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TOEPLITZ OPERATORS IN THE ANALYTIC GOURSAT PROBLEM

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0. Introduction - Goursat problem and the spectral condition.

Let $P(x,D)$ be a partial differential operator of order m,

$$P(x,D)=\sum_{|\alpha|\leq m} a_{\alpha}(x)\,D^{\alpha},$$

where the coefficients $a_{\alpha}(x)$ are assumed to be in $\mathcal{O}$, the set of holomorphic functions in a neighborhood of the origin $x=0$ of $\mathbf{C}^{n}$. Here we use the usual notation such as

$$x=(x_{1},x_{2},\cdots,x_{n})\in\mathbf{C}^{n},\quad D=(D_{1},D_{2},\cdots,D_{n})\ (D_{j}=\partial/\partial x_{j}).$$

Let $\gamma\in\mathbf{N}^{n}$ $(\mathbf{N}=\{0,1,2,\cdots\})$ be a given multi-index of nonnegative integers with $|\gamma|=m$. Then the Goursat problem $(P,\mathcal{O},\gamma)$ is formulated as follows.

$$(P,\mathcal{O},\gamma)\qquad P(x,D)u(x)=f(x)\in\mathcal{O},\quad u(x)-v(x)=O(x^{\gamma})\ \text{in}\ \mathcal{O},$$

where $w(x)=O(x^{\gamma})$ in $\mathcal{O}$ means that $w(x)x^{-\gamma}\in\mathcal{O}$, or equivalently, $w(x)\in x^{\gamma}\cdot\mathcal{O}$.

We say that the Goursat problem $(P,\mathcal{O},\gamma)$ is unquely solvable if for any $(f(x),v(x))\in\mathcal{O}\times\mathcal{O}$ there exists a unique solution $u(x)\in\mathcal{O}$ of the problem above.

The most general result on the unique solvability of the Goursat problem $(P,\mathcal{O},\gamma)$ is proved under the spectral condition which is stated by

$$(\mathrm{Sp})\qquad |a_{\gamma}(0)|>\sum_{|\alpha|=m,\alpha\neq\gamma}|a_{\alpha}(0)|\mathbf{w}^{\gamma-\alpha},\quad \text{for }\ {}^{\exists}\mathbf{w}=(w_{1},w_{2},\cdots,w_{n})\in\mathbf{R}_{+}^{n},$$

where $\mathbf{R}_{+}=(0,\infty)$, and $\mathbf{w}^{\gamma-\alpha}=w_{1}^{\alpha_{1}-\gamma_{1}}\cdots w_{n}^{\gamma_{n}-\alpha_{n}}$. (Cf. Gårding [G], Wagschal [W]).

The spectral condition enable us to employ the contraction mapping theorem or its variation by introducing a Banach space associated with the Goursat problem, and this condition seems to be best possible so far as we employ the contraction mapping theorem. In fact, while there are many variations of the Goursat problem, for example, there have been made some attempts for the generalization of function spaces and also for the method of proofs, in all those papers the spectral condition is posed as a fundamental assumption (see Persson [P], Miyake [M] and references cited there).

In a very recent paper by the author with M. Yoshino ([M-Y]), we have succeeded to relax this condition, and proved the spectral property of the Goursat problem by employing the Toeplitz operator method in two dimensional case, which is attempted to give a generalization of results by Leray ([L]) where a very simple example of operators was studied.

In this report, we shall give more general results for the Goursat problem in the function spaces, and remove the restriction on the dimension of x variables from the view point of the Toeplitz operators, which is a generalization of results in [M-Y].

We note that this is a joint work with M. Yoshino of Chuo University, and the detail will be published elsewhere.

1. Goursat problem in Gevrey space.

First, we introduce the Gevrey space where we consider the Goursat problem.

Let $\mathbf{s} = (s_1, s_2, \cdots, s_n) \in \mathbf{R}_+^n$ and $\mathbf{w} = (w_1, w_2, \cdots, w_n) \in \mathbf{R}_+^n$. Then an $\mathbf{s}$−Gevrey space $\mathcal{G}^{\mathbf{s}}(\mathbf{w})$ is defined by the following isomorphism which is called the formal Borel transformation.

$$(1) \qquad u(x) = \sum_{\alpha \in \mathbf{N}^n} u_\alpha \frac{x^\alpha}{\alpha!} \in \mathcal{G}^{\mathbf{s}}(\mathbf{w}) \underset{\text{def.}}{\Longleftrightarrow} v(z) := \sum_{\alpha \in \mathbf{N}^n} u_\alpha \frac{z^\alpha}{(\mathbf{s} \cdot \alpha)!} \in \mathcal{O}(|z| < \mathbf{w}^{\mathbf{s}}),$$

where $\{|z| < \mathbf{w}^{\mathbf{s}}\} := \prod_{j=1}^n \{|z_j| < w_j^{s_j}\}$, and $\mathcal{O}(\Omega)$ denotes the set of holomorphic functions on $\Omega \subset \mathbf{C}^n$. Here, $\mathbf{s} \cdot \alpha := \sum_{j=1}^n s_j \alpha_j$ and $(\mathbf{s} \cdot \alpha)! := \Gamma(\mathbf{s} \cdot \alpha + 1)$.

