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NONLINEAR EIGENVALUE PROBLEMS WITH SEVERAL PARAMETERS

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1. INTRODUCTION. We consider the nonlinear multiparameter problem with *indefinite* nonlinearities $f_k (1 \leq k \leq n)$:

$$\begin{aligned} & u''(r) + \frac{N-1}{r}u'(r) + \sum_{k=1}^{n} \mu_k f_k(r, u(r)) = \lambda g(r, u(r)), \quad 0 < r < 1, \\ & u(r) > 0, \quad 0 \leq r < 1, \\ & u'(0) = 0, \quad u(1) = 0. \end{aligned} \tag{1.1}$$

Here $N \geq 3$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in R_+^n$ $(n \geq 1), \lambda \in R$ are parameters. We know that the radial solution of the following elliptic equation

$$\begin{aligned} & \triangle u + \sum_{k=1}^{n} \mu_k f_k(|x|, u) = \lambda g(|x|, u) \quad \text{in } B := \{x \in R^N : |x| < 1\}, \\ & u > 0 \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B \end{aligned} \tag{1.2}$$

satisfies the equation (1.1). The typical example of the nonlinearities f_k and g is

$$f_k(r, u) := a_k(r)|u|^{p_k - 1}u, \;\; g(r, u) := a_0(r)u,$$

where $a_k(r) \in C^1([0, 1])$, and

$$a_k'(r) \leq 0, r \in [0, 1] \;\; (1 \leq k \leq n), \;\; a_i(0) > 0, \;\; a_0'(r) \geq 0, a_0(r) > 0, r \in [0, 1], \tag{1.3}$$

$$1 \leq p_1 \leq p_2 \leq \cdots p_{i-1} < p_i < p_{i+1} \leq \cdots \leq p_n < 1 + 4/N. \tag{1.4}$$

We emphasize that no sign conditions are imposed on $a_k(r)(k \neq i)$, and $a_i(r)$ may change sign. If f_k $(1 \leq k \leq n)$ and g are odd in u and satisfy suitable growth conditions, then by Ljusternik-Schnirelman (LS) theory, one can establish, given any $\alpha > 0$, the existence of variational eigenvalue $\lambda = \lambda(\mu, \alpha)$ for the equation (1.1) associated with eigenfunction $u_{\mu,\alpha} \in M_\alpha$, where

$$M_\alpha := \left\{ u \in X := W_0^{1,2}(B) : \Psi(u) := \frac{1}{\omega} \int_B \left(\int_0^{u(x)} g(|x|, s) ds \right) dx = \frac{1}{2}\alpha^2 \right\},$$

and ω be the measure of unit sphere S^{N-1} in R^N.

The aim of this paper is to study asymptotic behavior of $\lambda = \lambda(\mu, \alpha)$. More precisely, let an arbitrary $1 \le i \le n$ be fixed. Then we shall establish asymptotic formulas of $\lambda(\mu, \alpha)$, which is dominated by μ_i, as $\mu_i \to \infty$. Hence, we fix $1 \le i \le n$ throughout this paper.

Nonlinear elliptic multiparameter problems arises in many areas of applied mathematics including astrophysics, fluid mechanics, and especially, in the study of semilinear elliptic equations, which are, for example, derived from nonlinear Klein-Gordon equations in R^N

$$-\Delta u = f(u) - \lambda u \quad \text{in } R^N. \tag{1.5}$$

Indeed, in many cases, the nonlinearity f contains several parameters (see Berestycki and Lions [1]), and many interesting properties of solutions have been intensively investigated.

Another motivation comes from the study of "asymptotic direction" (limit of the ratio of the two eigenvalues) of the linear indefinite two-parameter Strum-Liouville problems

$$-(a(x)u(x)')' + \mu b(x)u(x) = \lambda c(x)u(x), \tag{1.6}$$

in which no sign conditions are imposed on $b(x)$ and $c(x)$. Asymptotic direction have played a fundamental role in the study of two-parameter eigenvalue problems, and has been studied extensively by many authors. Various references may be found in Faierman [6], Turyn [14] and the references therein for further informations. Our problem is regarded as the nonlinear version of finding asymptotic direction of eigenvalues and the variational approach seems effective to the problem (1.1). We note that the equation (1.1) has two variational structures to define variational eigenvalue.

Recently, Shibata [11] treated the simplest case of the equation (1.1), namely, one-dimensional two-parameter definite problem of the form

$$\begin{gathered} u''(x) + \mu u(x)^p = \lambda u(x)^q, \quad u > 0 \quad \text{for } 0 < x < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.7}$$

where $\mu, \lambda > 0$ are parameters and $1 \le q < p < q+2$ are constants. By using the LS-theory on general level set

$$N_{0,\mu,\beta} := \left\{ u \in W_0^{1,2}(I) : \frac{1}{2}\int_0^1 u'(x)^2 dx - \frac{1}{p+1}\mu \int_0^1 |u(x)|^{p+1} dx = -\beta \right\}, \quad (\beta > 0) \tag{1.8}$$

due to Zeidler [15], the variational eigenvalue $\lambda = \lambda_0(\mu, \beta)$ is defined, and precise asymptotic formula of $\lambda_0(\mu, \beta)$ as $\mu \to \infty$ for a fixed $\beta > 0$ was obtained:

$$\lambda_0(\mu, \beta) = C_1 \mu^{\frac{q+3}{p+3}} + o(\mu^{\frac{q+3}{p+3}}), \tag{1.9}$$

where

$$C_1 = \left\{ \left(\frac{q+1}{p+1}\right)^{\frac{q+3}{2(p-q)}} \frac{(p+3)(q+1)(p-q)\beta}{2(2q-p+3)} \sqrt{\frac{2}{\pi(q+1)}} \frac{\Gamma\left(\frac{p+3}{2(p-q)}\right)}{\Gamma\left(\frac{q+3}{2(p-q)}\right)} \right\}^{\frac{2(p-q)}{p+3}}. \tag{1.10}$$

The applications of this variational method are also applicable to our problem (1.1). More precisely, it was shown in Shibata [12] that on the general level set

$$N_{\mu,\beta} := \left\{ u \in X : \frac{1}{2}\int_B |\nabla u|^2 dx - \sum_{k=1}^n \mu_k \int_B \left(\int_0^{u(x)} f_k(|x|, s) ds \right) = -\beta\omega \right\},$$

where $\beta > 0$ is a parameter, the variational eigenvalue $\lambda = \lambda_0(\mu, \beta)$ is defined as Lagrange multiplier of the minimizing problem "*minimize* $\Psi(u)$ *under the constraint* $u \in N_{\mu,\beta}$." Under the appropriate conditions on f_k and g, the asymptotic formulas of $\lambda_0(\mu, \beta)$, which are the extension of (1.9) were established (see Remark 2.3 in Section 2).

In this paper, we adopt another variational method, namely, the LS theory due to Chiappinelli [3, 4], which is essentially developed in L^2-framework, and shall establish asymptotic formulas of variational eigenvalue $\lambda(\mu, \alpha)$, which are different from those of $\lambda_0(\mu, \beta)$. To obtain our results, the properties of the ground state solution w of the nonlinear scalar field equation

$$
\begin{gathered}
w''(s) + \frac{N-1}{s} w'(s) + w(s)^{p_i} - w(s) = 0, \quad s > 0, \\
w(s) > 0, \quad s \geq 0, \\
\lim_{s \to \infty} w(s) = 0.
\end{gathered} \tag{1.11}
$$

will also play important roles.

