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Normal Forms of Vector Fields and Diffeomorphisms

By

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Abstract

We shall show simultaneous normal forms of a system of vector fields and diffeomorphisms under Brjuno condition. These results are proved by a new scheme of a rapidly convergent iteration with high loss of derivatives such that for some $\varepsilon, 0<\varepsilon<1$, $\exp(\exp((\sigma-\sigma')^{-\varepsilon}))$, $0<\sigma'<\sigma$.

We solve an overdetermined system of equations arising in the study of normal forms and diffeomorphisms by this method.

1 Normal forms of vector fields

Let us consider a system of analytic vector fields $X^{\mu}(\mu=1,\cdots,d)$ in some neighborhood of the origin of $x=(x_{1},\cdots,x_{n})\in \boldsymbol{R}^{n}$,

$$X^{\mu}=\langle X^{\mu},\partial_{x}\rangle=\sum_{j=1}^{n}X_{j}^{\mu}(x)\partial_{x_{j}}, \quad 1\leq\mu\leq d, \tag{1.1}$$

with the convention that $\partial_{x}=(\partial_{x_{1}},\cdots,\partial_{x_{n}})$, $\partial_{x_{j}}=\partial/\partial x_{j}$. We assume

$$X^{\mu}\ (1\leq\mu\leq d) \text{ are singular i.e. } X^{\mu}(0)=0 \text{ for } 1\leq\mu\leq d. \tag{1.2}$$

The linear parts of $X^{\mu}(1\leq\mu\leq d)$ are semi-simple i.e.,

$$X^{\mu}(x)=(X_{1}^{\mu}(x),\cdots,X_{n}^{\mu}(x))=\Lambda^{\mu}x+R^{\mu}(x), \quad 1\leq\mu\leq d, \tag{1.3}$$

where

$$\Lambda^{\mu}=\begin{pmatrix}\lambda_{1}^{\mu} & & 0\\ & \ddots & \\ 0 & & \lambda_{n}^{\mu}\end{pmatrix}, \qquad \lambda_{j}^{\mu}\in \boldsymbol{C}$$

and where $R^{\mu}(x)$ are analytic at the origin and satisfy

$$R^{\mu}(0)=\partial_{x}R^{\mu}(0)=0, \quad 1\leq\mu\leq d.$$

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$X^\mu (1 \le \mu \le d)$ are pairwise commuting, i. e. $[X^\mu, X^\nu] = 0, \quad 1 \le \nu, \mu \le d.$ (1.4)

Set $\lambda^\mu = (\lambda_1^\mu, \cdots, \lambda_n^\mu), \quad (1 \le \mu \le d)$. We are interested in reduction of vector fields to normal forms. If $d = 1$ (single case), a normal form was obtained by Poincaré under the condition

$$(*) \qquad |\lambda\alpha| \ge c_0|\alpha| \quad \text{for } \alpha \in \boldsymbol{Z}_+^n, \ |\alpha| \gg 1$$

Roughly speaking, in order to find a change of variables which reduces a vector field to its normal form we must solve a nonlinear equation, a so-called homological equation. The condition (*) implies the existence of the bounded inverse of the linearized operator. The solvability of certain nonlinear equations under Poincaré condition was proved by Kaplan for more general equation.([6]).

The solvability of these nonlinear equations with unbounded inverse was proved by Siegel in case $d = 1$ ([12]) under a famous Siegel condition :

$$\exists c > 0, \exists \gamma > 0; |\lambda\alpha - \lambda_k| \ge c|\alpha|^{-\gamma} \quad \text{for } 1 \le k \le n, \ \alpha \in \boldsymbol{Z}_+^n. \tag{1.5}$$

Rüssman ([10]) generalized his idea and proved

Assume $d = 1$. Suppose (1.2), (1.3) and (1.5). Then the vector field (1.1) can be transformed to a normal form by a holomorphic change of variables.

By the studies of normal forms of mappings by Yoccoz ([13]) and M. Perez ([9]), it is natural to weaken the condition (1.5) to the following simultaneous Brjuno condition:

$\exists c > 0, \exists \gamma > 0$ such that

$$(Br) \qquad \max_{1\le\mu\le d} |\lambda^\mu\alpha - \lambda_k^\mu| \ge c\exp\left(-\frac{|\alpha|}{\log(2+|\alpha|)^{1+\gamma}}\right) \quad \forall\alpha \in \boldsymbol{Z}_+^n, \ 1 \le \forall k \le n.$$

We note that our condition is weaker because the bound from the below is exponentially small when $|\alpha| \to \infty$, and there is a maximum in μ in the left-hand side. Hence each vectors could be resonant and may not satisfy a Brjuno condition as a single equation, while they simultaneously satisfy (Br).

