_+|X}$ due to Kataoka [Kt1] and Schapira [S2]. Following [S2], let

$$\mathcal{C}_{Z_+|X} = \mu\mathrm{hom}(\mathbf{C}_{Z_+}, \mathcal{O}_X) \otimes \mathrm{or}_{M|X}[n].$$

Then all the cohomology groups $H^k(\mathcal{C}_{Z_+|X})$, $k \neq 0$, are zero and $H^0(\mathcal{C}_{Z_+|X})$ is an $\mathcal{E}_X$-module. We identify $\mathcal{C}_{Z_+|X}$ with its zero-th cohomology $H^0(\mathcal{C}_{Z_+|X})$. For the $\mathcal{E}_X$-module $\mathcal{C}_{N|X}$, refer to [KK], [KS2] and also [S1, S2]. (In this paper, we follow the definition of [KK, KS2] : $\mathcal{C}_{N|X} = H^n\mu_N(\mathcal{O}_X) \otimes \mathrm{or}_{N|X}$.) We prepare two lemmas.

Lemma 2.1.

(1) $\mathrm{R}\pi_*\mathcal{C}_{Z_+|X}|_M \cong \mathrm{R}\Gamma_{Z_+}\mathcal{B}_M$.

(2) $\mathrm{supp}(\mathcal{C}_{Z_+|X}) \cap T^*_N X \subset \overline{(T^*_N X)^+}$.

(3) *There is an $\mathcal{E}_X$-homomorphism $\mathcal{C}_{N|X} \otimes \mathrm{or}_{N|M} \to \mathcal{C}_{Z_+|X}$, and this is an isomorphism on $(T^*_N X)^+$.*

For the proof, see [Kt3, Sect.4] and [S2, S3].

Lemma 2.2. *If we assume* (1.1) *at a point p of $T^*_M X \cap T^*_N X$, we have*

$$\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T^*_N X} = 0$$

in a neighborhood of p.

Proof. (Cf. the proof of Corollary 3.3 of [SZ].) Let g be a real-valued smooth function defined on X such that $g|_M = f$. We set $h = g \circ \pi$, with $\pi : T^*X \to X$. From (1.1), we have

$$-H(dh) \notin C_p(\mathrm{Ch}(\mathcal{M}),\, Z_+ \times_M T^*_M X).$$

Hence we can find an open subset U of T^*X so that $U \cap \mathrm{Ch}(\mathcal{M}) = \varnothing$,

$$-H(dh) \notin C_p(T^*X \setminus U,\, Z_+ \times_M T^*_M X),$$

and $-H(dh) \notin C_p(T^*X\backslash U, U)$. Let $T^*_{Z_+}X$ denote the micro-support $\mathrm{SS}(\mathbf{C}_{Z_+})$ of the sheaf $\mathbf{C}_{Z_+}$ on X (cf. [KS1, Sect.5.1]). Since $T^*_{Z_+}X \subset Z_+ \times_M T^*_M X \cup U$ on a neighborhood of p, we have $-H(dh) \notin C_p(T^*X \setminus U, T^*_{Z_+}X)$. This yields

$$-H(dh) \notin C_p(\mathrm{Ch}(\mathcal{M}), T^*_{Z_+}X).$$

Since

$$\mathrm{SS}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})) \subset C(\mathrm{Ch}(\mathcal{M}), T^*_{Z_+}X),$$

it follows from the definition of micro-supports that

$$\mathrm{R}\Gamma_{\{h\geq 0\}}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_{\{h=0\}} = 0$$

in a neighborhood of p. Since $\mathcal{C}_{Z_+|X}$ is supported on $T^*_{Z_+}X$ and $T^*_{Z_+}X \subset \{h \geq 0\}$, we have

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_{\{h=0\}} \cong \mathrm{R}\Gamma_{\{h\geq 0\}}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_{\{h=0\}} \cong 0.$$

Q.E.D.

Since $\mathbf{C}_{Z_+}$ is cohomologically constructible, if we set

$$F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),$$

it follows from [KS1, Prop.4.4.2] that

$$\begin{aligned}
\mathrm{R}\pi_*\mathrm{R}\Gamma_{T^*_X X}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N &\cong \mathrm{R}\pi_*\mathrm{R}\Gamma_{T^*_X X}\mu\mathrm{hom}(\mathbf{C}_{Z_+}, F)|_N[n] \\
&\cong \mathrm{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{Z_+}, \mathbf{C}_X) \otimes F|_N[n] \\
&\cong F \otimes \mathbf{C}_{Z_+\backslash N}|_N \\
&\cong 0.
\end{aligned}$$