2. MAIN RESULTS. For simplicity, we denote by C the various positive constants independent of (μ, α). We explain notations. For $u, v \in X = W_0^{1,2}(B)$

$$
\|u\|_X^2 := \frac{1}{\omega} \int_B |\nabla u|^2 dx, \quad \|u\|_p^p = \frac{1}{\omega} \int_B |u(x)|^p dx, \quad (u, v) := \frac{1}{\omega} \int_B u(x) v(x) dx,
$$

$$
F_k(r, u) := \int_0^u f_k(r, s) ds, \quad G(r, u) := \int_0^u g(r, s) ds,
$$

$$
\Phi_k(u) := \frac{1}{\omega} \int_B F_k(|x|, u(x)) dx, \quad \Lambda_\mu(u) := \frac{1}{2} \|u\|_X^2 - \sum_{k=1}^n \mu_k \Phi_k(u).
$$

We assume the following conditions (A.1)-(A.3) on f_k and g:

(A.1) $f_k, g \in C^1([0,1]) \times R$ are odd in u.
(A.2)

$$
g(r, u) > 0, \quad \frac{\partial g(r, u)}{\partial r} \geq 0 \quad \text{for } (r, u) \in [0,1] \times R_+, \tag{2.1}
$$

$$
C^{-1} u \leq g(r, u) \leq Cu \quad \text{for } r \in [0,1] \text{ and } u \geq 0. \tag{2.2}
$$

(A.3) There exist constants $\{p_k\}_{k=1}^n, \{q_k\}_{k=1}^n$ satisfying (1.4) with $q_k \leq p_k (1 \leq k \leq n)$ such that

$$
|f_k(r, u)| \leq C(|u|^{p_k} + |u|^{q_k}) \quad \text{for } r \in [0,1], \ u \in R, \tag{2.3}
$$

$$
\frac{\partial f_k(r, u)}{\partial r} \leq 0 \quad \text{for } (r, u) \in [0,1] \times R_+. \tag{2.4}
$$

Furthermore, if $\Phi_i(u_0) \geq 0$ for $u_0 \in X$, then

$$
(f_i(r, u_0), u_0) - 2\Phi_i(u_0) \geq 0. \tag{2.5}
$$

Moreover, (A.4) (resp (A.5)) will be assumed in Theorem 2.1 (resp. Theorem 2.2).
(A.4) There exists $a_k(r) \in C^1([0,1])$ $(1 \le k \le n)$ such that

$$\frac{f_k(r,u)}{u^{p_k}} \to a_k(r), \quad \frac{g(r,u)}{u} \to a_0(r) \text{ as } u \to \infty \tag{2.6}$$

uniformly for $r \in [0,1]$, where $a_k(r)$ satisfies the condition (1.3). In addition,

$$\int_0^u \frac{\partial f_{k,0}(r,s)}{\partial r} ds \le 0, \quad \int_0^u \frac{\partial g_0(r,s)}{\partial r} ds \ge 0 \quad \text{for } r \in [0,1] \text{ and } u \ge 0, \tag{2.7}$$

where

$$f_{k,0}(r,u) := f_k(r,u) - a_k(r)u^{p_k}, \quad g_0(r,u) := g(r,u) - a_0(r)u. \tag{2.8}$$

(A.5) There exists $b_k(r) \in C^1([0,1])$ $(1 \le k \le n)$ such that

$$\frac{f_k(r,u)}{u^{q_k}} \to b_k(r), \quad \frac{g(r,u)}{u} \to b_0(r) \text{ as } u \downarrow 0 \tag{2.9}$$

uniformly for $r \in [0,1]$, where $b_k(r)$ satisfies the condition (1.3). In addition,

$$\int_0^u \frac{\partial f_{k,1}(r,s)}{\partial r} ds \le 0, \quad \int_0^u \frac{\partial g_1(r,s)}{\partial r} ds \ge 0 \quad \text{for } r \in [0,1] \text{ and } 0 \le u \ll 1, \tag{2.10}$$

where

$$f_{k,1}(r,u) := f_k(r,u) - b_k(r)u^{q_k}, \quad g_1(r,u) := g(r,u) - b_0(r)u. \tag{2.11}$$

The typical examples of f_k and g which satisfies (A.1) - (A.5) are:

$$\begin{gathered} f_k(r,u) = a_k(r)|u|^{p_k-1}u, \quad g(r,u) = a_0(r)u, \\ f_k(r,u) = (\cos \pi r)|u|^{p_k-1}u + |u|^{q_k-1}u, \quad g(r,u) = (1+r^2)u, \end{gathered} \tag{2.12}$$

where $a_k(r), \{p_k\}_{k=1}^n$ and $\{q_k\}_{k=1}^n$ satisfy (1.3) and (1.4).

For a given $(\mu, \alpha) \in R_+^{n+1}$, we say that $\lambda = \lambda(\mu,\alpha)$ is the variational eigenvalue if the associated eigenfunction $u_{\mu,\alpha} \in M_\alpha$ is radially symmetric and the conditions (B.1) - (B.2) are satisfied:

(B.1) $(\mu, \alpha, \lambda(\mu,\alpha), u_{\mu,\alpha}) \in R_+^{n+1} \times R \times M_\alpha$ satisfies (1.1).
(B.2)

$$2\Lambda_\mu(u_{\mu,\alpha}) = \beta(\mu,\alpha) := \inf_{u \in M_\alpha} 2\Lambda_\mu(u). \tag{2.13}$$

$\lambda(\mu,\alpha)$ is explicitly represented as follows:

$$\lambda(\mu,\alpha) = \frac{-\|u_{\mu,\alpha}\|_X^2 + \sum_{k=1}^n \mu_k (f_k(r,u_{\mu,\alpha}), u_{\mu,\alpha})}{(g(r,u_{\mu,\alpha}), u_{\mu,\alpha})}. \tag{2.14}$$

Indeed, multiply (1.1) by $u_{\mu,\alpha}$ and integrate it to obtain

$$-\|u_{\mu,\alpha}\|_X^2 + \sum_{k=1}^n \mu_k (f_k(r,u_{\mu,\alpha}), u_{\mu,\alpha}) = \lambda(\mu,\alpha)(g(r,u_{\mu,\alpha}), u_{\mu,\alpha}). \tag{2.15}$$

This implies (2.14). Unfortunately, the positivity of $\lambda(\mu,\alpha)$ does not follow from (2.14) directly.

We introduce (C-i) and (D-i) conditions for a sequence $\{(\mu,\alpha)\} \subset R_+^{n+1}$:

(C-i)

$$\alpha^{p_i-1}\mu_i \to \infty, \tag{2.16}$$

$$\alpha^{4/N}\mu_i \to \infty. \tag{2.17}$$

$$\mu_k \alpha^{\frac{4(p_k-p_i)}{4-N(p_i-1)}} \mu_i^{-\frac{4-N(p_k-1)}{4-N(p_i-1)}} \to 0 \quad (k \neq i). \tag{2.18}$$

(D-i)

$$\alpha^{q_i-1}\mu_i \to \infty, \tag{2.19}$$

$$\alpha^{4/N}\mu_i \to 0. \tag{2.20}$$

$$\mu_k \alpha^{\frac{4(q_k-q_i)}{4-N(q_i-1)}} \mu_i^{-\frac{4-N(q_k-1)}{4-N(q_i-1)}} \to 0 \quad (k \neq i). \tag{2.21}$$

Note that (2.19) and (2.20) occur when, for example, $\mu_i \to \infty$ and $\alpha = \mu_i^{-(N/4+\epsilon)}$, where $0 < \epsilon < \{4-N(q_i-1)\}/\{4(q_i-1)\}$. Finally, w denotes the ground state solution of (1.11), which uniquely exists, and W denotes the ground state of (1.11) with p_i replaced by q_i.