We note that (Br) implies that $\lambda_1^\mu, \ldots, \lambda_n^\mu$ are non simultaneous resonant, namely

$$\max_{1\le\mu\le d} |\lambda^\mu \cdot \alpha - \lambda_k^\mu| \ne 0, \quad \forall\alpha \in \mathbf{Z}_+^n, \ 1 \le k \le n. \tag{1.6}$$

Then we have

Theorem 1. 1 *Let $X^1(x), \ldots, X^d(x)$ be pairwise commuting holomorphic vector fields satisfying the conditions (1.2), (1.3) and (1.4). If $\lambda^1, \ldots, \lambda^d$ verify the Brjuno condition (Br) we can find a neighborhood Ω of the origin and a holomorphic change of the variables $x = y + u(y), y \in \Omega$ which transforms simultaneously $X^1(x), \ldots, X^d(x)$ into $\lambda^1 y\partial_y, \ldots, \lambda^d y\partial_y$, respectively. Moreover, u is a solution of the following equation*

$$\mathcal{L}_{\lambda^\mu} u - R^\mu(y + u) = 0, \qquad 1 \le \mu \le d. \tag{1.7}$$

1.1 Approximate solution to a homological equation

First we need to introduce some Banach spaces of holomorphic functions. Let Ω be an open ball containing the origin in $\mathbf{C}^n$ and let $\mathcal{O}(\Omega)$ be the set of holomorphic functions on Ω. Following [4] we define for $0 < T < diam(\Omega)/2$

$$H(T) = \{u(x) = \sum_{\alpha \in \mathbf{Z}^n} u_\alpha x^\alpha \in \mathcal{O}(\Omega) : |u|_T = \sum_{\alpha \in \mathbb{Z}^n} |u_\alpha| T^{|\alpha|} < \infty\} \tag{1.8}$$

Theorem 1. 2 *The following estimate is true*

$$|D^\beta u|_{T_1} \leq \frac{C}{(T - T_1)^{|\beta|}} |u|_T. \tag{1.9}$$

for all $0 < T_1 < T$.

We define

$$Mf = \sum_{\mu=1}^{d} \mathcal{L}_{\bar{\lambda^\mu}} \mathcal{L}_{\lambda^\mu} f, \qquad f \in (H(T))^n := H(T) \times \cdots \times H(T). \tag{1.10}$$

If we expand $f(x)$ into Taylor series $f(x) = \sum_\alpha f_\alpha x^\alpha$ and if we set $Mf = \sum M(\alpha) f_\alpha x^\alpha$ we can see that

$$M(\alpha) = \mathrm{diag}\left(M_1(\alpha), \ldots, M_n(\alpha)\right), \quad M_j(\alpha) = \sum_{\mu=1}^{d} |\lambda^\mu \cdot \alpha - \lambda_j^\mu|^2, \quad 1 \leq j \leq n. \tag{1.11}$$

Then we have

Lemma 1. 3 *Let $T_0 > 0$ be given. Suppose that (Br) is satisfied. Then, for any $0 < T' < T < T_0$ there exists an inverse $M^{-1} : (H(T'))^n \longrightarrow (H(T))^n$ and a constant $c_0 > 0$ such that the following estimate holds*

$$\|M^{-1}\|_{T' \to T} \leq \exp\left(2 \exp(c_0 (T - T')^{-1/(1+\tau)})\right), \quad T_0 < \forall T' < \forall T < 2T_0. \tag{1.12}$$

Proof. Set $\delta = T'/T$. We have, for $f = \sum f_\alpha x^\alpha \in (H(T))^n$

$$\begin{aligned} \|M^{-1} f\|_{T'} &= \sum_\alpha T'^{|\alpha|} |M^{-1}(\alpha) f_\alpha| \leq \\ &\leq \sum_\alpha \delta^{|\alpha|} \exp\left(2|\alpha| (ln(|\alpha| + 2))^{-\tau-1}\right) |f_\alpha| T^{|\alpha|}. \end{aligned} \tag{1.13}$$

Since

$$\sup_{|\alpha| \geq 1} \left(\delta^{|\alpha|} \exp\left(2|\alpha| (ln(|\alpha| + 2))^{-\tau-1}\right)\right) \leq \exp\left(\exp\left(c(ln \delta^{-1})^{-1/(1+\tau)}\right)\right) \tag{1.14}$$

for some $c > 0$ and since $ln(T/T') = ln(1 + (T - T')/T')$ is bounded by the constant times of $T - T'$ from the above and the below we have (1.12). □.