By the definition, $\mathcal{G}^{\mathbf{s}}(\mathbf{w})$ has a structure of Fréchet space induced from $\mathcal{O}(|z| < \mathbf{w}^{\mathbf{s}})$ equipped with uniformly convergent topology on every compact set of $\{|z| < \mathbf{w}^{\mathbf{s}}\}$.

Examples. Let $s = (s, s, \cdots, s)$ $(0 < s \leq 1)$.

[1]

$$\mathcal{G}^1(\mathbf{w}) = \mathcal{O}(\|x\|/\mathbf{w} < 1) = \bigcap_{\mathbf{w}' < \mathbf{w}} \mathcal{O}(\|x\|/\mathbf{w}' \leq 1),$$

where $\{\|x\|/\mathbf{w} < 1\} = \{|x_1|/w_1 + |x_2|/w_2 + \cdots + |x_n|/w_n < 1\}$, and $(w_1', \cdots, w_n') < (w_1, \cdots, w_n)$ means that $w_j' < w_j$ $(j = 1, 2, \cdots, n)$. Here, $\mathcal{O}(\|x\|/\mathbf{w} \leq 1) := \mathcal{O}(\|x\|/\mathbf{w} <$

$1)\cap C^0(\|x\|/\mathbf{w} \le 1)$ is a Banach space by the uniform convergence topology on $\{\|x\|/\mathbf{w} \le 1\}$.

[2] For $0 < s < 1$,

$$\mathcal{G}^s(\mathbf{w}) = \bigcap_{\mathbf{w}'<\mathbf{w}} \mathcal{A}^{1/(1-s)}(\mathbf{w}'),$$

where $\mathcal{A}^{1/(1-s)}(\mathbf{w})$ is a Banach space of entire functions of exponential order $1/(1-s)$ defined by

$$\mathcal{A}^{1/(1-s)}(\mathbf{w}) = \{u(x)\,; \quad |u(x)| \le C \exp\left[(1-s)s^{s/(1-s)}\left(\frac{\|x\|}{\mathbf{w}^s}\right)^{1/(1-s)}\right], \ ^\exists C = C(\mathbf{w}) \ge 0\},$$

where $\|x\|/\mathbf{w}^s = |x_1|/w_1^s + \cdots |x_n|/w_n^s$. Here, the norm, $\|u\|$ for $u(x) \in \mathcal{A}^{1/(1-s)}(\mathbf{w})$, is defined by the infimum of such C's in the above inequality.

Let $P(x, D)$ be a partial differential operator with polynomial coefficients and we write it by

$$P(x,D) = \sum_{\alpha,\beta\in\mathbf{N}^n}^{\text{finite sum}} a_{\alpha\beta}\, x^\beta\, D^\alpha, \quad a_{\alpha\beta} \in \mathbf{C}. \tag{2}$$

For a given multi-index $\gamma \in \mathbf{N}^n$, the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma)$ is formulated by

$$(P, \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma) \qquad P(x,D)u(x) = f(x) \in \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \qquad u(x) - v(x) = O(x^\gamma) \text{ in } \mathcal{G}^{\mathbf{s}}(\mathbf{w}),$$

where $f(x), v(x) \in \mathcal{G}^{\mathbf{s}}(\mathbf{w})$ are given functions, and $u(x)$ is the unknown function in $\mathcal{G}^{\mathbf{s}}(\mathbf{w})$. Here $w(x) = O(x^\gamma)$ in $\mathcal{G}^{\mathbf{s}}(\mathbf{w})$ means that $w(x)x^{-\gamma} \in \mathcal{G}^{\mathbf{s}}(\mathbf{w})$.

2. Gevrey filtlation and Assumptions.

Let $P(x, D)$ be the operator given by (2). For $\mathbf{s} \in \mathbf{R}_+^n$, the s–Gevrey order of the operator $P(x, D)$, denoted by $\mathrm{ord}_{\mathbf{s}}(P)$, is defined by

$$\mathrm{ord}_{\mathbf{s}}(P) = \max\{\mathbf{s}\cdot\alpha + (\mathbf{1}-\mathbf{s})\cdot\beta\,;\ a_{\alpha\beta} \ne 0\}. \tag{3}$$

Remark. The assumption that the operator $P(x, D)$ is of polynomial coefficients is made only to assure that $\mathrm{ord}_{\mathbf{s}}(P) < +\infty$ for every $\mathbf{s} \in \mathbf{R}_+^n$. In fact, in the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma)$ we may assume that the coefficients of the operator are homomorphic in a neighborhood of the origin in such variables x_j that $s_j \ge 1$.