Theorem 2.1. *Assume (A.1) - (A.4). Then the following asymptotic formula holds for $\{(\mu,\alpha)\} \subset R_+^{n+1}$ satisfying* (C-i):

$$\lambda(\mu,\alpha) = C_2(\alpha^{p_i-1}\mu_i)^{\frac{4}{4-N(p_i-1)}} + o\Big((\alpha^{p_i-1}\mu_i)^{\frac{4}{4-N(p_i-1)}}\Big), \tag{2.22}$$

where $C_2 = a_0(0)^{-1}a_i(0)^{\frac{4}{4-N(p_i-1)}}\|w\|_{L^2(R^N)}^{-\frac{4(p_i-1)}{4-N(p_i-1)}}$.

Theorem 2.2. *Assume (A.1) - (A.3) and (A.5). Then the following asymptotic formula holds for $\{(\mu,\alpha)\} \subset R_+^{n+1}$ satisfying* (D-i):

$$\lambda(\mu,\alpha) = C_3(\alpha^{q_i-1}\mu_i)^{\frac{4}{4-N(q_i-1)}} + o\Big((\alpha^{q_i-1}\mu_i)^{\frac{4}{4-N(q_i-1)}}\Big), \tag{2.23}$$

where $C_3 = b_0(0)^{-1}b_i(0)^{\frac{4}{4-N(q_i-1)}}\|W\|_{L^2(R^N)}^{-\frac{4(q_i-1)}{4-N(q_i-1)}}$.

Remark 2.3. In Shibata [12], the following asymptotic formulas of variational eigenvalue $\lambda = \lambda_0(\mu,\beta)$ on general level sets $N_{\mu,\beta}$ were obtained:

Theorem 2.4 ([12, Theorem 2.1]) *Assume (A.1) - (A.4). Furthermore, assume that*
(A.6) $\quad p_i - q_i \le p_k - q_k$ for $k < i$.
Suppose that a sequence $\{(\mu,\beta)\} \subset R_+^{n+1}$ satisfies

$$\beta\mu_i^{\frac{2}{p_i-1}}, \quad \beta\mu_i^{(N-2)/2} \to \infty, \quad \mu_k\beta^{\frac{2(p_k-p_i)}{N+2-p_i(N-2)}}\mu_i^{-\frac{N+2-p_k(N-2)}{N+2-p_i(N-2)}} \to 0 \ (k \neq i).$$

Then the following asymptotic formula holds:

$$\lambda_0(\mu,\beta) = C_4 a_0(0)^{-1}a_i(0)^{\frac{4}{N+2-p_i(N-2)}}(\beta\mu_i^{\frac{2}{p_i-1}})^{\frac{2(p_i-1)}{N+2-p_i(N-2)}} + o((\beta\mu_i^{\frac{2}{p_i-1}})^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}), \tag{2.24}$$

where $C_4 = \{(N+2-p_i(N-2))/((4+N-Np_i)\|w\|_{L^2(R^N)}^2)\}^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}$.

Theorem 2.5 ([12, Theorem 2.2]). *Assume (A.1) - (A.3), (A.5), and (A.6). Furthermore, suppose that a sequence* $\{(\mu, \beta)\} \subset R_+^{n+1}$ *satisfied*

$$\beta\mu_i^{\frac{2}{q_i-1}} \to \infty, \quad \beta\mu_i^{(N-2)/2} \to 0, \quad \mu_k\beta^{\frac{2(q_k-q_i)}{N+2-q_i(N-2)}}\mu_i^{-\frac{N+2-q_k(N-2)}{N+2-q_i(N-2)}} \to 0 \ (k \neq i).$$

Then the following asymptotic formula holds:

$$\lambda_0(\mu,\beta) = C_5 b_0(0)^{-1} b_i(0)^{\frac{4}{N+2-q_i(N-2)}} (\beta\mu_i^{\frac{2}{q_i-1}})^{\frac{2(q_i-1)}{N+2-q_i(N-2)}} + o((\beta\mu_i^{\frac{2}{q_i-1}})^{\frac{2(q_i-1)}{N+2-q_i(N-2)}}), \tag{2.25}$$

where $C_5 = \{(N+2-q_i(N-2))/((4+N-Nq_i)\|W\|^2_{L^2(R^N)})\}^{\frac{2(q_i-1)}{N+2-q_i(N-2)}}$.

By comparing Theorem 2.1 with Theorem 2.4 for a fixed $\alpha, \beta > 0$, we see that $\lambda(\mu, \alpha)$ tends to ∞ faster than $\lambda_0(\mu, \beta)$ as $\mu_i \to \infty$.

Remark 2.6. (1) In Theorem 2.1, if we assume the condition

$$\alpha^{\frac{2\{(N+2)-p_i(N-2)\}}{N(p_i-1)}} \mu_i \to \infty, \tag{2.26}$$

which is stronger than (2.17), then the technical condition (2.5) can be removed.
(2) The condition (2.5) can be weakened. Indeed, it is sufficient that (2.5) holds only for $u = u_{\mu,\alpha}$. The typical example

$$f_i(r,u) = a_i(r)|u|^{p_i-1}u + b_i(r)|u|^{q_i-1}u, \tag{2.27}$$

satisfying (1.3) and (1.4) with $q_i \leq p_i$, fulfills this weaker condition. Hence, we can also treat this nonlinearity by our arguments.

3. FUNDAMENTAL LEMMAS. Theorem 2.2 can be proved by the same arguments as those used to prove Theorem 2.1. Therefore, we show Theorem 2.1. Let $a_i(0) = a_0(0) = 1$ in what follows for simplicity. Furthermore, a subsequence of a sequence will be denoted by the same notation as that of original sequence for convenience. Existence of of variational eigenvalues $\lambda(\mu, \alpha)$ follows from a simple application of the result of Chiappinelli [4]. The aim of this section is to show:

Lemma 3.1. *Assume (C-i). Then*

$$\lambda(\mu,\alpha) \geq C(\alpha^{p_i-1}\mu_i)^{\frac{4}{4-N(p_i-1)}}.$$

As a consequence of (2.1), (2.4), Lemma 3.1, and the result of Gidas, Ni and Nirenberg [7, Theorem 1'], we obtain:

Corollary 3.2. *Assume (C-i). Then* $u_{\mu,\alpha}$ *is radially symmetric.*

To show Lemma 3.1, we prepare some inequalities and lemmas. By (2.2) and (2.3) we have for $u \in X$ and $1 \leq k \leq n$

$$|F_k(u)| \leq C(|u|^{p_k+1} + |u|^{q_k+1}), \quad |\Phi_k(u)| \leq C(\|u\|^{p_k+1}_{p_k+1} + \|u\|^{q_k+1}_{q_k+1}), \tag{3.1}$$

$$C^{-1}u^2 \leq G(r,u) \leq Cu^2, \quad C^{-1}\|u\|_2^2 \leq \Psi(u) \leq C\|u\|_2^2. \tag{3.2}$$

