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Asymptotic expansion of the Bergman kernel for weakly pseudoconvex tube domains

Joe KAMIMOTO (Univ. of Tokyo)

Let Ω be a domain with smooth boundary in $\mathbf{C}^n$. The Bergman space $B(\Omega)$ is the subspace consisting of holomorphic L^2-functions on Ω. The Bergman projection is the orthogonal projection $\mathbf{B}: L^2(\Omega)\to B(\Omega)$. We can write $\mathbf{B}$ as an integral operator

$$\mathbf{B}f(z)=\int_{\Omega}K(z,w)f(w)dV(w)\quad \text{for } f\in L^2(\Omega),$$

where $K:\Omega\times\Omega\to\mathbf{C}$ is the Bergman kernel of the domain Ω and dV is the Lebesgue measure on Ω. In this paper we restrict the Bergman kernel on the diagonal of the domain and study the boundary behavior of $K(z)=K(z,z)$.

Although there are many explicit computations for the Bergman kernels of the specific domains ([2],[8],[26],[9],[19],[5], [11],[15],[16],[27]), it seems difficult to express the Bergman kernel in closed form in general. Therefore appropriate formulas are necessary to know the boundary behavior of the Bergman kernel.

First we consider the case where Ω is a bounded strictly pseudoconvex domain. L. Hörmander [25] shows that the limit of $K(z)d(z-z^0)^{n+1}$ at $z^0\in\partial\Omega$ equals the determinant of the Levi form at z^0 times $n!/4\pi^n$, where

d is the Euclidean distance. Moreover C. Fefferman [14] and L. Boutet de Monvel and J. Sjöstrand [6] give the following asymptotic expansion of the Bergman kernal of Ω :

$$K(z) = \frac{\varphi(z)}{r(z)^{n+1}} + \psi(z) \log r(z), \tag{1}$$

where $r \in C^\infty(\bar{\Omega})$ is a defining function of Ω (i.e. $\Omega = \{r > 0\}$ and $|dr| > 0$ on the boundary) and φ, ψ are extended smoothly to the boundary. The functions φ, ψ can be expanded asymptotically with respect to r.

Next we consider the weakly pseudoconvex case. Many sharp estimates of the size of the Bergman kernel are obtained ([22],[42],[13],[7],[23],[12], [24],[37],[43], [18]). In particular D. Catlin [7] gives a complete estimate from above and below for domains of finite type in $\mathbf{C}^2$. Recently H. P. Boas, E. J. Straube and J. Yu [3] have computed a boundary limit in the sense of Hörmander for a large class of domains of finite type on a non-tangential cone. But asymptotic formulas are yet to be well understood. In this article we give an asymptotic expansion of the Bergman kernel for certain class of weakly pseudoconvex tube domains of finite type in $\mathbf{C}^2$. N. W. Gebelt [17] and F. Haslinger [21] have recently computed for the special cases, but our style of the expansion is different from theirs.

Our main idea on the analysis of the Bergman kernel is to introduce certain blowing-up transformation. Since the set of strictly pseudoconvex points are dense on the boundary of the domain of finite type, it is a serious problem to resolove the confusion arised by strictly pseudoconvex points near z^0. This confusion can be refused by restricting the argument on a non-tangential cone in the domain. We overcome this difficulty in the case of certain class of tube domains in $\mathbf{C}^2$ in the following. Our transformation blows up a weakly pseudoconvex point z^0 and the singularity can be expressed in the form of the direct product of two variables. By the way, asymptotic expansion of functions of several variables is studied by Y. Sibuya [44] and H. Majima [36]. We introduce Sibuya's style in our case. The expansion with respect to one variable has the form of Fefferman's expansion (1), so this expansion is induced by the strict pseudoconvexity. The characteristic influence of the weak pseudoconvexity appears in the expansion with respect to the other variable. Though the form of this expansion is similar to (1), we must use mth root of the defining function, i.e. $r^{\frac{1}{m}}$, as the expansion variable when z^0 is of type $2m$. We remark that a similar phenomenon is observed in the case of another class of domains in [17].

Our method of the computation is based on the studies [14],[6],[4],[41]. Our starting point is certain integral representation in [32],[38]. After introducing the blowing-up transformation into this representation, we compute the asymptotic expansion by using the stationary phase method. It is necessary for the above computation to localize the Bergman kernel near a weakly pseudoconvex point. This lacalization can be obtained in a similar fashion to the case of some class of Reinhardt domains ([41]).

Now we state our result.

