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MICROLOCAL STUDY OF TRICOMI EQUATIONS

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In this article we consider the initial value problems for Tricomi operators. For that purpose we employ a method by which we can simultaneously treat both hyperbolic operators and elliptic operators (It was at first employed in [4] only for hyperbolic operators). Our method will be applicable also for more general operators, but we do not discuss about further generalization here.

Let $P(x, D)$ be a microdifferential operator defined at $x^* = (0; 0, \cdots, 0, \sqrt{-1}) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^n$ of order m, written in the form

$$P(x,D) = D_1^2 - x_1 D_n^2 + \sum_{j=0,1} P_j(x, D')D_1^j,$$
$$\operatorname{ord} P_j \leq 1 - j.$$

Here we have written $D' = (D_2, \cdots, D_n)$, $D'' = (D_1, \cdots, D_{n-1})$, and $D''' = (D_2, \cdots, D_{n-1})$ as usual.

Let $x^{*\prime} = (0; 0, \cdots, 0, \sqrt{-1}) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1}$. We consider the Cauchy problem

$$\text{(1)} \qquad \begin{cases} Pu(x) = 0, \\ D_1^j u(0, x') = v_{j+1}(x'), \quad 0 \leq j \leq 1, \end{cases}$$

where $u(x) \in \mathcal{C}_{\mathbf{R}^n, x^*}$ and $v_j(x') \in \mathcal{C}_{\mathbf{R}^{n-1}, x^{*\prime}}$. We rewrite (1) in the following form:

$$\text{(2)} \qquad L\vec{u} = \vec{0}, \ \vec{u}(0, x') = \vec{v}(x').$$

Here we have written

$$L(x, D) = D_1 I_2 + \begin{pmatrix} 0, & -1, \\ Q_0(x, D'), & Q_1(x, D') \end{pmatrix}$$

with some $Q_j(x, D') \in \mathcal{E}_{x^*}$, $\operatorname{ord} Q_j \leq 2 - j$, and

$$\vec{u}(x) = \begin{pmatrix} u \\ D_1 u \end{pmatrix}, \qquad \vec{v}(x') = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

It is easy to see that for any $\vec{v} \in (\mathcal{C}_{\mathbf{R}^{n-1}, x^{*\prime}})^2$ there uniquely exists a formal solution $\vec{u}$, which is an analytic functional. In fact using the so-called pseudodifferential operators with finite velocity, we can construct the elementary solution for the above Cauchy problem (See [4]). Of course it is not a microfunction solution, and it does not have any meaning as it stands. Therefore we need to discuss when it becomes a microfunction.

For that purpose, we give the following

Theorem 1. *Let* $\omega \subset \sqrt{-1}\mathbf{S}^*\mathbf{R}^n$ *be a small neighborhood of* x^*, *and let* $\omega^\pm = \{(x,\xi)\infty \in \omega;\ \pm x_1 > 0\}$. *There exist* 2×2 *invertible matrices of holomorphic microlocal operators* $F^\pm(x,D')$, $G^\pm(x',D')$ *defined on (a complex neighborhood of)* $\omega^\pm$ *such that*

$$L(x,D) = F^\pm(x,D')\Lambda(x,D)G^\pm(x',D')$$

on $\omega^\pm$, *where*

$$\Lambda = \begin{pmatrix} D_1 - \sqrt{x_1}D_n, & 0 \\ 0, & D_1 + \sqrt{x_1}D_n \end{pmatrix}.$$

Here we take $\arg x_1 = 0$ (resp. $\arg x_1 = \pi$) on ω^+ (resp. ω^-), and we define $\arg\sqrt{x_1} = (\arg x_1)/2$. Holomorphic microlocal operators are some class of analytic pseudodifferential operators. They are defined by [6], and [1] gave a symbol theory for them. Note that they have microlocal property. Since $G^\pm$ does not contain x_1, it is in fact defined on the full neighborhood ω of x^*.