Furthermore, we know the interpolation inequalities (cf. Chiappinelli [4, Lemma 1].)

$$\|u\|_{p_k+1}^{p_k+1} \leq C\|u\|_X^{\frac{N(p_k-1)}{2}}\|u\|_2^{\frac{N+2-p_k(N-2)}{2}}, \quad \|u\|_{q_k+1}^{q_k+1} \leq C\|u\|_X^{\frac{N(q_k-1)}{2}}\|u\|_2^{\frac{N+2-q_k(N-2)}{2}}. \tag{3.3}$$

Lemma 3.3. *Let w_τ be the unique solution of the following equation for a given $\tau > 0$:*

$$\begin{aligned} &w_\tau''(s) + \frac{N-1}{s}w_\tau'(s) + w_\tau(s)^{p_i} - w_\tau(s) = 0, \quad 0 < s < \tau, \\ &w_\tau(s) > 0, \quad 0 \leq s < \tau, \\ &w_\tau'(0) = w_\tau(\tau) = 0. \end{aligned} \tag{3.4}$$

Then $w_\tau(|x|) \to w(|x|)$ not only uniformly on any compact sets on R^N, but also in $L^2(R^N)$ and $L^{p_k+1}(R^N)(1 \leq k \leq n)$ as $\tau \to \infty$.

This lemma can be shown by the same arguments as those which will be used in the proof of Lemma 4.1 and Lemma 4.8 proved later. Thus the proof is omitted.

The following properties of the ground state w of the equation (1.11) will play important roles to show Lemma 3.1. There uniquely exists the ground state w of (1.11) such that: w decreases for $s > 0, w \in C^2(R)$, and for some constant $\delta > 0$

$$w(s) \leq Ce^{-\delta s}, \quad s \geq 0, \tag{3.5}$$

$$\|w\|_{p_i+1,R,N}^{p_i+1} = \frac{2(p_i+1)}{N+2-p_i(N-2)}\|w\|_{2,R,N}^2, \quad \|w\|_{X,R,N}^2 = \frac{N(p_i-1)}{N+2-p_i(N-2)}\|w\|_{2,R,N}^2. \tag{3.6}$$

Here

$$\|w\|_{p,R,N}^p := \int_R s^{N-1}|w(s)|^p ds, \quad \|w\|_{X,R,N}^2 := \int_R s^{N-1}|w'(s)|^2 ds$$

and will be denoted by $\|w\|_p^p$ and $\|w\|_X^2$, respectively for simplicity. For these properties, we refer to Berestycki and Lions [2], Kwong [10] and Strauss [13].

Lemma 3.4. *Assume (C-i). Let $s_{\mu,\alpha} := \|V_{\mu,\alpha}\|_2^{-\frac{2(p_i-1)}{4-N(p_i-1)}}(\alpha^{p_i-1}\mu_i)^{\frac{2}{4-N(p_i-1)}}$, and $V_{\mu,\alpha}$ be the unique solution of (3.4) for $\tau = s_{\mu,\alpha}$. Furthermore, let $r := s_{\mu,\alpha}^{-1}s$, and*

$$v_{\mu,\alpha}(r) := \|V_{\mu,\alpha}\|_2^{-\frac{4}{4-N(p_i-1)}}(\alpha^4\mu_i^N)^{\frac{1}{4-N(p_i-1)}}d_{\mu,\alpha}V_{\mu,\alpha}(s),$$

where $d_{\mu,\alpha}$ is defined by the rule $\Psi(v_{\mu,\alpha}) = 1/2\alpha^2$. Then

$$C^{-1} \leq d_{\mu,\alpha} \leq C. \tag{3.7}$$

Furthermore,

$$\|v_{\mu,\alpha}\|_X^2 \leq C\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2\{N+2-p_i(N-2)\}}{4-N(p_i-1)}}, \tag{3.8}$$

$$\mu_i\|v_{\mu,\alpha}\|_{p_i+1}^{p_i+1} \leq C\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2\{N+2-p_i(N-2)\}}{4-N(p_i-1)}}, \tag{3.9}$$

$$\mu_i\|v_{\mu,\alpha}\|_{q_i+1}^{q_i+1} \leq C\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}, \tag{3.10}$$

$$\mu_k\|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1}, \mu_k\|v_{\mu,\alpha}\|_{q_k+1}^{q_k+1}, \mu_k|\Phi_k(v_{\mu,\alpha})| = o(1)\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}} \quad (k \neq i). \tag{3.11}$$

Proof. By definition of $v_{\mu,\alpha}$, we have $\|v_{\mu,\alpha}\|_2^2 = \alpha^2 d_{\mu,\alpha}^2$. This along with (3.2) implies

$$C^{-1}\alpha^2 d_{\mu,\alpha}^2 = C^{-1}\|v_{\mu,\alpha}\|_2^2 \le \Psi(v_{\mu,\alpha}) = \frac{1}{2}\alpha^2 \le C\|v_{\mu,\alpha}\|_2^2 = C\alpha^2 d_{\mu,\alpha}^2.$$

Thus, (3.7) is proved. Next, by (3.7), and Lemma 3.3, we obtain

$$\|v_{\mu,\alpha}\|_X^2 \le C\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}.$$

Thus, we obtain (3.8). (3.9) is also obtained by direct calculation. Since $v_{\mu,\alpha} \in M_\alpha$, we obtain by (3.3) and (3.8) that

$$\mu_k\|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1} \le C(\mu_k\alpha^{\frac{4(p_k-p_i)}{4-N(p_i-1)}}\mu_i^{-\frac{4-N(p_k-1)}{4-N(p_i-1)}})\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}},$$

$$\mu_k\|v_{\mu,\alpha}\|_{q_k+1}^{q_k+1} \le C(\mu_k\alpha^{\frac{4(p_k-p_i)}{4-N(p_i-1)}}\mu_i^{-\frac{4-N(p_k-1)}{4-N(p_i-1)}})(\alpha^4\mu_i^N)^{\frac{(q_k-p_k)}{4-N(p_i-1)}}\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}. \tag{3.12}$$

This along with the fact that $q_k \le p_k$, (2.17), (2.18), and (3.1) implies that

$$\mu_k|\Phi_k(v_{\mu,\alpha})| \le C\mu_k(\|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1} + \|v_{\mu,\alpha}\|_{q_k+1}^{q_k+1}) = o(1)\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}, \quad (k \ne i),$$

$$\mu_i|\Phi_i(v_{\mu,\alpha})| \le C\mu_i(\|v_{\mu,\alpha}\|_{p_i+1}^{p_i+1} + \|v_{\mu,\alpha}\|_{q_i+1}^{q_i+1}) \le C\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}. \tag{3.13}$$

Thus we obtain (3.10)–(3.11). □

To obtain Lemma 3.1, we need further observation of asymptotic property of $d_{\mu,\alpha}$. We put

$$F_{k,0}(u) := \int_0^u f_{k,0}(r,s)ds, \quad G_0(u) := \int_0^u g_0(r,s)ds.$$

Then, by (2.2) and (2.6) we have

$$|g_0(r,u)| \le Cu \quad \text{for } (r,u) \in [0,1] \times R_+,$$

$$\left|\frac{f_{k,0}(r,u)}{u^{p_k}}\right|, \left|\frac{F_{k,0}(r,u)}{u^{p_k+1}}\right|, \left|\frac{g_0(r,u)}{u}\right|, \left|\frac{G_0(r,u)}{u^2}\right| \to 0 \quad \text{unif. in } r \in [0,1] \text{ as } u \to \infty. \tag{3.14}$$

