数理解析研究所講究録
983 巻 1997 年 14-21

Polytopes, Invariants, and PDEs

岩崎 克則
KATSUNORI IWASAKI

Department of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153 Japan

1. POLYTOPES AND MEAN VALUE PROPERTY

The following theorem due to C.F. Gauss is fundamental in potential theory.

Theorem. (Gauss, 1840) *Let Ω be a domain in $\mathbb{R}^n$. Any function $f \in C^2(\Omega)$ is harmonic if and only if f belongs to $C(\Omega)$ and satisfies the mean value property:*

$$\text{(MVP)} \qquad f(x) = \frac{1}{\mu(V)} \int_V f(x+ry)\, d\mu(y),$$

for any $x \in \Omega$ and $0 < r < r_x$, where V is the unit ball (sphere) in $\mathbb{R}^n$ with center at the origin and μ is the volume (surface) element on V.

It is a natural problem to consider what happens if V is not the unit ball (sphere) but another figure in $\mathbb{R}^n$. Let $\mathcal{H}_V(\Omega)$ be the set of all functions $f \in C(\Omega)$ satisfying the mean value property (MVP) with respect to V. Then $\mathcal{H}_V(\Omega)$ is a linear space containing the constant functions. Our problem is to characterize the function space $\mathcal{H}_V(\Omega)$. In this talk we consider the case where V is a *polytope* in $\mathbb{R}^n$.

Let P be a polytope in $\mathbb{R}^n$, $P(k)$ the k-skeleton of P for $k = 0, 1, \dots, n$. We give a precise definition of polytope.

Definition. A *convex polytope* P is a finite intersection of closed half-spaces in $\mathbb{R}^n$ such that P is bounded and contains an interior point:

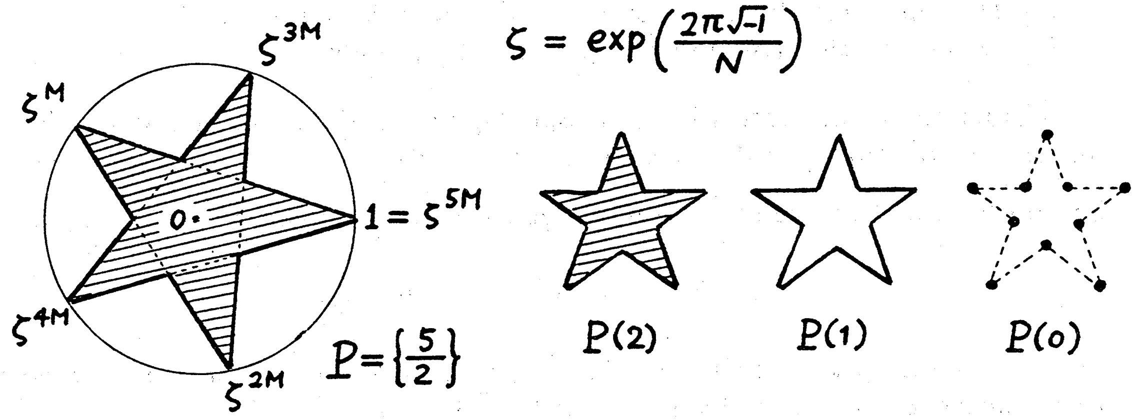
$$P = H_1^+ \cap H_2^+ \cap \cdots \cap H_s^+.$$

A *polytope* P is a finite union of convex polytopes in $\mathbb{R}^n$:

$$P = P_1 \cup P_2 \cup \cdots \cup P_t.$$

So a polytope is a solid, but not necessarily convex nor connected. In order to get a concrete image we give an example.

Example. (Regular star-polygons) The regular *star-polygon* $P = \{N/M\}$ for $(M, N) = 1$ and its skeletons $P(k)$ are given as follows.



In case $M = 1$, $\{N\}$ is a regular convex N-gon.

In 1962 A. Friedman and W. Littman proposed the following problem.