Given a function $f \in C^\infty(\mathbf{R})$ satisfying that

$$\begin{cases} f'' \geq 0 \text{ on } \mathbf{R} \text{ and } f \text{ has the form in some neighborhood of } 0: \\ f(x) = x^{2m} g(x) \text{ where } m = 2, 3, \ldots, \; g(0) > 0 \text{ and } x g'(x) \leq 0. \end{cases} \tag{2}$$

Let $\omega_f \subset \mathbf{R}^2$ be a domain defined by $\omega_f = \{(x, y); y > f(x)\}$. Let $\Omega_f \subset \mathbf{C}^2$ be the tube domain over ω_f, i.e.,

$$\Omega_f = \mathbf{R}^2 + i\omega_f.$$

Let $\pi : \mathbf{C}^2 \to \mathbf{R}^2$ be the projection defined by $\pi(z_1, z_2) = (\mathrm{Im} z_1, \mathrm{Im} z_2)$. It is easy to check that Ω_f is a pseudoconvex domain, moreover $z^0 \in \partial\Omega_f$, with $\pi(z^0) = O$, is a weakly pseudoconvex point of type $2m$ (or $2m-1$) in the sense of Kohn or D'Angelo and $\partial\Omega_f \setminus \pi^{-1}(O)$ is strictly pseudoconvex near z^0.

Now we introduce the map σ, which plays a key role on our analysis. Set $\Delta = \{(\tau, \varrho); 0 < \tau \leq 1, \; \varrho > 0\}$. The map $\sigma : \omega_f \to \Delta$ is defined by

$$\sigma : \begin{cases} \tau = \chi(1 - \frac{f(x)}{y}), \\ \varrho = y, \end{cases} \tag{3}$$

where the function $\chi \in C^\infty([0, 1))$ satisfies the conditions: $\chi'(u) \geq \frac{1}{2}$ on $[0, 1]$ and $\chi(u) = u$ for $0 \leq u \leq \frac{1}{3}$ or $\chi(u) = 1 - (1 - u)^{\frac{1}{2m}}$ for $1 - \frac{1}{3^{2m}} \leq u \leq 1$. Then $\sigma \circ \pi$ is the map from Ω to Δ.

The map σ induces an isomorphism of $\omega_f \cap \{x \geq 0\}$ (or $\omega_f \cap \{x \leq 0\}$) to Δ. The boundary of ω_f is transfered by σ in the following: $\sigma((\partial\omega_f) \setminus \{O\}) = \{(0, \varrho); \varrho > 0\}$ and $\sigma^{-1}(\{(\tau, 0); 0 \leq \tau \leq 1\}) = \{O\}$. This indicates that σ is a real blowing-up of $\partial\omega_f$ at O, so we may say that $\sigma \circ \pi$ blows up a weakly pseudoconvex point z^0. Moreover the pair of the variables (τ, ϱ) can be considered as the polar coordinates around O. We call τ the angular variable and ϱ the radial variable, respectively. Note that if z approaches some strictly (resp. weakly) pseudoconvex points, $\tau(\pi(z))$ (resp. $\varrho(\pi(z))$) tends to 0 on the coordinates (τ, ϱ).

The next theorem asserts that the singularity of the Bergman kernel of Ω_f at z^0, with $\pi(z^0) = O$, can be essentially expressed in terms of the polar coordinates (τ, ϱ).

Theorem 1 *The Bergman kernel of Ω_f has the form in some neighborhood of z^0:*

$$K(z) = \frac{\Phi(\tau, \varrho^{\frac{1}{m}})}{\varrho^{2+\frac{1}{m}}} + \tilde{\Phi}(\tau, \varrho^{\frac{1}{m}}) \log \varrho^{\frac{1}{m}}, \tag{4}$$

where $\Phi \in C^\infty((0,1] \times [0, \varepsilon))$ and $\tilde{\Phi} \in C^\infty([0,1] \times [0, \varepsilon))$, with some $\varepsilon > 0$.

Moreover Φ is written in the form on the set $\{\tau > \alpha \varrho^{\frac{1}{2m}}\}$ with some $\alpha > 0$: for every nonnegative integer μ_0

$$\Phi(\tau, \varrho^{\frac{1}{m}}) = \sum_{\mu=0}^{\mu_0} c_\mu(\tau) \varrho^{\frac{\mu}{m}} + R_{\mu_0}(\tau, \varrho^{\frac{1}{m}}) \varrho^{\frac{\mu_0}{m} + \frac{1}{2m}}, \tag{5}$$

where

$$c_\mu(\tau) = \frac{\varphi_\mu(\tau)}{\tau^{3+2\mu}} + \psi_\mu(\tau) \log \tau, \tag{6}$$

for $\varphi_\mu, \psi_\mu \in C^\infty([0,1])$, φ_0 is positive on $[0,1]$ and R_{μ_0} satisfies $|R_{\mu_0}(\tau, \varrho^{\frac{1}{m}})| \leq C_{\mu_0}[\tau - \alpha \varrho^{\frac{1}{2m}}]^{-4-2\mu_0}$ for some positive constant C_{μ_0}.