For the later use we define the approximate inverse to a homological operator as follows:

$$P(D)\vec{f} = \sum_{\mu=1}^{d} \mathcal{L}_{\overline{\lambda^\mu}} M^{-1} f_\mu, \qquad \vec{f} = (f_1, \ldots, f_d) \in (H(T))^{nd}. \tag{1.15}$$

We observe that

$$\mathcal{L}_{\lambda^\mu} P(D)\vec{f} = f_\mu + M^{-1} \sum_{\nu=1,\nu\neq\mu}^{d} \mathcal{L}_{\overline{\lambda^\nu}} (\mathcal{L}_{\lambda^\mu} f_\nu - \mathcal{L}_{\lambda^\nu} f_\mu) \quad 1 \le \forall\mu \le d. \tag{1.16}$$

1.2 Rapidly convergent iteration scheme

Now we will prove Theorem 2.1. We shall find $u(x)$ such that

$$\left(1 + \frac{\partial u}{\partial x}\right)^{-1} X^\mu(x + u(x)) = \Lambda^\mu x = {}^t(\lambda_1^\mu x_1, \ldots, \lambda_n^\mu x_n) \tag{1.17}$$

for $1 \le \mu \le d$. The equation (1.17) is equivalent to solving the following overdetermined system of equations

$$\mathcal{L}_{\lambda^\mu} u(x) = R_\mu(x + u(x)), \qquad 1 \le \mu \le d. \tag{1.18}$$

We set

$$v_0(x) = \sum_{\nu=1}^{d} \mathcal{L}_{\overline{\lambda^\nu}} M^{-1} R_\nu^0(x), \qquad R_\nu^0(x) = R_\nu(x). \tag{1.19}$$

By a scale change of variables we may assume that $|v_0|_T \ll 1$. Then we consider the change of the variables $x + v_0(x)$ and obtain the new system of vector fields

$$X^{\mu,1}(x) = \left(1 + \frac{\partial v_0}{\partial x}\right)^{-1} X^\mu(x + v_0(x)) \equiv \Lambda^\mu x + R_\mu^1(x), \quad 1 \le \mu \le d. \tag{1.20}$$

Straightforward calculations show (multiplying by $(1 + \partial v_0/\partial x)$ from the left and recalling that $X^\mu(x + v_0(x)) = \Lambda^\mu x + \Lambda^\mu v_0(x) + R_\mu^0(x + v_0(x))$)

$$\Lambda^\mu x + \Lambda^\mu v_0(x) + R_\mu^0(x + v_0(x)) = \left(1 + \frac{\partial v_0}{\partial x}\right) \Lambda^\mu x + \left(1 + \frac{\partial v_0}{\partial x}\right) R_\mu^1(x),$$

i.e.,

$$\begin{aligned} \left(1 + \frac{\partial v_0}{\partial x}\right) R_\mu^1(x) &= -\frac{\partial v_0}{\partial x} \Lambda^\mu x + \Lambda^\mu v_0(x) + R_\mu^0(x + v_0(x)) \\ &= -\mathcal{L}_{\lambda^\mu} v_0 + R_\mu^0(x + v_0(x)) \end{aligned} \tag{1.21}$$

$$
\begin{aligned}
&= -R_\mu^0(x) + \sum_{\nu=1}^{d} \mathcal{L}_{\overline{\lambda^\nu}} M^{-1}\left(\mathcal{L}_{\lambda^\nu} R_\mu^0 - \mathcal{L}_{\lambda^\mu} R_\nu^0\right) + R_\mu^0(x+v_0(x)) \\
&= v_0(x)\int_0^1 R_\mu^{0\prime}(x+tv_0(x))dt + \sum_{\nu=1}^{d} \mathcal{L}_{\overline{\lambda^\nu}} M^{-1}\left(R_\mu^0 \partial_x R_\nu^0 - R_\nu^0 \partial_x R_\mu^0\right),
\end{aligned}
$$

where we have used

$$
\mathcal{L}_{\lambda^\mu} R^\nu(x) - \mathcal{L}_{\lambda^\nu} R^\mu(x) = \partial R^\mu(x) R^\nu(x) - \partial R^\nu(x) R^\mu(x), \quad 1 \le \nu, \mu \le d.
$$

This is equivalent to $[X^\mu, X^\nu] = 0$. Therefore, R_μ^1 is estimated quadratically.