Problem. (Friedman-Littman, 1962) *Is $\mathcal{H}_{P(k)}(\Omega)$ finite-dimensional ?*

In fact they assumed that P is convex and $k = n - 1, n$. This problem has been open since they posed the question (except for only a few specific polytopes). Recently I was able to settle it affirmatively for any polytope P and any $k = 0, 1, \ldots, n$.

Main Theorem. *Let P be any (not necessarily convex nor connected) solid polytope in $\mathbb{R}^n$, $G \subset O(n)$ the symmetry group of P. Then for any $k = 0, 1, \ldots, n$, $\mathcal{H}_{P(k)}(\Omega)$ enjoys the following properties:*

(1) *$\mathcal{H}_{P(k)}(\Omega)$ is a finite-dimensional linear space of polynomials,*
(2) *a basis of $\mathcal{H}_{P(k)}(\Omega)$ can be taken from homogeneous polynomials,*
(3) *$\mathcal{H}_{P(k)}(\Omega)$ admits a structure of $\mathbb{C}[\partial]$-module, and*
(4) *if G acts on $\mathbb{R}^n$ irreducibly then $\mathcal{H}_{P(k)}(\Omega)$ is a finite-dimensional linear space of harmonic polynomials.*

Remark. It can be shown that the restriction map $\mathcal{H}_{P(k)}(\mathbb{R}^n) \to \mathcal{H}_{P(k)}(\Omega)$ is isomorphic; hence $\mathcal{H}_{P(k)}(\Omega)$ is independent of Ω. This allows us to write $\mathcal{H}_{P(k)}(\Omega) = \mathcal{H}_{P(k)}$. The theorem implies that $\mathcal{H}_{P(k)}$ is completely different from the classical space $\mathcal{H}_V(\Omega)$ with $V = B^n$ or S^{n-1}; the latter is always infinite-dimensional and the restriction map $\mathcal{H}_V(\Omega_1) \to \mathcal{H}_V(\Omega_2)$ for $\Omega_1 \supset \Omega_2$ is not (always) isomorphic.

Following the remark we use the simplified notation $\mathcal{H}_{P(k)} = \mathcal{H}_{P(k)}(\Omega)$. The second assertion of the main theorem implies that there exists a direct sum decomposition:

$$\mathcal{H}_{P(k)} = \bigoplus_{m \geq 0}^{\text{finite}} \mathcal{H}_{P(k)}(m),$$

where $\mathcal{H}_{P(k)}(m)$ is the linear space of all homogeneous polynomials of degree m satisfying the $P(k)$-mean value property. Several problems naturally arise. For example the following problems are interesting from a combinatorial point of view.

Problem.

(1) Determine $\dim \mathcal{H}_{P(k)}$ and construct a basis of $\mathcal{H}_{P(k)}$.
(2) Determine $\dim \mathcal{H}_{P(k)}(m)$ and construct a basis of $\mathcal{H}_{P(k)}(m)$.
(3) Determine the structure of $\mathcal{H}_{P(k)}$ as a $\mathbb{C}[\partial]$-module.

In order to demonstrate what is happening we consider the case where P is a regular star-polygon $\{N/M\}$ with center at the origin.

Example. *Let P be a regular star-polygon $\{N/M\}$ with center at the origin.*

$$\dim \mathcal{H}_{P(k)} = 2N \qquad (k = 0, 1, 2).$$

Let (x, y) be an orthonormal coordinate system of $\mathbb{R}^2$ such that P is symmetric with respect to the x-axis. We set $z = x + \sqrt{-1}y$. Then,

$$\mathcal{H}_{P(k)}(m) = \begin{cases} \mathbb{C} & (m = 0), \\ \mathbb{C}z^m \oplus \mathbb{C}\bar{z}^m & (1 \le m \le N-1), \\ \mathbb{C}\operatorname{Im}(z^N) & (m = N), \\ \{0\} & (m \ge N+1). \end{cases}$$

As a $\mathbb{C}[\partial]$-module $\mathcal{H}_{P(k)}$ is generated by the single element $\operatorname{Im}(z^N)$.

2. Characterization in Terms of Partial Differential Equations

The classical mean value property (with respect to B^n or S^{n-1}) is characterized by the Laplace equation $\Delta f = 0$. The $P(k)$-mean value property can also be characterized in terms of partial differential equations, though, not by a single equation but by a *system* of infinitely many equations.