Considering the meaning of the variables τ, ϱ, we may say that each asymptotic expansion of K with respect to τ or $\varrho^{\frac{1}{m}}$ is induced by the strict or weak pseudoconvexity, respectively. By (5),(6), in order to see the characteristic influence on the singularity of K by the weak pseudoconvexity, it is sufficient to restrict the argument on the region

$$\mathcal{U}_\alpha = \{z \in \mathbf{C}^2; \tau \circ \pi(z) > \frac{1}{\alpha}\} \quad (\alpha > 1).$$

This is because $\mathcal{U}_\alpha$ is the *widest* region where the coefficients $c_\mu(\tau)$'s are bounded. We call $\mathcal{U}_\alpha$ an admissible approach region of the Bergman kernel of Ω_f at z^0. The region $\mathcal{U}_\alpha$ seems deeply connected with the admissible approach regions studied in [33],[34],[1],[35], etc. We remark that on the region $\mathcal{U}_\alpha$, the exchange of the expansion variable $\varrho^{\frac{1}{m}}$ for $r^{\frac{1}{m}}$, where r is a defining function of Ω_f (e.g. $r(x,y) = y - f(x)$), gives no influence on the form of the expansion.

Now let us compare the asymptotic expansion (4) on $\mathcal{U}_\alpha$ and that of Fefferman (1). Then the essential difference of them only appears in the expansion variable (i.e. $r^{\frac{1}{m}}$ or r). A similar phenomenon can be obserbed in subelliptic estimates for the $\bar{\partial}$-Neumann problem. It is well known that the finite-type condition is necessary and sufficient to satisfy a subelliptic estimate :

$$|||\phi|||_\epsilon^2 \leq C(||\bar{\partial}\phi||^2 + ||\bar{\partial}^*\phi||^2 + ||\phi||^2) \quad (\epsilon > 0),$$

(refer to [31] for the details). The difference of the above estimate in the strictly and weakly pseudoconvex case appears in the value of ϵ. In two dimensional case, the estimate holds for $\epsilon = \frac{1}{2}$ (resp. $\epsilon = \frac{1}{2m}$) but for no larger value in the strictly pseudoconvex case (resp. in the weakly pseudoconvex and of type $2m$ case). In this viewpoint, our expansion on $\mathcal{U}_\alpha$ seems to be a natural generalization of the strictly pseudoconvex case.

Remarks 1. The idea of the transformation σ is originally introduced in the study ([27]) of the Bergman kernel of the domain $\mathcal{E}_m = \{z \in \mathbf{C}^n; \sum_{j=1}^n |z_j|^{2m_j} < 1\}$ ($m_j \in \mathbf{N}$, $m_n \neq 1$). Since $\mathcal{E}_m$ has high homogeneity, the asymptotic expansion of the radial variable does not appear.

2. If we consider the Bergman kernel on the region $\mathcal{U}_\alpha$, then we can remove the condition $xg'(x) \leq 0$ in (2). Namely even if the condition $xg'(x) \leq 0$ does not satisfy, we can obtain (4),(5) in the theorem where c_μ's are bounded on $\mathcal{U}_\alpha$. But the condition $xg'(x) \leq 0$ is necessary to see the asymptotic expansion with respect to τ.

3. ¿From the definition of asymptotic expansion of functions of several variables in [44],[36], the expansion in the theorem is not complete. In order to get a complete asymptotic expansion, we must blow up the point $(\tau, \varrho) = (0, 0)$ again. The transformation $(\tau, \varrho) \mapsto (\tau, \varrho\tau^{-2m})$ is sufficient for this purpose.

4. The limit of $\varrho^{2+\frac{1}{m}} K(z)$ at z^0 is $c_0(\tau)$. Note that the boundary limit depends on the angular variable τ. But this limit is determined uniquely ($c_0(0)$) on a non-tangential cone in Ω_f (see [3]).

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E-mail : kamimoto@ms.u-tokyo.ac.jp