By using Lemma 3.4 and a direct calculation, we obtain:

Lemma 3.5. *Assume (C-i). Then* $d_{\mu,\alpha} \to 1$. □

Furthermore, by Lemmas 3.3 - 3.5 we also obtain

$$\mu_i\Phi_i(v_{\mu,\alpha}) = \frac{2}{N+2-p_i(N-2)}(1+o(1))\|w\|_2^{\frac{4(1-p_i)}{4-N(p_i-1)}}\mu_i^{\frac{4}{4+N-Np_i}}\alpha^{\frac{2(N+2-p_i(N-2))}{4+N-Np_i}}. \tag{3.15}$$

Lemma 3.6. *Assume (C-i). Then*

$$\|u_{\mu,\alpha}\|_X^2 \le C\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}, \tag{3.16}$$

$$\mu_i|\Phi_i(u_{\mu,\alpha})|, \mu_i\|u_{\mu,\alpha}\|_{p_i+1}^{p_i+1}, \mu_i\|u_{\mu,\alpha}\|_{q_i+1}^{q_i+1} \le C\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}. \tag{3.17}$$

$$\mu_k|\Phi_k(u_{\mu,\alpha})|, \mu_k\|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1}, \mu_k\|u_{\mu,\alpha}\|_{q_k+1}^{q_k+1} = o(1)\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2)}{4-N(p_i-1)}} \quad (k \ne i). \tag{3.18}$$

Proof. (2.13) along with the fact that $v_{\mu,\alpha} \in M_\alpha$ implies

$$\Lambda_\mu(u_{\mu,\alpha}) = \frac{1}{2}\|u_{\mu,\alpha}\|_X^2 - \sum_{k=1}^n \mu_k \Phi_k(u_{\mu,\alpha}) \le \Lambda_\mu(v_{\mu,\alpha}) = \frac{1}{2}\|v_{\mu,\alpha}\|_X^2 - \sum_{k=1}^n \mu_k \Phi_k(v_{\mu,\alpha}); \tag{3.19}$$

this implies that

$$\frac{1}{2}\|u_{\mu,\alpha}\|_X^2 \le \sum_{k=1}^n \mu_k|\Phi_k(u_{\mu,\alpha})| + \frac{1}{2}\|v_{\mu,\alpha}\|_X^2 + \sum_{k=1}^n \mu_k|\Phi_k(v_{\mu,\alpha})|. \tag{3.20}$$

Here we recall the inequality

$$ab \le a^{\beta_1}/\beta_1 + b^{\beta_2}/\beta_2 \quad (a,b \ge 0,\ \ 1/\beta_1 + 1/\beta_2 = 1). \tag{3.21}$$

Since $v_{\mu,\alpha} \in M_\alpha$, we obtain by (3.3) and (3.21) that for $0 < \epsilon \ll 1$ and $1 \le k \le n$

$$\begin{aligned} &\mu_k\|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1} \le C\epsilon^{-\frac{N(p_k-1)}{4-N(p_k-1)}} \left(\mu_k \alpha^{\frac{4(p_k-p_i)}{4-N(p_i-1)}} \mu_i^{-\frac{4-N(p_k-1)}{4-N(p_i-1)}}\right)^{\frac{4}{4-N(p_k-1)}} \\ &\times \mu_i^{\frac{4}{4-N(p_i-1)}} \alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}} + C\epsilon\|u_{\mu,\alpha}\|_X^2, \\ &\mu_k\|u_{\mu,\alpha}\|_{q_k+1}^{q_k+1} \le C\epsilon^{-\frac{N(q_k-1)}{4-N(q_k-1)}} \left(\mu_k \alpha^{\frac{4(p_k-p_i)}{4-N(p_i-1)}} \mu_i^{-\frac{4-N(p_k-1)}{4-N(p_i-1)}}\right)^{\frac{4}{4-N(q_k-1)}} \\ &\times \mu_i^{\frac{4}{4-N(p_i-1)}} \alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}} (\alpha^4 \mu_i^N)^{\frac{4(q_k-p_k)}{(4-N(p_i-1))(4-N(q_k-1))}} + C\epsilon\|u_{\mu,\alpha}\|_X^2. \end{aligned} \tag{3.22}$$

By (2.17), (2.18), (3.1), and (3.22), we obtain for $k \ne i$

$$\begin{aligned} &\mu_k|\Phi_k(u_{\mu,\alpha})| \le C(\|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1} + \|u_{\mu,\alpha}\|_{q_k+1}^{q_k+1}) \\ &\le o(1)\mu_i^{\frac{4}{4-N(p_i-1)}} \alpha^{\frac{2(N+2-p_i(N-2)}{4-N(p_i-1)}} + C\epsilon\|u_{\mu,\alpha}\|_X^2, \\ &\mu_i|\Phi_i(u_{\mu,\alpha})| \le C(\|u_{\mu,\alpha}\|_{p_i+1}^{p_i+1} + \|u_{\mu,\alpha}\|_{q_i+1}^{q_i+1}) \le C\mu_i^{\frac{4}{4-N(p_i-1)}} \alpha^{\frac{2(N+2-p_i(N-2)}{4-N(p_i-1)}} + C\epsilon\|u_{\mu,\alpha}\|_X^2; \end{aligned} \tag{3.23}$$

this along with (3.8), (3.13) and (3.20) implies that

$$\|u_{\mu,\alpha}\|_X^2 \le C\mu_i^{\frac{4}{4-N(p_i-1)}} \alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}} + C\epsilon\|u_{\mu,\alpha}\|_X^2. \tag{3.24}$$

Thus, (3.16) follows immediately from (3.24), and (3.17) follows from (3.16) and (3.23). Since $\epsilon > 0$ is arbitrary in (3.23), (3.18) follows from (3.16) and (3.23). □

By Lemma 3.3, Lemma 3.5 and (3.6), we obtain:

Lemma 3.7. *Assume (C-i). Then*

$$\begin{aligned} &2\sum_{k=1}^n \mu_k \Phi_k(v_{\mu,\alpha}) - \|v_{\mu,\alpha}\|_X^2 \\ &= \frac{4-N(p_i-1)}{N+2-p_i(N-2)}(1+o(1))\|w\|_2^{\frac{4(1-p_i)}{4-N(p_i-1)}} \mu_i^{\frac{4}{4-N(p_i-1)}} \alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}. \end{aligned} \tag{3.25}$$

Now, we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. By (2.3) and (3.18), for $k \neq i$ we have

$$\mu_k|(f_k(|x|, u_{\mu,\alpha}), u_{\mu,\alpha})| \le C\mu_k(\|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1} + \|u_{\mu,\alpha}\|_{q_k+1}^{q_k+1}) = o(1)\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}. \tag{3.26}$$

By (3.11), (3.18), (3.19), and Lemma 3.7

$$\mu_i\Phi_i(u_{\mu,\alpha}) \ge \sum_{k=1}^{n}\mu_k\Phi_k(v_{\mu,\alpha}) - \frac{1}{2}\|v_{\mu,\alpha}\|_X^2 - \sum_{k\neq i}^{n}\mu_k\Phi_k(u_{\mu,\alpha}) \ge C\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}. \tag{3.27}$$