In order to describe the system we introduce some notation. Let $\{P_{i_j}\}_{i_j \in I_j}$ be the set of j-dimensional faces of P, H_{i_j} the j-dimensional affine subspace of $\mathbb{R}^n$ containing P_{i_j}, $\pi_{i_j} : \mathbb{R}^n \to H_{i_j}$ the orthogonal projection from $\mathbb{R}^n$ down to the subspace H_{i_j}. Let $p_{i_j} \in \mathbb{R}^n$ be the vectors (or points) in $\mathbb{R}^n$ defined by

$$p_{i_j} = \pi_{i_j}(0) \in H_{i_j}.$$

We remark that $P_{i_0} = H_{i_0} = \{p_{i_0}\}$ for any $i_0 \in I_0$ and that $H_{i_n} = \mathbb{R}^n$ and $p_{i_n} = 0$ for any $i_n \in I_n$. For $i_j \in I_j$ and $i_{j+1} \in I_{j+1}$ we write $i_j \prec i_{j+1}$ if P_{i_j} is a face of $P_{i_{j+1}}$. For $i_j \prec i_{j+1}$ let $\mathbf{n}_{i_j i_{j+1}}$ be the outer unit normal vector of $\partial P_{i_{j+1}}$ in $H_{i_{j+1}}$ at the face P_{i_j}. The vector $p_{i_j} - p_{i_{j+1}}$ is parallel to $\mathbf{n}_{i_j i_{j+1}}$, so that we can define the *incidence number* $[i_j : i_{j+1}] \in \mathbb{R}$ by the relation

$$p_{i_j} - p_{i_{j+1}} = [i_j : i_{j+1}]\mathbf{n}_{i_j i_{j+1}}.$$

Let $I(k)$ be the index set defined by

$$I(k) = \{i = (i_0, i_1, \ldots, i_k)\ ;\ i_j \in I_j,\ i_0 \prec i_1 \prec \cdots \prec i_k\}.$$

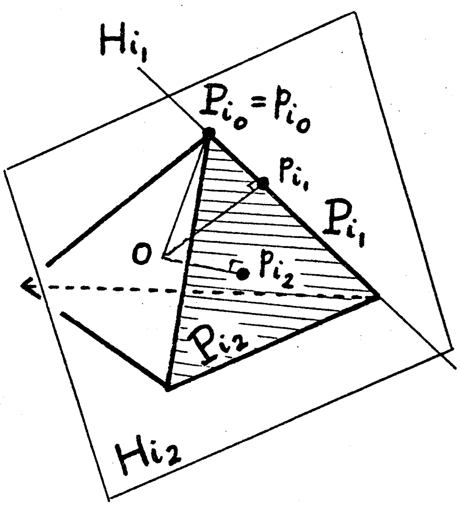
For $i = (i_0, i_1, \dots, i_k) \in I(k)$ we set

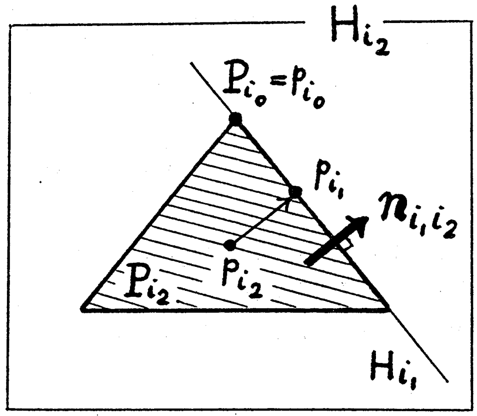
$$[i] = \begin{cases} 1 & (k = 0), \\ [i_0 : i_1][i_1 : i_2] \cdots [i_{k-1} : i_k] & (k = 1, 2, \dots, n). \end{cases}$$

Let $h_m^{(j)}(\xi)$ be the complete symmetric polynomial of degree m in j-variables:

$$h_m^{(j)}(\xi_1, \dots, \xi_j) = \sum_{m_1 + \cdots + m_j = m} \xi_1^{m_1} \xi_2^{m_2} \cdots \xi_j^{m_j},$$

where the summation is taken over all j-tuples $(m_1, \dots, m_j)$ of nonnegative integers satisfying the indicated condition. Finally we set $\langle \xi, \eta \rangle = \xi_1 \eta_1 + \cdots + \xi_n \eta_n$ for two vectors $\xi = (\xi_1, \dots, \xi_n), \eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^n$.