In the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma)$, the multi-index γ is assumed to be taken so that

$$\mathbf{s} \cdot \gamma = \mathrm{ord}_{\mathbf{s}}(P). \tag{4}$$

Also, as a fundamental assumption we suppose that

$$\overset{\circ}{P}_{\mathbf{s}}(D) := \sum_{\mathbf{s}\cdot\alpha=\mathbf{s}\cdot\gamma} a_{\alpha 0}\, D^{\alpha} \neq 0. \tag{5}$$

For the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma)$, we define a function $f_{\mathbf{s},\gamma}(z)$ $(z \in \mathbf{C}^n)$ by

$$f_{\mathbf{s},\gamma}(z) := \overset{\circ}{P}_{\mathbf{s}}(z^{-1})\, z^{\gamma} = \sum_{\mathbf{s}\cdot\alpha=\mathbf{s}\cdot\gamma} a_{\alpha 0}\, z^{\gamma-\alpha} \tag{6}$$

which is called the Toeplitz symbol associated with the Goursat problem. Here, $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$ and $z^{\alpha} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha \in \mathbf{Z}^n$.

3. Theorems.

Under the preparations as above, we can state our theorems as follows.

Theorem 1. *Suppose that*

$$0 \notin \mathrm{Convex-hull}\, \{f_{s,\gamma}(z)\, ;\; |z| = \mathbf{w}^{\mathbf{s}}\}, \tag{7}$$

where $\{|z| = \mathbf{w}^{\mathbf{s}}\} = \prod_{j=1}^{n} \{|z_j| = w_j^{s_j}\}$. *Then Goursat problem* $(P, \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$ *is uniquely solvable for sufficiently small* $\rho > 0$.

Furthermore, suppose that the s*−principal part of* $P(x, D)$ *is of constant coefficients, i.e.,* $\mathbf{s} \cdot \alpha + (\mathbf{1} - \mathbf{s}) \cdot \beta < \mathrm{ord}_{\mathbf{s}}(P)$ *if* $\beta \neq 0$ *for* $a_{\alpha\beta} \neq 0$. *Then the Goursat problem* $(P, \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$ *is uniquely solvable for any* $\rho > 0$.

In the theorem, we do not assume a priori that $a_{\gamma 0} \neq 0$ in the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma)$. Concerning this we can prove the following,

Corollary. *In the Goursat problem* $(P, \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma)$, *the condition (7) implies that* $a_{\gamma 0} \neq 0$.

Proof. The condition (7) implies that the Goursat problem $(\overset{\circ}{P}_{\mathbf{s}}(D), \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma)$ is uniquely solvable. On the other hand, if $a_{\gamma 0} = 0$ then $u(x) = x^{\gamma}$ satisfies $\overset{\circ}{P}_{\mathbf{s}}(D)u(x) = 0,\ u(x) = O(x^{\gamma})$, which menas that $u(x)$ is a non trivial solution with 0 Goursat data which is a contradiction to the unique solvability.

Theorem 2. *Let $n = 2$. Suppose $f_{\mathbf{s},\gamma}(z) \neq 0$ on $\{|z| = \mathbf{w}^{\mathbf{s}}\}$, and suppose*

$$\oint_{|z_1|=w_1^{s_1}} d\log f_{\mathbf{s},\gamma}(z_1, z_2) = 0, \tag{8}$$

for any fixed z_2 ($|z_2| = w_2^{s_2}$). Then the index of the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$ is equal to 0 for sufficiently small ρ. Furthermore, if the condition (7) is satisfied on $\{|z| = \mathbf{r}^{\mathbf{s}}\}$ for some suitable choice of $\mathbf{r} = (r_1, r_2) \in \mathbf{R}_+^2$, then the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$ is uniquely solvable for sufficiently small ρ. In the case where the s*−Gevrey principal part is of constant coefficients, the assertions as in Theorem 1 hold for any $\rho > 0$.*

It is obvious that the condition (8) is replaced by

$$\oint_{|z_2|=w_2^{s_2}} d\log f_{\mathbf{s},\gamma}(z_1, z_2) = 0, \tag{8'}$$

for any fixed z_1 ($|z_1 = w_1^{s_1}$).

Examples.

[1] Let $P(D) = D_1 - D_2^2$ be the heat operator, and consider the following two Cauchy problems,

$$(1) \quad (P, \mathcal{G}^{(2s,s)}(w_1, w_2), (1,0)), \qquad (2) \quad (P, \mathcal{G}^{(2s,s)}(w_1, w_2), (0,2)),$$

where $s > 0$. The Toeplitz symbols corresponding to each Cauchy problem are given by

$$(1) \quad f_{s,(1,0)}(z) = 1 - \frac{z_1}{z_2^2}, \qquad (2) \quad f_{s,(0,2)}(z) = \frac{z_2^2}{z_1} - 1,$$

where $|z_1| = w_1^{2s}, |z_2| = w_2^s$. Hence the problem (1) is uniquely solvable if $w_1 < w_2$, and the problem (2) is uniquely solvable if $w_1 > w_2$.