Therefore, by (2.5)

$$(f_i(|x|, u_{\mu,\alpha}), u_{\mu,\alpha}) - 2\Phi_i(u_{\mu,\alpha}) \ge 0. \tag{3.28}$$

Now, by (2.2), (2.14), (3.18), (3.26), (3.28) and Lemma 3.7

$$\begin{aligned}
\lambda(\mu,\alpha) &= \frac{-\|u_{\mu,\alpha}\|_X^2 + \sum_{k=1}^{n}\mu_k(f_k(u_{\mu,\alpha}), u_{\mu,\alpha})}{(g(u_{\mu,\alpha}), u_{\mu,\alpha})} \\
&\ge C\alpha^{-2}\left\{\sum_{k=1}^{n}\mu_k\{(f_k(|x|, u_{\mu,\alpha}), u_{\mu,\alpha}) - 2\Phi_k(u_{\mu,\alpha})\} + 2\sum_{k=1}^{n}\mu_k\Phi_k(v_{\mu,\alpha}) - \|v_{\mu,\alpha}\|_X^2\right\} \\
&\ge C\alpha^{-2}\{(f_i(|x|, u_{\mu,\alpha}, u_{\mu,\alpha}) - 2\Phi_i(u_{\mu,\alpha})) + (2\mu_i\Phi_i(v_{\mu,\alpha}) - \|v_{\mu,\alpha}\|_X^2) \\
&+ o(1)\mu_i^{\frac{4}{4-N(p_i-1)}}\alpha^{\frac{2(N+2-p_i(N-2))}{4-N(p_i-1)}}\} \ge C(\alpha^{p_i-1}\mu_i)^{\frac{4}{4-N(p_i-1)}}.
\end{aligned}$$

Thus the proof is complete. □

4. THE LIMITING PROCEDURE. To prove Theorem 2.1, we follow the arguments used in Shibata [11, 12]. We put

$$\xi_{\mu,\alpha} := (\lambda(\mu,\alpha)/\mu_i)^{\frac{1}{p_i-1}}, \quad w_{\mu,\alpha}(s) := \xi_{\mu,\alpha}^{-1}u_{\mu,\alpha}(r), \quad s := \sqrt{\lambda(\mu,\alpha)}r. \tag{4.1}$$

Then (1.1) implies that $w_{\mu,\alpha}(s)$ satisfies the following equation (4.2):

$$\begin{aligned}
&w''_{\mu,\alpha}(s) + \frac{N-1}{s}w'_{\mu,\alpha}(s) + a_i(\lambda(\mu,\alpha)^{-1/2}s)w_{\mu,\alpha}(s)^{p_i} - a_0(\lambda(\mu,\alpha)^{-1/2}s)w_{\mu,\alpha}(s) \\
&\quad + \sum_{k=1,k\neq i}^{n}\lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{p_k-1}a_k(\lambda(\mu,\alpha)^{-1/2}s)w(\mu,\alpha,s)^{p_k} \\
&\quad + \sum_{k=1}^{n}\lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{-1}f_{k,0}(\lambda(\mu,\alpha)^{-1/2}s, \xi_{\mu,\alpha}w_{\mu,\alpha}(s)) \qquad (4.2)\\
&\quad - \xi_{\mu,\alpha}^{-1}g_0(\lambda(\mu,\alpha)^{-1/2}s, \xi_{\mu,\alpha}w_{\mu,\alpha}(s)) = 0, \quad s \in I_{\mu,\alpha} := (0, \sqrt{\lambda(\mu,\alpha)}), \\
&w_{\mu,\alpha}(s) > 0, \quad 0 \le s < \sqrt{\lambda(\mu,\alpha)}, \\
&w'_{\mu,\alpha}(0) = w_{\mu,\alpha}(\sqrt{\lambda(\mu,\alpha)}) = 0.
\end{aligned}$$

Therefore, we expect that the limit equation of (4.2) should be (1.11), and the first aim of this section is to show the following Lemma 4.1:

Lemma 4.1 *Let $w = w(s)$ be the ground state of (1.11). Assume (C-i). Then $w_{\mu,\alpha}(s) \to w(s)$ uniformly on any compact subsets on R.*

By the transformation and change of variable of (4.1), we have

$$\|w_{\mu,\alpha}\|^2_{X,\mu,\alpha} := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w'_{\mu,\alpha}(s)^2 ds = \lambda(\mu,\alpha)^{\frac{N-2}{2}} \xi^{-2}_{\mu,\alpha} \|u_{\mu,\alpha}\|^2_X, \tag{4.3}$$

$$\|w_{\mu,\alpha}\|^{p_k+1}_{p_k+1,\mu,\alpha} := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w_{\mu,\alpha}(s)^{p_k+1} ds = \lambda(\mu,\alpha)^{\frac{N}{2}} \xi^{-(p_k+1)}_{\mu,\alpha} \|u_{\mu,\alpha}\|^{p_k+1}_{p_k+1}, \tag{4.4}$$

$$\|w_{\mu,\alpha}\|^2_{2,\mu,\alpha} := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w_{\mu,\alpha}(s)^2 ds = \lambda(\mu,\alpha)^{\frac{N}{2}} \xi^{-2}_{\mu,\alpha} \|u_{\mu,\alpha}\|^2_2. \tag{4.5}$$

We may abbreviate $\|w_{\mu,\alpha}\|_{X,\mu,\alpha}$, $\|w_{\mu,\alpha}\|_{p,\mu,\alpha} (p = 2, p_k + 1)$ to $\|w_{\mu,\alpha}\|_X$, $\|w_{\mu,\alpha}\|_p$, respectively. To show Lemma 4.1, we prepare some lemmas. By Gidas, Ni and Nirenberg [7] and Corollary 3.2, we know that $u_{\mu,\alpha}$ satisfies the following properties:

$$u'_{\mu,\alpha}(r) < 0, \;\; r \in (0,1), \quad u'_{\mu,\alpha}(0) = 0, \quad \sigma_{\mu,\alpha} := \max_{0 \le r \le 1} u_{\mu,\alpha}(r) = u_{\mu,\alpha}(0). \tag{4.6}$$

Lemma 4.2. *For a solution $u_{\mu,\alpha}$ of (1.1), the following equality holds for $r \in [0,1]$:*

$$\begin{aligned} &\frac{1}{2} u'_{\mu,\alpha}(r)^2 + \int_0^r \frac{N-1}{s} u'_{\mu,\alpha}(s)^2 ds + J(\mu,\alpha,r,u_{\mu,\alpha}(r)) - \sum_{k=1}^n \mu_k B_k(\mu,\alpha,r) \\ &+ \lambda(\mu,\alpha) B_0(\mu,\alpha,r) = J(\mu,\alpha,0,\sigma_{\mu,\alpha}) \\ &= \frac{1}{2} u'_{\mu,\alpha}(1)^2 + \int_0^1 \frac{N-1}{s} u'_{\mu,\alpha}(s)^2 dr - \sum_{k=1}^n \mu_k B_k(\mu,\alpha,1) + \lambda(\mu,\alpha) B_0(\mu,\alpha,1) > 0, \end{aligned} \tag{4.7}$$

where

$$J(\mu,\alpha,r,u) := \sum_{k=1}^n \mu_k F_k(r,u) - \lambda(\mu,\alpha) G(r,u), \tag{4.8}$$

$$B_k(\mu,\alpha,r) := \int_0^r \left\{ \int_0^{u_{\mu,\alpha}(t)} \frac{\partial f_k(t,s)}{\partial t} ds \right\} dt \le 0, \quad r \in [0,1], \tag{4.9}$$

$$B_0(\mu,\alpha,r) := \int_0^r \left\{ \int_0^{u_{\mu,\alpha}(t)} \frac{\partial g(t,s)}{\partial t} ds \right\} dt \ge 0, \quad r \in [0,1]. \tag{4.10}$$