The following theorem gives a characterization of the $P(k)$-mean value property in terms of a system of partial differential equations.

Theorem 1. *Any $f \in \mathcal{H}_{P(k)}(\Omega)$ is smooth in Ω and satisfies the system of partial differential equations:*

$$(*) \qquad \tau_m^{(k)}(\partial) f = 0 \qquad (m = 1, 2, 3, \dots),$$

where $\tau_m^{(k)}(\xi)$ is the homogeneous polynomial of degree m defined by

$$\tau_m^{(k)}(\xi) = \sum_{i \in I(k)} [i] h_m^{(k+1)}(\langle p_{i_0}, \xi \rangle, \langle p_{i_1}, \xi \rangle, \dots, \langle p_{i_k}, \xi \rangle),$$

Conversely any distribution solution of $(*)$ is real-analytic and belongs to $\mathcal{H}_{P(k)}(\Omega)$.

Here a crucial observation is that the system $(*)$ is *holonomic*. The holonomicity follows from the geometry and combinatorics of the polytope P.

Theorem 2. *The system $(*)$ is holonomic.*

These two theorems play an essential role in establishing our main theorem.

3. Polytopes with Symmetry

Our problem is of particular interest if P admits symmetry. Let $G \subset O(n)$ be the symmetry group of P. Then we have a lower bound of the dimension of $\mathcal{H}_{P(k)}$ in terms of G.

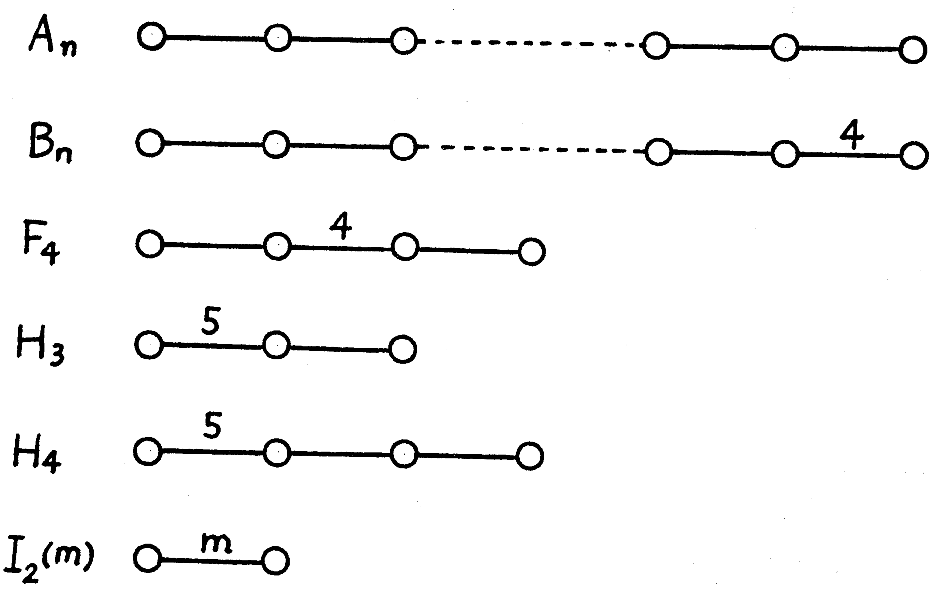
Theorem 3. $\dim \mathcal{H}_{P(k)} \geq |G|$.

For any regular convex polytope P in $\mathbb{R}^n$ with center at the origin, we are able to determine the function space $\mathcal{H}_{P(k)}$ explicitly. The symmetry group $G \subset O(n)$ of P is an irreducible finite reflection group. All irreducible finite reflection groups are classified in terms of connected Coxeter graphs. Hence we have the following diagram.