We conditnue this process. Suppose that we have constructed $v_0(x), \ldots, v_{k-1}(x)$ such that after a change of variables

$$
(1+v_0)\circ(1+v_1)\circ\cdots\circ(1+v_{k-1})(x)
$$

we have obtained

$$
X^{\mu,k}(x) = \Lambda^\mu x + R_\mu^k(x), \qquad 1 \le \mu \le n. \tag{1.22}
$$

Next we define

$$
v_k(x) = \sum_{\nu=1}^{d} \mathcal{L}_{\overline{\lambda^\nu}} M^{-1} R_\nu^k(x), \tag{1.23}
$$

and

$$
\begin{aligned}
X^{\mu,k+1}(x) &= (1+\partial_x v_k)^{-1} X^{\mu,k}(x+v_k(x)) \qquad (1.24)\\
&= (1+\partial_x v_k)^{-1}(1+\partial_x v_{k-1})^{-1}\cdots(1+\partial_x v_0)^{-1} \\
&\times\ X^\mu((1+v_k)\circ(1+v_{k-1})\circ\cdots\circ(1+v_0)(x)) \\
&= \Lambda^\mu x + R_\mu^{k+1}(x).
\end{aligned}
$$

As before we get

$$
\begin{aligned}
&(1+\partial_x v_k) R_\mu^{k+1}(x) = -\mathcal{L}_{\lambda^\mu} v_k + R_\mu^k(x+v_k(x)) \\
&= -R_\mu^k(x) + \sum_{\nu=1}^{d} \mathcal{L}_{\overline{\lambda^\nu}} M^{-1}\left(\mathcal{L}_{\lambda^\nu} R_\mu^k - \mathcal{L}_{\lambda^\mu} R_\nu^k\right) + R_\mu^k(x+v_k(x)) \\
&= v_k(x)\int_0^1 R_\mu^{k\prime}(x+tv_k(x))dt + \sum_{\nu=1}^{d} \mathcal{L}_{\overline{\lambda^\nu}} M^{-1}\left(R_\mu^k \partial_x R_\nu^k - R_\nu^k \partial_x R_\mu^k\right). \qquad (1.25)
\end{aligned}
$$

Hence, there exist $c > 0$ and $c_1 >$ such that

$$
\begin{aligned}
|R_\mu^{k+1}|_T &\le c\Big(\left|(1+\partial_x v_k)^{-1}\right|_T |v_k|_T \frac{c_1}{T-T'} |DR_\mu^k|_{T'} + \qquad (1.26)\\
&+\ \left|(1+\partial_x v_k)^{-1}\right|_T \exp\left(2\exp(c(T-T')^{-1/(1+\tau)})\right) |R^k|_{T'} |DR_\mu^k|_{T'}\Big).
\end{aligned}
$$

By using the estimate for composition of maps we see that

$$
(1+v_0)\circ(1+v_1)\circ\cdots\circ(1+v_k)(x) \longrightarrow 1+u \quad \text{in } H(T) \text{ as } r\to\infty \tag{1.27}
$$

and $|R^k|_T \to 0$, as $k \to \infty$. It follows that

$$
(1+\partial u_k)^{-1} X^\mu(x+u_k(x)) \longrightarrow (1+\partial u)^{-1} X^\mu(x+u(x)) = \Lambda^\mu x. \tag{1.28}
$$

1.3 Normal forms in the case of Jordan blocks

Now we shall remove the restriction (1.2). Namely,

$$X^\mu(x) = {}^t(X_1^\mu(x), \dots, X_n^\mu(x)) = J^\mu x + R^\mu(x), \qquad 1 \le \mu \le d,$$

where the matrices J^μ are not necessarily semi-simple and we use the same notations as before. We define the homological operator $\mathcal{L}_{\lambda^\mu}$ by

$$\mathcal{L}_{\lambda^\nu} u = \partial u J^\mu x - J^\mu u, \qquad (\mu = 1, \dots d), \qquad u \in (H(T))^n. \tag{1.29}$$

The commutativity of $X^\mu(x)$ imply that the matrices J^μ commute each other, namely $[J^\mu, J^\nu] = 0$ for every μ and ν. This determines J^μ up to their Jordan blocks if we fix one Jordan normal form of some J^μ. In the following, we may assume that all J^μ are diagonalized up to their Jordan blocks.