Hence, from Lemma 2.1, we have

$$\begin{aligned}
\mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N &\cong \mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N \\
&\xrightarrow{\sim} \mathrm{R}\pi_*\mathrm{R}\Gamma_{T^*X\backslash X}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N \\
&\cong \mathrm{R}\pi'_*\left(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T^*_N X\backslash N}\right),
\end{aligned}$$

where $\pi' : T_N^*X \setminus N \to N$. It then follows from Lemma 2.1(2), (3) and 2.2 that

$$\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T_N^*X\setminus N} \cong \mathrm{R}\Gamma_{(T_N^*X)^+}(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T_N^*X\setminus N}) \\ \cong \mathrm{R}\Gamma_{(T_N^*X)^+}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X}) \otimes \mathrm{or}_{N|M}\,.$$

Thus we have

$$\text{(2.0)}\quad \mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M} \\ \cong \mathrm{R}\pi'_*\mathrm{R}\Gamma_{(T_N^*X)^+}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X}).$$

Since $T_Y^*X \cap \mathrm{Supp}(\widetilde{\mathcal{M}}) \subset T_X^*X$, we have

$$\text{the right-hand side of (2.0)} \\ \cong \mathrm{R}\dot{\pi}_{N*}\left[\mathrm{R}\rho_*\mathrm{R}\Gamma_{(T_N^*X)^+}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{T_N^*Y\setminus N}\right] \\ = \mathrm{R}\dot{\pi}_{N*}\left[\mathrm{R}\rho_*^+\left(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T_N^*X)^+}\right)|_{T_N^*Y\setminus N}\right],$$

where we denote by $\rho^+ : (T_N^*X)^+ \to T_N^*Y$ the restriction of ρ. Hence, in summary, we have[2]

$$\text{(2.1)}\quad \mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M} \\ \cong \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\rho_*^+\left(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T_N^*X)^+}\right).$$

In the rest of this section, we prove

$$\text{(2.2)}\qquad \mathrm{R}\rho_*^+\left(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T_N^*X)^+}\right)[1] \cong \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N)$$

on $T_N^*Y \setminus N$. Combining (2.1) and (2.2), we get isomorphism (1.3).

We prepare two lemmas for the second part of the proof. Lemma 1.1 follows from the following Lemma 2.3 with $I = T_N^*Y \setminus N$.

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[2] Takeuchi also proves (2.1) in the case where (B.1) and (B.2) are fulfilled; see K. Takeuchi : Edge of the wedge type theorems for hyperfunction solutions, preprint (Jan. 1996). If we assume (B.2), $M \hookrightarrow X$ is non characteristic for F on $N^+(\subset T^*M)$, and we immediately obtain (2.0) by applying Theorem 6.7.1 of [KS1] (see also Corollary 6.7.3).

Lemma 2.3. *Let I be a conic open subset of $T_N^*Y \setminus N$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on a conic neighborhood of $\rho^{-1}(I)$, with $\rho : T_N^*X \to T_N^*Y$. Assume the following :*

(a.1) *$\varphi : Y \to X$ is non characteristic for $\mathcal{M}$ on a neighborhood of I in the sense of [SKK, II, Def.3.5.4].*

(a.2) *For a conic neighborhood U of $\rho^{-1}(I) \cap T_M^*X$,*

$$U \cap (T_N^*X)^+ \cap \operatorname{Supp}(\mathcal{M}) = \varnothing.$$

Then (1) ρ is finite on $\rho^{-1}(I) \cap (T_N^*X)^+ \cap \operatorname{Supp}(\mathcal{M})$. (2) If we set

$$\mathcal{N}^+ = \rho_*((\mathcal{E}_{Y\to X} \otimes_{\mathcal{E}_X} \mathcal{M}) \otimes \mathbf{C}_{(T_N^*X)^+}),$$

$\mathcal{N}^+$ is a coherent $\mathcal{E}_Y|_I$-module.

(We omit the proof. Cf. [SKK, II, Thm.3.5.3].)

Lemma 2.4. *Let $\mathcal{M}$, $\mathcal{N}^+$ be as in Lemma 2.3. Then there exists a commutative diagram on I*

$$\begin{array}{ccc} \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{E}_Y) & \xrightarrow{\sim} & \mathrm{R}\rho_*^+(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X\leftarrow Y})|_{(T_N^*X)^+})[1] \\ \uparrow & & \uparrow \\ \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\varphi^*\mathcal{M}, \mathcal{E}_Y) & \xrightarrow{\sim} & \mathrm{R}\rho_*(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X\leftarrow Y})|_{T_N^*X})[1] \end{array}$$

and every horizontal arrow is an isomorphism, where $\rho^+ = \rho|(T_N^*X)^+$.

Proof. This follows from the definition of $\mathcal{N}^+$ and [SKK, II, Thm.3.5.6]. Q.E.D.