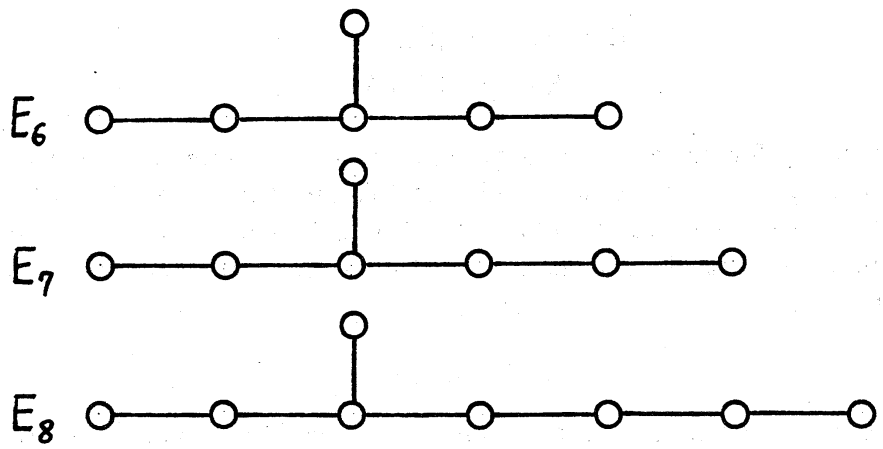
Diagram.

$$\begin{array}{ccl}
\{\text{regular convex polytopes}\} & P & \\
\downarrow & \downarrow & (\text{symmetry group}) \\
\{\text{irreducible finite reflection groups}\} & G & \\
\updownarrow & \updownarrow & (\text{one-to-one}) \\
\{\text{connected Coxeter graphs}\} & \Gamma &
\end{array}$$

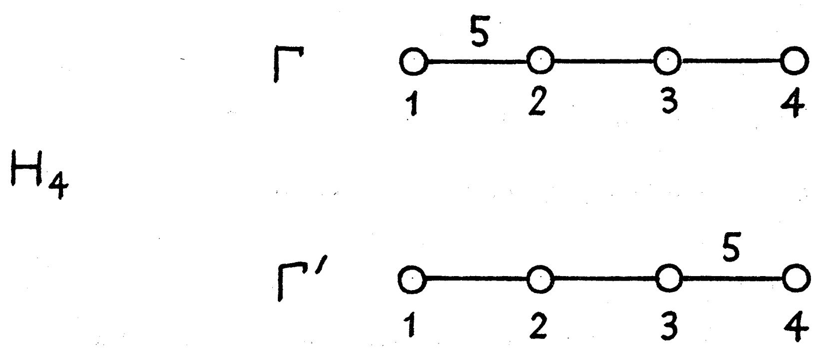
An irreducible finite reflection group G is the symmetry group of a regular convex polytope P if and only if the Coxeter graph Γ of G has no node. Therefore all admissible graphs are precisely those of types A, B, F, H and I.



Graphs of types D and E do not correspond to any regular convex polytope.



A regular convex polytope P and its reciprocal P^* correspond to the same Coxeter graph Γ, but no other regular convex polytopes correspond to Γ. Moreover $P = P^*$ if and only if $\Gamma = \Gamma'$, where Γ' is the reversed graph of Γ.



Accordingly we have the classification of regular convex polytopes:

Classification of regular convex polytopes. ($n = \dim P$)

(1) A_n (regular simplexes[1]),
(2) B_n (cross polytopes and measure polytopes[2]),
(3) H_3 (icosahedron and dodecahedron),
(4) H_4 (600-cells and 120-cells),
(5) F_4 (24-cells), and
(6) $I_2(m)$ (regular m-gon).