[2] (Example by Leray [L]) Let $P(D) = \lambda D_1 D_2 - D_1^2 - D_2^2$, where $\lambda \in \mathbf{C}$ is a complex parameter. Let $\mathbf{w} = (w, 1)$ ($w > 0$) and $\gamma = (1, 1)$, and consider the Goursat problem $(P(D), \mathcal{O}(||x||/\mathbf{w} < \rho), \gamma)$. Then this Goursat problem is uniquely solvable if

$$\begin{cases} \dfrac{(\operatorname{Re}\lambda)^2}{(1+w)^2} + \dfrac{(\operatorname{Im}\lambda)^2}{(1-w)^2} > 1, & w \neq 1 \\ \lambda \notin [-2, 2], & w = 1 \end{cases}$$

[3] Let

$$P(D) = -\frac{9}{4}D_1^3 + 3D_1 D_2^2 + \frac{4}{3}D_2^3.$$

Let $\mathbf{w} = (w, 1)$ and $\gamma = (2, 1)$, and consider the Goursat problem $(P(D), \mathcal{O}(||x||/\mathbf{w} < \rho), \gamma)$, where $D_1^2 D_2$ is absent from the operator $P(D)$.

Then for any w with $3/4 < w < 5/2$, this Goursat problem has an index 0 with 1 dimensional kernel and cokernel. In fact, in this case the Toeplitz symbol is given by

$$f_{\mathbf{1},\gamma}(z_1, 1) = -\frac{9}{4} z_1^{-1} + 3z_1 + \frac{4}{3} z_1^2 = \left(\frac{4}{3} - \frac{1}{z_1}\right)\left(z_1 + \frac{3}{2}\right)^2,$$

and the condition (8) is satisfied only for $3/4 < w < 3/2$, but the condition (7) can not be satisfied on $\{|z| = \mathbf{r}^{\mathbf{s}}\}$ for any $\mathbf{r} = (r_1, r_2)$. It is easily seen that $u(x) = x_1^2 x_2$ is the base of the kernel, and $(v(x), f(x)) = (0, 1)$ is the base of the cokernel of the Goursat problem.

4. Toeplitz operators.

Theorems are proved by employing the following elementary results on the Toeplitz operators.

Let $T^n = \prod_{j=1}^n \{|z_j| = 1\}$ be the $n-$dimensional torus. We denote by $L^2(T^n)$ the set of square integrable functions on T^n, that is, the set of functions $u(z) = \sum_{\alpha \in \mathbf{Z}^n} u_\alpha z^\alpha$ with finite norm $||u|| := \sum_{\alpha \in \mathbf{Z}^n} |u_\alpha|^2 < \infty$. Let $H^2(T^n)$ be the Hardy space on T^n, that is, the set of functions $u(z) \in L^2(T^n)$ such that $u_\alpha = 0$ for $\alpha \notin \mathbf{N}^n$. Let $\pi : L^2(T^n) \to H^2(T^n)$ be the natural projection. Then for $f(z)$ continuous on T^n, the Toeplitz operator $T_f : H^2(T^n) \to H^2(T^n)$ is defined by

$$T_f(u) = \pi(f(z)u(z)), \quad u(z) \in H^2(T^n). \tag{9}$$

Here $f(z)$ is called the Toeplitz symbol of the operator T_f.

Now we have,

Proposition 1. (i) *Let $\sigma(T_f)$ be the spectrum of $T_f : H^2(T^n) \to H^2(T^n)$. Then,*

$$\sigma(T_f) \subset \text{Convex} - \text{hull}\{f(z)\,;\, z \in T^n\} \underset{\text{put}}{=} \Gamma(f). \tag{10}$$

For $\lambda \notin \Gamma(f)$, the operator norm of $\lambda I - T_f$ is estimated by

$$||\lambda I - T_f|| \geq \text{dist}(\lambda, \Gamma(f)). \tag{11}$$

(ii) *Let $n = 1$. Then if $f(z) \neq 0$ $(|z| = 1)$, T_f is a Fredholm operator with an index $\chi(T_f)$ given by*

$$\chi(T_f) = \frac{-1}{2\pi i} \oint_{|z|=1} d \log f(z). \tag{12}$$

Next we introduce the notion of finite section Toeplitz operators.

Suppose that $n = 1$. For $N \geq 0$, we define

$$H^2_N(T) := H^2(T) \cap \mathcal{P}_N,$$

where $\mathcal{P}_N$ denotes the set of polynomials of z of degree at most N. Let $\pi_N : L^2(T) \to H^2_N(T)$ be the natural projection. Then an N−th finite section Toeplitz operator $T_f(N)$: $H^2_N(T) \to H^2_N(T)$ is defined by

$$T_f(N)(u) = \pi_N(f(z)u(z)), \quad u(z) \in H^2_N(T). \tag{13}$$

Proposition 2. *Let $f(z) \in \mathbf{C}[z, z^{-1}]$. Then we have:*

(i) *In order that there exists $N_0 \geq 0$ such that*

$$T_f(N) \,:\, H^2_N(T) \to H^2_N(T)$$

is invertible for all $N \geq N_0$ with the uniform norm estimate from below

$$||T_f(N)|| \geq d > 0$$

for some constant d independent of $N \geq N_0$, it is necessary and sufficient that $f(z) \neq 0$ and $\chi(T_f) = 0$.