Since $\mathcal{N}^+$ is coherent over $\mathcal{E}_Y|_{T_N^*Y}$ and ρ^+ is finite on $\operatorname{Supp}(\widetilde{\mathcal{M}}) \cap (T_N^*X)^+$, by Lemma 2.4, we have

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N) &\cong \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{E}_Y) \otimes^L_{\mathcal{E}_Y} \mathcal{C}_N \\ &\cong \rho_*^+[\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{E}_{X\leftarrow Y})|_{(T_N^*X)^+}] \otimes^L_{\mathcal{E}_Y} \mathcal{C}_N[1] \\ &\cong \rho_*^+[\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{E}_{X\leftarrow Y})|_{(T_N^*X)^+} \otimes^L_{\rho^{-1}\mathcal{E}_Y} \rho^{-1}\mathcal{C}_N][1]. \end{aligned}$$

Using the $\mathcal{E}_X$-homomorphism $\mathcal{E}_{X\leftarrow Y} \otimes_{\rho^{-1}\mathcal{E}_Y} \rho^{-1}\mathcal{C}_N \to \mathcal{C}_{N|X}$ [KK, II], we have

$$\begin{aligned} &\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N) \\ &\to \rho_*^+[\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{E}_{X\leftarrow Y} \otimes^L_{\rho^{-1}\mathcal{E}_Y} \rho^{-1}\mathcal{C}_N)|_{(T_N^*X)^+}][1] \\ &\to \rho_*^+[\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T_N^*X)^+}][1]. \end{aligned} \tag{2.3}$$

Let $q \in T^*_N Y \setminus N$. For $k \in \mathbf{Z}$, looking at the stalk on q, we have from (2.3)

$$\mathcal{E}xt^k_{\mathcal{E}_Y}(\mathcal{N}^+_q, \mathcal{C}_{Nq}) \to \bigoplus_{p \in (T^*_N X)^+ \cap \mathrm{Supp}(\widetilde{\mathcal{M}}) \cap \rho^{-1}(q)} \mathcal{E}xt^{k+1}_{\mathcal{E}_X}(\widetilde{\mathcal{M}}_p, (\mathcal{C}_{N|X})_p).$$

It follows from the division theorem for the $\mathcal{E}_X$-module $\mathcal{C}_{N|X}$ [KK, II, Prop.3; KS2, 6.3.1] and the definition of $\mathcal{N}^+$ that this is an isomorphism for any $k \in \mathbf{Z}$; therefore (2.3) is an isomorphism in $\mathrm{D^b}(T^*_N Y \setminus N)$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. If φ is micro-hyperbolic for $\widetilde{\mathcal{M}}$ at $p \in T^*_M X \times_M N$, we have [KS2]

$$\mathrm{R}\Gamma_{\pi_M^{-1}(Z_+)} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_M)_p = 0.$$

Since this holds at all $p \in (T^*_M X \setminus M) \times_M N$ by assumption (B.2), we have an isomorphism

$$\mathrm{R}\Gamma_{Z_+} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)|_N \xrightarrow{\sim} \mathrm{R}\Gamma_{Z_+} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N.$$

Combining this and (1.3), we get (1.4). Q.E.D.

3. Application

Let $M_+ = Z_+ \setminus N$. Isomorphism (1.3) gives a description of the structure of the sheaf $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M_+}\mathcal{B}_M)|_N$ in terms of a system of micro-differential equations on the boundary.

Theorem 3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Assume* (A.1) *and* (A.2). *Assume moreover $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) = 0$ for all $k > 0$. Then*

$$(3.1) \quad \mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M_+}\mathcal{B}_M)|_N \cong \mathrm{Ker}(\mathcal{H}om_{\mathcal{D}_Y}(\varphi^*\mathcal{M}, \mathcal{B}_N) \to \dot{\pi}_{N*}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N)),$$

where $\varphi^*\mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{\varphi^{-1}\mathcal{D}_X} \varphi^{-1}\mathcal{M}$ and $\mathcal{N}^+$ is the coherent $\mathcal{E}_Y$-module on $T^*_N Y \setminus N$ given in Lemma 1.1, and

$$(3.2) \quad \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M_+}\mathcal{B}_M)|_N \cong H^k \mathrm{R}\dot{\pi}_{N*} \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\varphi^*\widetilde{\mathcal{M}}/\mathcal{N}^+, \mathcal{C}_N)$$

for $k \neq 0$.