We next give an application. In fact the author does not have given the complete proof for the rest of this article, although in the symposium he reported as if he had proved even a stronger result.

Let $\omega \subset \sqrt{-1}\mathbf{S}^*\mathbf{R}^n$ (resp. $\omega' \subset \sqrt{-1}\mathbf{S}^*\mathbf{R}^{n-1}$) be a small neighborhood of x^* (resp. $x^{*\prime}$), and let

$$\omega_0(r) = \{(x,\xi) \in \omega; |\xi''| \le r|x_1|\mathrm{Im}\ \xi_n\},$$
$$\omega^\pm{}_0(r) = \omega^\pm \cap \omega_0(r),$$
$$\omega_0' = \{(x',\xi') \in \omega'; \xi''' = 0\}$$

for $r > 0$ (here $x^{*\prime} = (0;0,\cdots,\sqrt{-1}) \in \sqrt{-1}\mathbf{S}^*\mathbf{R}^{n-1}$). We define

$$\mathcal{C}_1 = \varinjlim_{r,\omega} \Gamma_{\omega_0(r)}(\omega,\mathcal{C}),$$
$$\mathcal{C}_1{}^\pm = \varinjlim_{r,\omega} \Gamma_{\omega^\pm{}_0(r)}(\omega^\pm,\mathcal{C}),$$
$$\mathcal{C}_1{}' = \varinjlim_{\omega'} \Gamma_{\omega_0'}(\omega',\mathcal{C}).$$

Let us consider (2) in the special case $\vec{v} \in (\mathcal{C}_1{}')^2$, and discuss if the formal solution $\vec{u}$ belongs to $(\mathcal{C}_1)^2$. We first consider $\vec{u}$ in the hyperbolic region ω^+. Multiplying (2) by $(F^+(x,D'))^{-1}$ from the left, we obtain

$$\Lambda(x,D)\vec{u}^+ = \vec{0},\ \vec{u}^+(0,x') = \vec{v}^+(x'), \tag{3}$$

where $\vec{u}^\pm(x) = (F^\pm(x,D'))^{-1}\vec{u}(x)$ and $\vec{v}^\pm(x') = G^\pm(x',D')\vec{v}(x')$. It is easy to see that the formal solution $\vec{u}^+$ (and thus $\vec{u}$) always becomes a microfunction here. In fact, denoting $\vec{u}^\pm = {}^t(u_1^\pm, u_2^\pm)$ and $\vec{v}^\pm = {}^t(v_1^\pm, v_2^\pm)$, we have

$$u_1^+ = \exp\left(\frac{2}{3}x^{3/2}D_n\right)v_1^+(x'),$$
$$u_2^+ = \exp\left(-\frac{2}{3}x^{3/2}D_n\right)v_2^+(x')$$

on ω^+. Here $\exp(\pm\frac{2}{3}x^{3/2}D_n)$ are nothing but classical Fourier integral operators on ω^+, because the exponent of their complete symbols $\exp(\pm\frac{2}{3}x^{3/2}\xi_n)$ are pure imaginary on ω^+. In fact we have

$$\exp(\pm\frac{2}{3}x^{3/2}D_n) : \mathcal{C}_1{}^+ \ni w(x) \longrightarrow w(x'', x_n \pm \frac{2}{3}x_1^{3/2}) \in \mathcal{C}_1{}^+. \tag{4}$$

We next consider the formal solution $\vec{u}$ on the elliptic region ω^-. Multiplying (2) by $(F^-(x,D'))^{-1}$ from the left, we obtain

$$\Lambda(x,D)\vec{u}^- = \vec{0},\ \vec{u}^-(0,x') = \vec{v}^-(x'), \tag{5}$$

In this case formally we have

$$u_1^- = \exp(\frac{2}{3}x^{3/2}D_n)v_1^-(x'),$$
$$u_2^- = \exp(-\frac{2}{3}x^{3/2}D_n)v_2^-(x')$$

on ω^-. Here $\exp(\pm\frac{2}{3}x^{3/2}D_n)$ are defined similarly in the sense of analytic functionals, but this time they are not real Fourier integral operators. In fact, employing the definition (4) it is easy to see that

(i) $v_1^- \neq 0 \Longrightarrow u_1^- \notin \mathcal{C}_1{}^-$,

(ii) for any $v_2^- \in \mathcal{C}_1{}'$ we have $u_2^- = 0$.