Theorem 4. *Let P be any regular convex polytope in $\mathbb{R}^n$ with center at the origin. Let $G \subset O(n)$ be the symmetry group of G, $\Delta(x)$ the fundamental alternating*

[1]tetrahedron for $n = 3$
[2]octahedron and cube for $n = 3$

polynomial of G, i.e., the product of the linear forms whose zeros define a reflecting hyperplane of G. Then $\mathcal{H}_{P(k)}$ is independent of $k = 0, 1, \ldots, n$, and

$$\dim \mathcal{H}_{P(k)} = |G|,$$
$$\mathcal{H}_{P(k)} = \mathbb{C}[\partial]\Delta(x).$$

Invariant theory for finite reflection groups, as well as systems of invariant differential equations, plays an essential role in establishing this theorem. In the course of the proof, we were able to introduce a distinguished basis of G-invariant polynomials (canonically attached to the invariant differential equations) for each finite reflection group G.

4. Problem

We say that a polytope P admits *ample symmetry* if the symmetry group $G \subset O(n)$ of P acts on $\mathbb{R}^n$ irreducibly. Recall that if P is a polytope with ample symmetry then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear subspace of the harmonic polynomials. The following problem seems to be very interesting.

Problem. (Exhaustion of harmonic polynomials by P-harmonic polynomials) Is there any infinite sequence $P_1, P_2, P_3, \ldots, P_m, \ldots$ of polytopes in $\mathbb{R}^n$ with ample symmetry such that

$$P_m \to B^n \quad \text{as} \quad m \to \infty,$$
$$\mathcal{H}_{P_1(k)} \subset \mathcal{H}_{P_2(k)} \subset \mathcal{H}_{P_3(k)} \subset \cdots \subset \mathcal{H}_{P_m(k)} \subset \cdots, \quad \text{and}$$
$$\bigcup_{m=1}^{\infty} \mathcal{H}_{P_m(k)} = \{\text{harmonic polynomials in } n\text{-variables}\}$$

for any/some $k = 0, 1, \ldots, n$?

In the case of two-dimensions we know that the answer is *yes*. Indeed we can take P_m to be a regular convex m-gon ($m \geq 3$). At least to the speaker, however, this problem becomes quite difficult if the dimension n is greater than two. The difficulty lies in the fact that there are only finitely many regular polytopes in $\mathbb{R}^n$ for each $n \geq 3$, or more precisely that there are only finitely many finite subgroups of $O(n)$ for each $n \geq 3$. Thus in order to tackle this problem we should pay our attention not only to the symmetry of polytopes but also to the combinatorics or geometry of them.

References

1. Beckenbach, E.F. and Reade, M.O., *Mean values and harmonic polynomials*, Trans. Amer. Math. Soc. **53** (1943), 230–238.
2. Beckenbach, E.F. and Reade, M.O., *Regular solids and harmonic polynomials*, Duke Math. J. **12** (1945), 629–644.
3. Coxeter, H.M.S., *Regular polytopes*, 3rd ed., Dover, New York, 1973.
4. Flatto, L., *Functions with a mean value property*, J. Math. Mech. **10** (1961), 11–18.
5. Flatto, L., *Functions with a mean value property, II*, Amer. J. Math. **85** (1963), 248–270.
6. Flatto, L. and Wiener, M.M., *Regular polytopes and harmonic polynomials*, Canad. J. Math. **22** (1970), 7–21.

7. Friedman, A., *Mean values and polyharmonic polynomials*, Michigan Math. J. **4** (1957), 67–74.
8. Friedman, A. and Littman, W., *Functions satisfying the mean value property*, Trans. Amer. Math. Soc. **102** (1962), 167–180.
9. Iwasaki, K., *Polytopes and the mean value property*, to appear in Discrete & Comput. Geometry.
10. Iwasaki, K., *Basic invariants of finite reflection groups*, preprint.
11. Iwasaki, K., *Regular simplexes and the mean value property*, preprint.
12. Iwasaki, K., *Cross polytopes, measure polytopes and the mean value property*, in preparation.
13. Steinberg, R., *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc. **112** (1964), 392–400.
14. Walsh, J.L., *A mean value theorem for polynomials and harmonic polynomials*, Bull. Amer. Math. Soc. **42** (1936), 923–930.
15. Zalcman, L., *Mean values and differential equations*, Israel J. Math. **14** (1973), 339–352.