We assume the following simultaneous Poincaré condition; there exists $c > 0$ such that

$$\max_{1 \le \mu \le d} |\lambda^\mu \alpha - \lambda_k^\mu| \ge c|\alpha| \quad \forall \alpha \in \mathbf{Z}_+^n \ 1 \le \forall k \le n. \tag{1.30}$$

We note that this condition is stronger than the usual Poincaré condition if $d = 1$ because we have a nonresonance condition. Then we have the following

Theorem 1. 4 *Let $X^1(x), \dots, X^d(x)$ be pairwise commuting holomorphic vector fields as above. If $\lambda^1, \dots, \lambda^d$ verify (1.30) we can find a neighborhood Ω of the origin and a holomorphic change of the variables $x = y + u(y), y \in \Omega$ which transforms simultaneously $X^1(x), \dots, X^d(x)$ into their linear parts $J^1 y \partial_y, \dots, J^d y \partial_y$, respectively. Moreover, u is a solution of (1.7).*

Remark. The novelity of the theorem lies in the case $d \ge 2$ under natural extension of a usual Poincaré condition for a single equation.

In order to prove Theorem 2.4 we define the operator M by (1.10). Then we have

Lemma 1. 5 *Suppose that (1.30) is satisfied. Then the followings holds;*

i) There exists a scale change of variables $x_j = \rho_j y_j$ $(\rho_j > 0, j = 1, \dots, n)$ and $T_0 > 0$ such that in the new coordinate y, the inverse $M^{-1} : (H(T))^n \longrightarrow (H(T))^n$ exists as a continuous linear operator for any $0 < T < T_0$.

ii) We have

$$[M^{-1}, \mathcal{L}_{\lambda^\mu}] = [M^{-1}, \mathcal{L}_{\overline{\lambda^\mu}}] = 0 \qquad \textit{for every} \quad 1 \le \mu \le d. \tag{1.31}$$

Especially, the operator $P(D)$, (1.15) satisfies (1.16).

Proof. We write $\mathcal{L}_{\lambda^\mu} = \mathcal{L}'_{\lambda^\mu} + \mathcal{L}''_{\lambda^\mu}$, where $\mathcal{L}'_{\lambda^\mu}$ and $\mathcal{L}''_{\lambda^\mu}$ correspond to semi-simple and nilpotent part of J^μ, respectively. Because the change of variables in the lemma transforms $x_{\nu+\ell}\partial_{x_\nu}$ into $\rho_{\nu+\ell}\rho_\nu^{-1} y_{\nu+\ell}\partial_{y_\nu}$, it follows that $(\sum_{\mu=1}^d \mathcal{L}'_{\overline{\lambda^\mu}} \mathcal{L}'_{\lambda^\mu})^{-1} \mathcal{L}''_{\lambda^\mu}$ can be made arbitrarily small by an appropriate choice of ρ_j. It follows that M has the representation $M = \sum_{\mu=1}^{d} \mathcal{L}'_{\lambda^\mu}\mathcal{L}'_{\overline{\lambda^\mu}} + \varepsilon$, where ε has small norm while the first term is invertible by (1.30). This proves i).

In order to prove ii) it is sufficient to show that $[M, \mathcal{L}_{\lambda^\mu}] = 0$ for every $1 \le \mu \le d$. In view of the definition of M we shall show the commutativity of $\mathcal{L}_{\lambda^\mu}$ and $\mathcal{L}_{\lambda^\nu}$. Noting that $[J^\mu, J^\nu] = 0$ and the symmetry of $\partial^2 u$ we have, for $u \in (H(T))^n$

$$\begin{aligned}[][\mathcal{L}_{\lambda^\mu}, \mathcal{L}_{\lambda^\nu}]u &= \partial_x(\partial_x u J^\nu x - J^\nu u)J^\mu x - J^\mu(\partial_x u J^\nu x - J^\nu u) \\ -\partial_x(\partial_x u J^\mu x - J^\mu u)J^\nu x &+ J^\nu(\partial_x u J^\mu x - J^\mu u) = {}^tJ^\mu x\partial^2 u J^\nu x - {}^tJ^\nu x \partial^2 u J^\mu x = 0.\end{aligned}$$

This ends the proof.