Moreover, if $0 \notin \Gamma(f)$, then we can take $N_0 = 0$.

(ii) *The Toeplitz operator T_f is invertible if and only if $f(z) \neq 0$ $(|z| = 1)$ and $\chi(T_f) = 0$. And the operator norm of T_f is estimated by $||T_f|| \geq d > 0$, where d depends on* $\min\{|z - z_j|\,;\, |z| = 1, f(z_j) = 0\}$.

5. Sketch of the proof.

We shall give an outline of the proofs deviding several steps.

5.1. Reduction to an integro-differetial equation.

Let $u(x) = \sum_{\alpha \in \mathbf{N}^n} u_\alpha x^\alpha/\alpha! \in \mathbf{C}[[x]]$, the set of formal power series of $x-$ variables over $\mathbf{C}$. Then an integration $D^{-\gamma}u(x)$ for $\gamma \in \mathbf{N}^n$ is defined by

$$D^{-\gamma}u(x) = \sum_{\alpha \in \mathbf{N}^n} u_\alpha \frac{x^{\alpha+\gamma}}{(\alpha+\gamma)!}. \tag{14}$$

Then by the definition we have that $D^{\alpha}D^{-\gamma} = D^{\alpha-\gamma}$ for all $\alpha, \gamma \in \mathbf{N}^n$. Especially we have $D^{\gamma}D^{-\gamma} = id.$ on $\mathbf{C}[[x]]$, and $D^{-\gamma}D^{\gamma} = id.$ on $x^{\gamma} \cdot \mathbf{C}[[x]]$ for all $\gamma \in \mathbf{N}^n$.

It is easily proved that

$$D^{-\gamma} : \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}) \to x^{\gamma} \cdot \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w})$$

is isomorphic.

We notice that in the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$, we may assume that $v(x) = 0$ for the Goursat data (i.e., $u(x) = O(x^{\gamma})$ in $\mathcal{G}^{\mathbf{s}}(\rho\mathbf{w})$), by taking $w(x) = u(x) - v(x)$ as a new unknown function. Hence the unique solvability of the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$ is equivalent to the bijectivity of the mapping

$$P(x, D) : x^{\gamma} \cdot \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}) \longrightarrow \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}).$$

These observations show that the study of the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$ is reduced to the study of the following mapping of integro-differential operator

$$L(x, D) \equiv P(x, D)\, D^{-\gamma} : \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}) \longrightarrow \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}). \tag{15}$$

We notice that

$$L(x, D) = \sum_{\alpha,\beta}^{\text{finite sum}} a_{\alpha\beta} x^{\beta} D^{\alpha-\gamma} \underset{\text{put}}{=} \sum_{\delta,\beta} a_{\delta\beta} x^{\beta} D^{\delta}, \quad (\delta \in \mathbf{Z}^n)$$

satisfies

$$a_{\delta\beta} \neq 0 \Longrightarrow \mathbf{s} \cdot \delta + (\mathbf{1} - \mathbf{s}) \cdot \beta \leq 0, \quad \text{i.e.,} \quad \mathrm{ord}_{\mathbf{s}}(L) \leq 0.$$

Also by the assumption (5) we have

$$\overset{\circ}{L}_{\mathbf{s}}(D) := \sum_{\mathbf{s}\cdot\delta=0} a_{\delta 0} D^{\delta} = \overset{\circ}{P}_{\mathbf{s}}(D)\, D^{-\gamma} \neq 0.$$

To end this section, we remark that the Fredholm property of the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\mathbf{w}), \gamma)$ means that of the mapping (15) and an index of the Goursat problem is just given by that of the the mapping (15).

5.2. Hilbert space $G^{\mathbf{s}}(\rho\mathbf{w})$.

We define a Hilbert space $G^{\mathbf{s}}(\rho\mathbf{w})$ by

$$u(x) = \sum_{\alpha\in\mathbf{N}^n} u_{\alpha} \frac{x^{\alpha}}{\alpha!} \in G^{\mathbf{s}}(\rho\mathbf{w})$$

$$\underset{\text{def.}}{\Longleftrightarrow} v(z) := \sum_{\alpha\in\mathbf{N}^n} \frac{u_{\alpha}}{(\mathbf{s}\cdot\alpha)!} ((\rho\mathbf{w})^{\mathbf{s}} z)^{\alpha} \in H^2(T^n),$$

where $(\rho\mathbf{w})^{\mathbf{s}}z = ((\rho w_1)^{s_1}z_1, \cdots, (\rho w_n)^{s_n}z_n)$. Then it is easily seen that

$$\mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}) = \bigcap_{\kappa<\rho} G^{\mathbf{s}}(\kappa\mathbf{w}) \ (= \operatorname{proj}\lim_{\kappa\nearrow\rho} G^{\mathbf{s}}(\kappa\mathbf{w})).$$

Hence by taking the procedure of projective limit the Goursat problem $(P, \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$ is reduced to the following mapping instead of (15),