Proof. Let us first recall that, if $\varphi : Y \to X$ is non characteristic for a $\mathcal{D}_X$-module $\mathcal{M}$, we have a canonical isomorphism

$$\mathrm{R}\Gamma_N \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M}[1] \cong \mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\varphi^*\mathcal{M}, \mathcal{B}_N)$$

[SKK, II, Cor.3.5.8]. By the proof of Theorem 1.2, the following diagram is commutative :

$$\begin{array}{ccc} \mathrm{R}\Gamma_N \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M}[1] & \longrightarrow & \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\varphi^*\widetilde{\mathcal{M}}, \mathcal{C}_N) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{Z_+} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M}[1] & \underset{(1.3)}{\xrightarrow{\sim}} & \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N). \end{array}$$

Hence, from the Mayer-Vietoris cohomological sequence, we have a long exact sequence

$$\cdots \to \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M_+}\mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M} \to \mathcal{E}xt^k_{\mathcal{D}_Y}(\varphi^*\mathcal{M}, \mathcal{B}_N)$$
$$\to H^k\mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N) \to \cdots,$$

where the second arrow is factorized as follows :

$$\mathcal{E}xt^k_{\mathcal{D}_Y}(\varphi^*\mathcal{M}, \mathcal{B}_N) \xrightarrow{\alpha} H^k\mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\varphi^*\widetilde{\mathcal{M}}, \mathcal{C}_N)$$
$$\xrightarrow{\beta} H^k\mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N).$$

Since $\mathcal{E}xt^k_{\mathcal{D}_Y}(\varphi^*\mathcal{M}, \mathcal{A}_N) = 0$ for $k > 0$ by assumption, α is surjective for all $k \in \mathbf{Z}$ and is an isomorphism for $k > 0$. On the other hand, since $\mathcal{N}^+$ is a direct summand of $\varphi^*\widetilde{\mathcal{M}}$ as an $\mathcal{E}_Y$-module, β is surjective and

$$\mathrm{Ker}(\beta) = H^k\mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\varphi^*\widetilde{\mathcal{M}}/\mathcal{N}^+, \mathcal{C}_N).$$

Hence, using an isomorphism $\mathrm{or}_{N|M} \cong \mathbf{C}_N$ (see Remark 1 below), we obtain (3.1) and (3.2). Q.E.D.

Remark 1. The following diagram is commutative and every vertical arrow is an isomorphism :

$$\begin{array}{ccc} \mathbf{C}_{M_+} & \longrightarrow & \mathbf{C}_{Z_+} \\ \downarrow & & \downarrow \\ \mathrm{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{Z_+}, \mathbf{C}_M) & \longrightarrow & \mathrm{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{M_+}, \mathbf{C}_M). \end{array}$$

Hence we have an isomorphism $\eta : \mathbf{C}_N \to \mathrm{or}_{N|M}$ such that

$$\begin{array}{ccccc} \mathbf{C}_{Z_+} & \longrightarrow & \mathbf{C}_N & \longrightarrow & \mathbf{C}_{M_+}[1] \\ \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow \\ \mathrm{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{M_+}, \mathbf{C}_M) & \longrightarrow & \mathrm{or}_{N|M} & \longrightarrow & \mathrm{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{Z_+}, \mathbf{C}_M)[1] \end{array}$$

becomes commutative. (This corresponds to choosing a non-degenerate section df of T_N^*M as positive orientation.) Note that the following diagram is then commutative for $F \in \mathrm{Ob}(\mathrm{D^b}(M))$:

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{M_+}F|_N & \xrightarrow{1\otimes\eta} & \mathrm{R}\Gamma_{M_+}F|_N \otimes \mathrm{or}_{N|M} & \longrightarrow & \\
\downarrow\cong & & & & \\
\mathrm{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{M_+}, F)|_N & \longrightarrow & \mathrm{R}\mathcal{H}om_{\mathbf{C}}(\mathrm{or}_{N|M}[-1], F)|_N & \xrightarrow{\cong} & \\
& & & \longrightarrow & \mathrm{R}\Gamma_N F|_N[1] \otimes \mathrm{or}_{N|M} \\
& & & & \downarrow\cong \\
& & & \xrightarrow{\cong} & \mathrm{R}\Gamma_N F|_N[1] \otimes \mathrm{or}^\vee_{N|M}
\end{array}$$

with $\mathrm{or}^\vee_{N|M} = \mathcal{H}om_{\mathbf{C}}(\mathrm{or}_{N|M}, \mathbf{C}_N)$, which is canonically isomorphic to $\mathrm{or}_{N|M}$. (The topological boundary value morphism for F is defined [S2, S3] as anti-clockwise composition of morphisms, from $\mathrm{R}\Gamma_{M_+}F|_N$ to $\mathrm{R}\Gamma_N F|_N[1] \otimes \mathrm{or}^\vee_{N|M}$, in this diagram.)

Remark 2. For single differential equations, Oaku [O, Sect.3] extends (3.1) to the case where condition (A.2) is satisfied locally on T_N^*Y. If $\mathcal{N}^+ = 0$ in that case, this has been first treated by Kaneko [Kn].

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