The proof of Theorem 2.4 can be proved by the same argument as in Theorem 2.1.

2 Normal forms of commuting holomorphic diffeomorphisms

We consider d pairwise commuting local biholomorphic maps $\Phi_\mu : \mathbb{C}^n \to \mathbb{C}^n$, $\mu = 1, \dots, d$ in a neighbourhood Ω of the fixed point 0. Hence we can write

$$\Phi_\mu(x) = \Lambda_\mu x + \varphi_\mu(x), \qquad \mu = 1, \dots, d \tag{2.1}$$

where $\Lambda_\mu \in GL(n : \mathbb{C})$ and $\varphi_\mu \in (\mathbb{C}_1\{x\})^n$, i.e.

$$\varphi_\mu(x) = O(|x|^2), \ |x| \to 0, \qquad \mu = 1, \dots, d \tag{2.2}$$

The commuting relation $\Phi_\mu \circ \Phi_\nu = \Phi_\nu \circ \Phi_\mu$ implies $\Lambda_\mu \circ \Lambda_\nu = \Lambda_\nu \circ \Lambda_\mu$ for all $\mu, \nu = 1, \dots, d$. Without loss of generality (after a linear change of the variables) we may assume that all matrices are in the Jordan normal forms with identical block structures. We set $\lambda_\mu = (\lambda_{\mu 1}, \dots, \lambda_{\mu n})$ to be the vector consisting of the eigenvalues of the matrix Λ_μ, $\mu = 1, \dots, d$.

Our result for (simultaneous) analytic equvalence of the maps to their linear parts will be proved under the additional requirement that all matrices are semisimple. Therefore we can set

$$\Lambda_\mu = diag\{\lambda_{\mu 1}, \dots, \lambda_{\mu 1}\}, \qquad \lambda_{\mu j} \in \mathbb{C}, \ j = 1, \dots, n, \ \mu = 1, \dots, d. \tag{2.3}$$

We suppose that the vectors λ_μ, $\mu = 1, \dots, d$ are nonresonant, namely

$$\lambda_\mu^\alpha = \lambda_{\mu 1}^{\alpha_1} \dots \lambda_{\mu n}^{\alpha_n} \neq \lambda_{\mu j}, \qquad \alpha \in \mathbb{Z}_+^n, |\alpha| \ge 2, \ j = 1, \dots, n, \ \mu = 1, \dots, d. \tag{2.4}$$

In fact, we will impose a simultaneous Brjuno type condition: there exist two positive constants c_0 and τ such that

$$\min_{1 \le j \le n} \max_{1 \le \mu \le d} \left|\lambda_\mu^\alpha - \lambda_{\mu j}\right| \ge c_0 \exp\left(-\frac{|\alpha|}{(\ln |\alpha|)^{1+\tau}}\right), \qquad \alpha \in \mathbb{Z}_+^n, |\alpha| \ge 2. \tag{2.5}$$

Furthermore, in the case of more than one diffeomorphism (i.e. $d \geq 2$) we impose the following restriction: there exist a constant $C_0 > 0$ such that

$$\frac{\sum_{\mu=1}^{d} |\lambda^{\mu\gamma}|}{1 + \sum_{\mu=1}^{d} |\lambda^{\mu\alpha}|} \leq C_0, \quad \forall \alpha, \gamma \in \mathbf{Z}_+^n, \ |\gamma| \geq 2, \ \gamma \leq \alpha. \tag{2.6}$$

We note that in the case of a single map ($d = 1$) the vector λ_1 belongs to the Poincaré domain if either $\min_{j=1,\ldots,d} |\lambda_{1j}| > 1$ or $\max_{j=1,\ldots,d} |\lambda_{1j}| < 1$ (cf. [1, p. 311]). In that case the condition (2.6) is always fulfilled. One checks easily that it is also true when the space dimension is one ($n = 1$) while d is arbitrary positive integer. Then we have

Theorem 2. 1 *Let $\Phi_1, \ldots, \Phi_d$ be pairwise commuting local biholomorphic maps preserving the origin and satisfying the conditions (2.5) and (2.6). Then we can find a neighborhood B of the origin and a holomorphic change of the variables $y \to x = u(y)$ which transforms simultaneously $\Phi_1, \ldots, \Phi_d$ into their linear parts $\Lambda_1 x, \ldots, \Lambda_d x$, respectively.*

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