It follows that

$$\begin{aligned} \vec{u} \in (\mathcal{C}_1)^2 &\Longleftrightarrow \vec{u}^- \in (\mathcal{C}_1{}^-)^2 \\ &\Longleftrightarrow u_1^- \in \mathcal{C}_1{}^- \\ &\Longleftrightarrow v_1^- = 0 \\ &\Longleftrightarrow g_{(1,1)}(x',D')v_1(x') + g_{(1,2)}(x',D')v_2(x') = 0, \end{aligned} \tag{6}$$

where

$$G^-(x',D') = \begin{pmatrix} g_{(1,1)}(x',D'), & g_{(1,2)}(x',D') \\ g_{(2,1)}(x',D'), & g_{(2,2)}(x',D') \end{pmatrix}.$$

Therefore we obtain the following

Theorem 2. *Let $\vec{v} \in (\mathcal{C}_1{}')^2$. Then we have $\vec{u} \in (\mathcal{C}_1)^2$ if, and only if,*

$$g_{(1,1)}(x',D')v_1(x') + g_{(1,2)}(x',D')v_2(x') = 0.$$

We have another representation. Since $G^-(x',D')$ is invertible, we have $\vec{v} = (G^-(x',D'))^{-1}\vec{w}(x')$, where $\vec{w} = G^-(x',D')\vec{v}(x')$. If $\vec{v}(x')$ satisfies (6), the first component $w_1(x')$ must vanish, and thus we have

$$\vec{v}(x') = \begin{pmatrix} h_{(1,1)}(x',D'), & h_{(1,2)}(x',D') \\ h_{(2,1)}(x',D'), & h_{(2,2)}(x',D') \end{pmatrix} \begin{pmatrix} 0 \\ w_2(x') \end{pmatrix} = \begin{pmatrix} h_{(1,2)}(x',D')w_2(x') \\ h_{(2,2)}(x',D')w_2(x') \end{pmatrix},$$

where

$$(G^-(x',D'))^{-1} = \begin{pmatrix} h_{(1,1)}(x',D'), & h_{(1,2)}(x',D') \\ h_{(2,1)}(x',D'), & h_{(2,2)}(x',D') \end{pmatrix}.$$

Let $A = \{\vec{v}(x') \in (\mathcal{C}_1{}')^2;$ the Cauchy problem (2) is well defined$\}(\subset (\mathcal{C}_1{}')^2)$. We have proved that A is one-dimensional, i.e.,

Corollary. *Let* $\vec{v} \in (\mathcal{C}_1{}')^2$*. Then we have* $\vec{u} \in (\mathcal{C}_1)^2$ *if, and only if,*

$$v_1(x') = h_{(1,2)}(x', D')w_2(x'),$$
$$v_2(x') = h_{(2,2)}(x', D')w_2(x').$$

References

[1] T. Aoki, *Symbols and formal symbols of pseudodifferential operators*, Advanced Studies in Pure Mathematics **4** (1984), 181–208.

[4] M. Kashiwara and T. Kawai, *Microhyperbolic pseudodifferential operators I*, J. Math. Soc. Japan **27** (1975), 359–404.

[6] M. Sato, T. Kawai, and M. Kashiwara, Microfunctions and pseudo-differential equations Lect. Notes in Math. 287, Springer, 1973, pp. 265–529.

[4] K. Uchikoshi, *Stokes operators for microhyperbolic equations*, to appear.