(16) $$L(x, D) \,:\, G^{\mathbf{s}}(\kappa\mathbf{w}) \longrightarrow G^{\mathbf{s}}(\kappa\mathbf{w}), \quad 0 < \kappa < \rho.$$

Now we decompose the operator $L(x, D)$ in the form

$$\begin{aligned} L(x, D) &= \mathring{L}_{\mathbf{s}}(D) + \sum_{\mathbf{s}\cdot\delta+(\mathbf{1}-\mathbf{s})\cdot\beta=0;\beta\neq 0} a_{\delta\beta}\, x^{\beta} D^{\delta} + \sum_{\mathbf{s}\cdot\delta+(\mathbf{1}-\mathbf{s})\cdot\beta<0} a_{\delta\beta}\, x^{\beta} D^{\delta} \\ &\underset{\text{put}}{=} \mathring{L}_{\mathbf{s}}(D) + Q(x, D) + R(x, D). \end{aligned}$$

Then we have

Proposition 3. (1) *$\mathring{L}_{\mathbf{s}}(D)$ is a bounded operator on $G^{\mathbf{s}}(\rho\mathbf{w})$ and the operator norm is estimated independently on $\rho > 0$ by*

$$||\mathring{L}_{\mathbf{s}}(D)|| \leq \sum_{\mathbf{s}\cdot\delta=0} |a_{\delta}|\mathbf{w}^{\mathbf{s}\delta},$$

where $\mathbf{w}^{\mathbf{s}\delta} = w_1^{s_1\delta_1} \cdots w_n^{s_n\delta_n}$.

(2) *$Q(D)$ is a bonded operator on $G^{\mathbf{s}}(\rho\mathbf{w})$, and its operator norm is evaluated by $||Q(x, D)|| = o(1)$ as $\rho \searrow 0$.*

(3) *$R(x, D)$ is a compact operator on $G^{\mathbf{s}}(\rho\mathbf{w})$, and its operator norm is evaluated by $||R(x, D)|| = o(1)$ as $\rho \searrow 0$.*

As an immediate corollary to this proposition, we get the spectral condition as follows.

Corollary. *Suppose for $\mathbf{w}$ the following condition is satisfied*

(Sp) $$|a_{\gamma 0}| > \sum_{\mathbf{s}\cdot\alpha=\mathbf{s}\cdot\gamma;\alpha\neq\gamma} |a_{\alpha 0}|\mathbf{w}^{\mathbf{s}(\gamma-\alpha)}.$$

Then the Goursat problem $(P(x, D), \mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}), \gamma)$ is uniquely solvable for sufficiently small ρ.

Furthermore, if the s*−Gevrey principal part of* $P(x,D)$ *is of constant coefficients, i.e.,* $Q(x,D)=0$ *in the decomposition of* $L(x,D)$*, then the Goursat problem* $(P,\mathcal{G}^{\mathbf{s}}(\rho\mathbf{w}),\gamma)$ *is uniquely solvable for any* $\rho>0$.

Proof. As the considerations above the problem is reduced to the invertibility of the integro-differential operator $L(x,D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ for all small $\rho>0$. Since

$$\mathring{L}_s(D)=a_{\gamma 0}+\sum_{\mathbf{s}\cdot\alpha=\mathbf{s}\cdot\gamma;\alpha\neq\gamma} a_{\alpha 0}D^{\alpha-\gamma},$$

the operator norm on $G^{\mathbf{s}}(\rho\mathbf{w})$ is estimated by

$$||\mathring{L}_{\mathbf{s}}(D)||\geq|a_{\gamma 0}|-\sum_{\mathbf{s}\cdot\alpha=\mathbf{s}\cdot\gamma;\alpha\neq\gamma}|a_{\alpha 0}|\mathbf{w}^{\mathbf{s}(\gamma-\alpha)}>0,$$

which implies the invertibility of $\mathring{L}_{\mathbf{s}}(D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ with uniform norm estimates on $\rho>0$ for $\mathring{L}_{\mathbf{s}}(D)^{-1}$. Now the result follows from the operator norms for $Q(xD)$ and $R(x,D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ as $\rho\to 0$ in the above proposition.

To prove the latter half we notice that $R(x,D)$ is a compact operator on $G^{\mathbf{s}}(\rho\mathbf{w})$ for all $\rho>0$. Hence $L(x,D)$ is a compact perturbation from the invertible operator $\mathring{L}_{\mathbf{s}}(D)$, and therefore the index of $L(x,D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ is equal to 0. Hence it is sufficient to show the injectivity of $L(x,D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ for all $\rho>0$. By the definition we easily see that the inclusion $G^{\mathbf{s}}(\rho\mathbf{w})\to G^{\mathbf{s}}(\rho'\mathbf{w})$ is injective for any $\rho'<\rho$. The invertibility of $L(x,D)$ on $G^{\mathbf{s}}(\rho'\mathbf{w})$ for sufficiently small ρ' implies the injectivity of $L(x,D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$, which proves the assertion.