Lemma 4.3. *Assume (C-i). Then $\sigma_{\mu,\alpha} \to \infty$.*

Proof. Assume that there exists a subsequence of $\{\sigma_{\mu,\alpha}\}$ such that $\sigma_{\mu,\alpha} \le C$. By (2.18) we have for $k \ne i$

$$\mu_i^{-1} \mu_k = o(1) (\alpha^4 \mu_i^N)^{\frac{p_i - p_k}{4 - N(p_i - 1)}}; \tag{4.11}$$

this along with (3.1), (3.2), (4.7) and Lemma 3.1 implies that

$$\begin{aligned} C(\alpha^4\mu_i^N)^{\frac{p_i-1}{4-N(p_i-1)}} &\le \frac{\lambda(\mu,\alpha)}{\mu_i} \le C\sum_{k=1}^{n}\mu_k F_k(\sigma_{\mu,\alpha})\mu_i^{-1}G(\sigma_{\mu,\alpha})^{-1} \\ &\le C\sum_{k=1}^{n}\mu_k(\sigma_{\mu,\alpha}^{q_k-1}+\sigma_{\mu,\alpha}^{p_k-1})\mu_i^{-1} \le C\sum_{k=1}^{n}\mu_k\mu_i^{-1} \le C(1+o(1)(\alpha^4\mu_i^N)^{\frac{p_i-p_k}{4-N(p_i-1)}}). \end{aligned} \tag{4.12}$$

This is a contradiction, since we assume (2.17). Thus the proof is complete. □

Lemma 4.4 *Assume (C-i). Then* $\mu_k\sigma_{\mu,\alpha}^{p_k-1} \le C\mu_i\sigma_{\mu,\alpha}^{p_i-1}$ *for* $1 \le k \le n$.

Proof. Since $q_k \le p_k$, by (3.1), (3.2), (4.8), Lemma 3.1, and Lemma 4.3, we obtain

$$\lambda(\mu,\alpha) \le C\sum_{k=1}^{n}\mu_k F_k(0.\sigma_{\mu,\alpha})G(0,\sigma_{\mu,\alpha})^{-1} \le C\sum_{k=1}^{n}\mu_k(\sigma_{\mu,\alpha}^{p_k-1}+\sigma_{\mu,\alpha}^{q_k-1}) \le C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k-1}. \tag{4.13}$$

Since $q_k \le p_k$, we obtain by (3.1), (4.7), (4.9), (4.10) and (4.13) that for $0 \le r \le 1$

$$\frac{1}{2}u'_{\mu,\alpha}(r)^2 \le J(\mu,\alpha,r,\sigma_{\mu,\alpha}) - J(\mu,\alpha,r,u_{\mu,\alpha}) \le C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k+1}. \tag{4.14}$$

Let $r_1 := r_{1,\mu,\alpha} \in [0,1)$ satisfy $u_{\mu,\alpha}(r_1) = 1/2\sigma_{\mu,\alpha}$. Since $u_{\mu,\alpha}(r)$ is decreasing in r and $u_{\mu,\alpha} \in M_\alpha$, by (3.2) we have

$$\frac{1}{2}\alpha^2 = \Psi(u_{\mu,\alpha}) \ge C\|u_{\mu,\alpha}\|_2^2 \ge C\int_0^{r_1} r^{N-1}u_{\mu,\alpha}(r)^2dr \ge C\sigma_{\mu,\alpha}^2 r_1^N. \tag{4.15}$$

For each (μ,α), let $1 \le j(\mu,\alpha) \le n$ satisfy $\mu_{j(\mu,\alpha)}\sigma_{\mu,\alpha}^{p_{j(\mu,\alpha)}+1} = \max_{1\le k\le n}\mu_k\sigma_{\mu,\alpha}^{p_k+1}$. Then there exists a subsequence of $\{(\mu,\alpha)\}$ and $1 \le j \le n$ such that $j = j(\mu,\alpha)$ for all (μ,α) commonly. Along this subsequence we have

$$\mu_k\sigma_{\mu,\alpha}^{p_k+1} \le \mu_j\sigma_{\mu,\alpha}^{p_j+1} \tag{4.16}$$

for any $1 \le k \le n$. We fix this subsequence of (μ,α) and j. Then by mean value theorem, (4.14), and (4.16) we obtain

$$\frac{\sigma_{\mu,\alpha}}{2r_1} = \left|\frac{u_{\mu,\alpha}(0)-u_{\mu,\alpha}(r_1)}{r_1}\right| \le C\sqrt{\mu_j\sigma_{\mu,\alpha}^{p_j+1}}; \tag{4.17}$$

this implies that $\sigma_{\mu,\alpha}^{\frac{1-p_j}{2}}\mu_j^{-\frac{1}{2}} \le Cr_1$. This along with (4.15) yields

$$\sigma_{\mu,\alpha} \le C(\alpha^2\mu_j^{N/2})^{\frac{2}{4-N(p_j-1)}}. \tag{4.18}$$

By (4.13), (4.16) and (4.18)

$$\lambda(\mu,\alpha) \le C\mu_j\sigma_{\mu,\alpha}^{p_j-1} = C(\alpha^{p_j-1}\mu_j)^{\frac{4}{4-N(p_j-1)}}. \tag{4.19}$$

Since (2.18) implies that for $k \neq i$

$$(\alpha^{p_k-1}\mu_k)^{\frac{4}{4-N(p_k-1)}}(\alpha^{p_i-1}\mu_i)^{-\frac{4}{4-N(p_i-1)}} \to 0,$$

it follows from Lemma 3.1, and (4.19) that the inequality (4.16) never occurs for some $j \neq i$. Namely, we find that $j(\mu,\alpha) = i$ for all (μ,α) except finite members of (μ,α). Thus the proof is complete. □

Lemma 4.5. *Assume (C-i). Then*

$$\lambda(\mu,\alpha) \leq C(\alpha^{p_i-1}\mu_i)^{\frac{4}{4-N(p_i-1)}}. \tag{4.20}$$

Proof. By (4.13) and the arguments of (4.14)-(4.18), we see that (4.16) and (4.18) are valid for $j = i$, that is,

$$\mu_k\sigma_{\mu,\alpha}^{p_k-1} \leq \mu_i\sigma_{\mu,\alpha}^{p_i-1}, \quad \sigma_{\mu,\alpha} \leq C(\alpha^2\mu_i^{N/2})^{\frac{2}{4-N(p_i-1)}}. \tag{4.21}$$

Substituting (4.21) into (4.13), we obtain our assertion. □

By Lemma 3.1 and Lemma 4.5, we obtain:

Lemma 4.6. *Assume (C-i). Then* $\lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{p_k-1} \to 0$ *for* $k \neq i$. □

By using the idea of Dancer [5], we can prove:

Lemma 4.7. *Assume (C-i). Let* $\eta_{\mu,\alpha} := \max_{s\in I_{\mu,\alpha}} w_{\mu,\alpha}(s)(= w_{\mu,\alpha}(0)) = \xi_{\mu,\alpha}^{-1}\sigma_{\mu,\alpha}$. *Then* $C^{-1} \leq \eta_{\mu,\alpha} \leq C$. □

Now we are ready to prove Lemma 4.1.