5.3. Reduction to the theory of Toeplitz operators.

Following the argument above, we study the following mapping,

$$\mathring{L}_{\mathbf{s}}(D)\,:\,G^{\mathbf{s}}(\rho\mathbf{w})\to G^{\mathbf{s}}(\rho\mathbf{w}). \tag{17}$$

Let $\delta\in\mathbf{Z}^n$ satisfy $\mathbf{s}\cdot\delta=\sum_{j=1}^n s_j\delta_j=0$. Then the following commutative diagram is examined easily.

$$\begin{array}{ccc}
\displaystyle\sum_{\alpha\geq 0}u_\alpha\frac{x^\alpha}{\alpha!} & \xrightarrow{\text{Borel transf.}} & \displaystyle\sum_{\alpha\geq 0}\frac{u_\alpha}{(\mathbf{s}\cdot\alpha)!}((\rho\mathbf{w})^{\mathbf{s}}z)^\alpha \\
D^\delta\big\downarrow & & \big\downarrow T_{((\rho\mathbf{w})^{\mathbf{s}}z)^{-\delta}} \\
\displaystyle\sum_{\alpha-\delta\geq 0}u_\alpha\frac{x^{\alpha-\delta}}{(\alpha-\delta)!} & \xrightarrow[\text{Borel transf.}]{} & \displaystyle\sum_{\alpha-\delta\geq 0}\frac{u_\alpha}{(\mathbf{s}\cdot\alpha)!}((\rho\mathbf{w})^{\mathbf{s}}z)^{\alpha-\delta}.
\end{array}$$

Here $T_{((\rho\mathbf{w})^{\mathbf{s}}z)^{-\delta}}$ is the Toeplitz operator on $H^2(T^n)$ with symbol $((\rho\mathbf{w})^{\mathbf{s}}z)^{-\delta}$. By the assumption that $\mathbf{s}\cdot\delta = 0$, we have $((\rho\mathbf{w})^{\mathbf{s}}z)^{-\delta} = (\mathbf{w}^{\mathbf{s}}z)^{-\delta}$, and hence

$$T_{((\rho\mathbf{w})^{\mathbf{s}}z)^{-\delta}} = T_{(\mathbf{w}^{\mathbf{s}}z)^{-\delta}}, \quad z \in T^n.$$

Here, we recall that $f_{\mathbf{s},\gamma}(z) \equiv \overset{\circ}{P}_{\mathbf{s}}(z^{-1})\, z^{\gamma} = \overset{\circ}{L}_{\mathbf{s}}(z^{-1})$.

Thus we have shown that the mapping (17) is equivalent to the following Toeplitz operator with a symbol $f_{\mathbf{s},\gamma}(\mathbf{w}^{\mathbf{s}}z)$ ($z \in T^n$) which is independent of the paramerter $\rho > 0$.

$$T_{f_{\mathbf{s},\gamma}(\mathbf{w}^{\mathbf{s}}z)} : H^2(T^n) \longrightarrow H^2(T^n). \tag{18}$$

5.4. Proof of Theorems.

Proof of Theorem 1. The assumption (6) is equivalent that

$$0 \notin \text{Convex-hull}\,\{f_{\mathbf{s},\gamma}(\mathbf{w}^{\mathbf{s}}z)\,;\, |z_j| = 1\} \underset{\text{put}}{=} \Gamma(\mathbf{w}).$$

Hence by Proposition 1 the Toeplitz operator $T_{f_{\mathbf{s},\gamma}(\mathbf{w}^{\mathbf{s}}z)}$ is invertible and its operator norm is estimated by

$$||T_{f_{\mathbf{s},\gamma}(\mathbf{w}^{\mathbf{s}}z)}|| \geq \text{dist}\,(0, \Gamma(\mathbf{w})),$$

which is independent of $\rho > 0$. This shows that $\overset{\circ}{L}_{\mathbf{s}}(D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ is invertible and the operator norm of inverse operator is estimated from above uniformly on $\rho > 0$. Hence by Proposition 3, we can conclude that the integro-differential operator $L(x, D)$ is invertible on $G^{\mathbf{s}}(\rho\mathbf{w})$ for suffciently small $\rho > 0$, which proves the theorem.

Consider the case where the $\mathbf{s}-$principal part of $P(x, D)$ is of constant coefficients. In this case, $Q(x, D) \equiv 0$ in the decomposition of $L(x, D)$, and hence $L(x, D)$ is a compact perturbation from the invertible operator $\overset{\circ}{L}_{\mathbf{s}}(D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ for any $\rho > 0$. Now the reasoning below is the same with that of Corollary to Proposition 3, and the proof is completed.

Proof of Theorem 2.

The assumption (8) says that

$$\oint_{|z_1|=1} d\log f_{\mathbf{s},\gamma}(w_1^{s_1}z_1, w_2^{s_2}) = 0.$$

We shall prove an index of the operator of $L(x,D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ is equal to 0 for sufficiently small $\rho > 0$ under this condition.

For this purpose we first consider the following mapping,

$$\overset{\circ}{L}_{\mathbf{s}}(D) \,:\, G^{\mathbf{s}}(\rho\mathbf{w}) \longrightarrow G^{\mathbf{s}}(\rho\mathbf{w}).$$

In the following, we consider only the case where s_1/s_2 is rational for $\mathbf{s} = (s_1, s_2) \in \mathbf{R}^2_+$, since if otherwise we know that $\overset{\circ}{L}_{\mathbf{s}}(D) \equiv$ Const. which means that the above mapping is invertible, and the conclusion is obvious.

Let $s_1/s_2 = q/p$ be the irreducible fraction. We recall that the Toeplitz symbol $g(z) \underset{\text{put}}{=} f_{\mathbf{s},\gamma}(\mathbf{w}^s z) \in \mathbf{C}[z, z^{-1}]$ is s−quasi homogeneous of degree 0, that is, $g(c^{s_1}z_1, c^{s_2}z_2) = g(z_1, z_2)$ for $c \neq 0$ $(c \in \mathbf{C})$. Then $g(z)$ is written in the form,

$$g(z) = \sum_{j=\ell}^{m} b_j z_1^{pj} z_2^{-qj}, \quad {}^{\exists}\ell \leq {}^{\exists}m \in \mathbf{Z}.$$

As we have shown the above mapping is equivalent to the following Toeplitz operator,

$$T_g \,:\, H^2(T^2) \longrightarrow H^2(T^2).$$

Let us show that this mapping is decomposed into a direct sum of finite section Toeplitz operators as follows.

Let $\mathcal{P}^{\mathbf{s}}(k) = \{u(z) = \sum_{\mathbf{s}\cdot\alpha=k} u_\alpha z^\alpha\}$ be the set of s−quasi homogeneous polymonial of degree $k \in \mathbf{Q}_+$ equipped with L^2−norm on $|z| = 1$. Then the s−quasi homogeneity of zero of $g(z)$ implies that the Teoplitz poerator T_g becomes an operator on $\mathcal{P}^{\mathbf{s}}(k)$. Hence by decomposing the space $H^2(T^2)$ into the direct sum of s−quasi homogeneous spaces $\{\mathcal{P}^{\mathbf{s}}(k)\}$ $(k \in \mathbf{Q}_+)$, the Toeplitz operator T_g on $H^2(T^2)$ is decomposed into the operators on s−quasi homogeneous spaces.

Let $\dim_\gamma \mathcal{P}^{\mathbf{s}}(k) = N$, and put

$$h(z_1) = \sum_{j=-\ell}^{m} b_j z_1^j \; (= g(z_1^{1/p}, 1)).$$

Then we can easily see that

$$T_g \,:\, \mathcal{P}^{\mathbf{s}}(k) \longrightarrow \mathcal{P}^{\mathbf{s}}(k)$$

is equivalent to

$$T_h(N) \,:\, H^2_N(T) \longrightarrow H^2_N(T).$$

The assumption (8) shows that $\oint_{|z|=1} d\log h(z) = 0$. Hence by Proposition 2, there exists N_0 such that for all $N \geq N_0$, $T_h(N)$ is invertible on $H_N^2(T)$, and the norm inequalities $||T_h(N)|| \geq d > 0$ hold by a positive constant d independent of $N \geq N_0$. This implies that $\chi(T_g) = \chi(\overset{\circ}{L}_{\mathbf{s}}(D), G^{\mathbf{s}}(\rho\mathbf{w})) = 0$ ($^\forall \rho > 0$), and the first half of Theorem 2 follows from Proposition 3.

Recall the assumption of the latter half is that there eixsts $\mathbf{r} = (r_1, r_2) \in \mathbf{R}_+$ such that $0 \notin$ Convex-hull$\{f_{\mathbf{s},\gamma}(\mathbf{r}^{\mathbf{s}}z) \, |z_j| = 1\}$. Hence, $\overset{\circ}{L}_{\mathbf{s}}(D)$ is invertible on $G^s(\mathbf{r})$. This shows that for any $f(x) \in \mathbf{C}[[x]]$, the equation $\overset{\circ}{L}_{\mathbf{s}}(D)u(x) = f(x)$ has a unique solution $u(x) \in \mathbf{C}[[x]]$. In fact, by decomposing $\mathbf{C}[[x]]$ into the direct sum of s$-$quasi homogeneous polynomials, we see that on each space of s$-$quasi homogeneous polynomials $\mathcal{P}^{\mathbf{s}}(k)$ ($k \in \mathbf{Q}_+$) $\overset{\circ}{L}_{\mathbf{s}}(D)$ is invertible. Hence $\overset{\circ}{L}_{\mathbf{s}}(D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ is injective, and therefore is invertible for any $\rho > 0$ with uniform norm estimates on ρ for the inverse operator $\overset{\circ}{L}_{\mathbf{s}}(D)^{-1}$. Now by the same reasoning as above we can conclude the invertibility of $L(x, D)$ on $G^{\mathbf{s}}(\rho\mathbf{w})$ for sufficiently small $\rho > 0$.

The case where the s$-$principal part is of constant coefficients is the same as in Theorem 1. Thus the proof is completed.

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