Proof of Lemma 4.1. It follows from (4.14) and (4.21) that

$$\frac{1}{2}\xi^2\lambda(\mu,\alpha)w'_{\mu,\alpha}(s)^2 = \frac{1}{2}u'_{\mu,\alpha}(r)^2 \leq C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k+1} + C\lambda(\mu,\alpha)\sigma_{\mu,\alpha}^2 \leq \mu_i\sigma_{\mu,\alpha}^{p_i+1};$$

this along with Lemma 4.7 yields

$$w'_{\mu,\alpha}(s)^2 \leq C\mu_i\sigma_{\mu,\alpha}^{p_i+1}\lambda(\mu,\alpha)^{-1}\xi^{-2} \leq C.$$

Therefore, $|w'_{\mu,\alpha}| \leq C$. This together with (4.2) and Lemma 4.7 yields $|w''_{\mu,\alpha}| \leq C$. Now we choose a subsequence of $\{w_{\mu,\alpha}\}$ and w_∞ such that $w_{\mu,\alpha} \to w_\infty$, $w'_{\mu,\alpha} \to w'_\infty$ uniformly on any compact sets in R. By a standard limiting procedure and regularity argument, we see that $w_\infty = w_\infty(s) \in C^2(R)$ satisfies (1.11). Moreover, since $\|u_{\mu,\alpha}\|_2 = O(\alpha)$, it follows from Lemma 3.1 that

$$\|w_{\mu,\alpha}\|_2^2 = \lambda(\mu,\alpha)^{N/2}\xi_{\mu,\alpha}^{-2}\|u_{\mu,\alpha}\|_2^2 \leq C\lambda(\mu,\alpha)^{-\frac{4-N(p_i-1)}{2(p_i-1)}}\mu_i^{\frac{2}{p_i-1}}\alpha^2 \leq C;$$

this along with Fatou's lemma yields

$$\|w_\infty\|_2^2 \leq \liminf \|w_{\mu,\alpha}\|_2^2 \leq C. \tag{4.22}$$

Since $w_\infty(s)$ is decreasing in s, it follows from (4.22) that $w_\infty(s)$ satisfies (1.11). Lemma 4.7 implies that $w_\infty \not\equiv 0$. The positivity of w_∞ follows from the uniqueness theorem of ODE. We, therefore, find that w_∞ is exactly the ground state solution w of (1.11). Now, full assertion follows from a standard compactness argument. □

The following lemma is a variant of Shibata [11, Lemma 4.7].

Lemma 4.8 ([12, Lemma 6.6]) *Assume (C-i). Then there exists* $Y_0(x) = Y_0(|x|) \in L^2(R^N) \cap L^{p_k+1}(R^N)$ $(1 \le k \le n)$ *such that* $w_{\mu,\alpha}(|x|) \le Y_0(|x|)$ *for* $x \in R^N$.

5. PROOF OF THEOREM 2.1. By Lemma 4.1, Lemma 4.8, and Lebesgue's convergence theorem

$$\|w_{\mu,\alpha}\|_2 \to \|w\|_2, \quad \|w_{\mu,\alpha}\|_{p_k+1} \to \|w\|_{p_k+1} \quad (1 \le k \le n). \tag{5.1}$$

For $1 \le k \le n$, by (4.5) and the same argument as that used in Lemma 3.5 we obtain

$$(f_{k,0}(\lambda(\mu,\alpha)^{-1/2}s, \xi_{\mu,\alpha}w_{\mu,\alpha}), w_{\mu,\alpha}) = o(1)\xi_{\mu,\alpha}^{p_k}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}, \tag{5.2}$$

$$(g_0(\lambda(\mu,\alpha)^{-1/2}s, \xi_{\mu,\alpha}w_{\mu,\alpha}), w_{\mu,\alpha}) = o(1)\xi_{\mu,\alpha}^2\|w_{\mu,\alpha}\|_2^2. \tag{5.3}$$

Moreover, by (5.3) and the same argument used in Lemma 3.5, we have

$$\begin{aligned} \frac{1}{2}\alpha^2 = \Psi(u_{\mu,\alpha}) &= \frac{1}{2}(1+o(1))\|u_{\mu,\alpha}\|_2^2, \\ (g(r,u_{\mu,\alpha}), u_{\mu,\alpha}) &= (1+o(1))\|u_{\mu,\alpha}\|_2^2 = (1+o(1))\alpha^2. \end{aligned} \tag{5.4}$$

Multiply (4.2) by $w_{\mu,\alpha}$. Then integration by parts together with Lemma 4.6 and (5.1)–(5.3) yields

$$\begin{aligned} \|w_{\mu,\alpha}\|_X^2 &= \|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1} - \|w_{\mu,\alpha}\|_2^2 + \sum_{k\ne i}^{n} \lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{p_k-1}(a_k(\lambda(\mu,\alpha)^{-1/2}s)w_{\mu,\alpha}^{p_k}, w_{\mu,\alpha}) \\ &+ \sum_{k=1}^{n} \lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{-1}(f_{k,0}(\lambda(\mu,\alpha)^{-1/2}s, \xi_{\mu,\alpha}w_{\mu,\alpha}), w_{\mu,\alpha}) \\ &- \xi_{\mu,\alpha}^{-1}(g_0(\lambda(\mu,\alpha)^{-1/2}s, \xi_{\mu,\alpha}w_{\mu,\alpha}), w_{\mu,\alpha}) \to \|w\|_{p_i+1}^{p_i+1} - \|w\|_2^2. \end{aligned} \tag{5.5}$$

Then by (2.14) and (5.1) - (5.5)

$$\begin{aligned} \lambda(\mu,\alpha)(g(r,u_{\mu,\alpha}), u_{\mu,\alpha}) &= (1+o(1))\lambda(\mu,\alpha)\alpha^2 = \lambda(\mu,\alpha)^{\frac{2-N}{2}}\xi_{\mu,\alpha}^2\{\|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1} - \|w_{\mu,\alpha}\|_X^2 \\ &+ \sum_{k\ne i}^{n} \lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{p_k-1}(a_k(\lambda(\mu,\alpha)^{-1/2}s)w_{\mu,\alpha}^{p_k}, w_{\mu,\alpha}) \\ &+ \sum_{k=1}^{n} \lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{-1}(f_{k,0}(\lambda(\mu,\alpha)^{-1/2}s, \xi_{\mu,\alpha}w_{\mu,\alpha}), w_{\mu,\alpha}) \\ &- \xi_{\mu,\alpha}^{-1}(g_0(\lambda(\mu,\alpha)^{-1/2}s, \xi_{\mu,\alpha}w_{\mu,\alpha}), w_{\mu,\alpha})\}. \end{aligned} \tag{5.6}$$

That is,

$$(1+o(1))\lambda(\mu,\alpha)\alpha^2 = (1+o(1))\lambda(\mu,\alpha)^{\frac{2-N}{2}}\xi_{\mu,\alpha}^2\|w\|_2^2. \tag{5.7}$$

This implies

$$\frac{\lambda(\mu,\alpha)}{(\alpha^{p_i-1}\mu_i)^{\frac{4}{4-N(p_i-1)}}} \to \|w\|_2^{-\frac{4(p_i-1)}{4-N(p_i-1)}}. \tag{5.8}$$

This proves our theorem. □

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