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Global Asymptotic Properties of a Delay SIR Epidemic Model with Varying Population Size and Finite Incubation Times

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Abstract. This paper concerns the global asymptotic properties of the disease free equilibrium and the endemic equilibrium of a delay SIR epidemic model with varying population size and a finite incubation time. It is shown that, while the endemic equilibrium does not exist, the disease free equilibrium is always globally attractive, i.e. the disease will eventually disappear. If the endemic equilibrium exists, it is shown that it is globally asymptotically stable, i.e. the disease always remains endemic, as long as the incubation time is short enough and the product of the contact rate and the birth rate of the population is relatively large (or the sum of the recovery and death rate of infectives is small enough).

Key words: SIR epidemic model, time delay, global attractivity, global asymptotic stability.

1 Introduction

In this paper, we shall analyze the following well known SIR epidemic model with delay

$$\left\{\begin{array}{ll} \dot{S}(t)= & -\beta S(t)\int_0^h f(s)I(t-s)ds-\mu_1 S(t)+b \\ \dot{I}(t)= & \beta S(t)\int_0^h f(s)I(t-s)ds-(\mu_2+\lambda)I(t) \\ \dot{R}(t)= & \lambda I(t)-\mu_3 R(t), \end{array}\right. \quad (1)$$

where $h,\ \beta, b, \lambda,\ \mu_1,\ \mu_2$ and μ_3 are positive constants; $f(s)$ is a nonnegative and continuous function such that $\int_0^h f(s)ds=1$.

By a biological meaning, the initial condition of (1) is given as

$$S(t_0+s)=\varphi_1,\ I(t_0+s)=\varphi_2,\ R(t_0+s)=\varphi_3,\quad -h\le s\le 0, \tag{2}$$

where $t_0\in R$, $\varphi=(\varphi_1,\varphi_2,\varphi_3)^T\in C$ such that $\varphi_i\ge 0$ and $\varphi_i(0)>0$ for $i=1,2,3$, C denotes the Banach space $C([-h,0],R^3)$ of continuous functions mapping the interval $[-h,0]$ into R^3.

It is easy to check that the solution $(S(t),I(t),R(t))$ of (1) with the initial condition (2) exists and is unique for all $t\ge t_0$ (see [7] or [9]). Also it is trivial that $S(t)>0, I(t)>0$ and $R(t)>0$ for all $t\ge t_0$.

Clearly, for any parameters h, β, b, λ, μ_1, μ_2 and μ_3, (1) always has a disease free equilibrium

$$E_0=(\frac{b}{\mu_1},\ 0,\ 0).$$

If

$$\frac{b}{\mu_1}>S^*\equiv\frac{\mu_2+\lambda}{\beta}, \tag{3}$$

then (1) also has an endemic equilibrium

$$E_+=(S^*,\ I^*,\ R^*)\equiv(\frac{\mu_2+\lambda}{\beta},\ \frac{b-\mu_1S^*}{\beta S^*},\ \frac{\lambda(b-\mu_1S^*)}{\mu_3\beta S^*})$$

In (1), $S(t)$, $I(t)$ and $R(t)$ denote the numbers of a population susceptible to the disease, of infective members and of members who have been removed from the possibility of infection through full immunity, respectively. It is assumed that all newborns are susceptible. The μ_1, μ_2 and μ_3 represent the death rates of susceptibles, infectives and recovered, respectively. The b and λ represent the birth rate of the population and the recovery rate of infectives, respectively. The β is the average number of contacts per infective per day. Thus, the term

$$\beta S(t)\int_0^h f(s)I(t-s)ds$$

can be considered as the force of infection at time t, where $f(s)$ is the fraction of vector population in which the time taken to become infectious is h.

As pointed out in [1] and [8], the *SIR* epidemic model (1) is appropriate for viral agent diseases such as measles, mumps and smallpox. If we use $\beta S(t)I(t)$ as the force of infection at time t instead of $\beta S(t)\int_0^h f(s)I(t-s)ds$, and assume that a population has a constant size and equal birth and death rates (that is $S(t)+I(t)+R(t)$ =constant and $\mu_1 = \mu_2 = \mu_3 = b$), then, (1) is reduced to the *SIR* epidemic model which was proposed and considered by Hethcote [8]. It is found that there exists a threshold (i.e., $\delta \equiv \beta/(\mu_2+\lambda)$) for an epidemic to occur [8]. By using $\beta S(t)I(t-\tau)$ as the force of infection at time t for some positive constant τ, Cooke [6] formulated a vector disease *SIR* model with a discrete time delay. Beretta, Capasso and Rinaldi [3] introduced an infinite delay to Hethcote's *SIR* epidemic model, and Beretta and Takeuchi [4] studied the global asymptotic stability of the disease free equilibrium and the local asymptotic stability of the endemic equilibrium. Recently, Beretta and Takeuchi [5] further considered the *SIR* epidemic model (1), which is clearly more realistic to describe disease transmission.

For the *SIR* epidemic model (1), the following results are known [5]:

(i) *The disease free equilibrium E_0 is globally asymptotically stable whenever $b/\mu_1 < S^* = (\mu_2+\lambda)/\beta$ (thus, the endemic equilibrium E_+ does not exist)*;

(ii) *When the endemic equilibrium E_+ exists (that is if (3) holds), it is locally asymptotically stable. An attractive region of E_+ which is explicitly given by the parameters was also obtained (see [5]).*

(iii) *If the average incubation time $T \equiv \int_0^h sf(s)ds$ is small enough (more exactly, if $T < (\beta\bar{S})^{-1}$), then there exist some solution $(S(t), I(t), R(t))$ of (1) and some time $\bar{t} \geq t_0$ such that $S(\bar{t}) \leq \bar{S}$, where $\bar{S} = [(\lambda+\mu_2)^2 + b\beta]/[\beta(\mu_1+\mu_2+\lambda)]$.*

The present paper shall further consider the global asymptotic properties of the disease free equilibrium E_0 and the endemic equilibrium E_+ of (1), and is organized as follows.

In Section 2, we consider the global attractivity of the disease free equilibrium E_0 based on the calssical Liapunov-LaSalle invariance principle in the case of $b/\mu_1 = (\mu_2 + \lambda)/\beta$. It is shown that E_0 is still globally attractive in this case. This extends the above result (i). In Section 3, we consider the global asymptotic stabality of the endemic equilibrium E_+ based on the some difference inequality and the construction of Liapunov functionals. This problem actually was proposed as an open problem in [4] and [5]. Our results show that, while E_+ exists, it is globally asymptotically stable as long as the delay h is short enough and βb is relatively large (or $\mu_2+\lambda$ is small enough). A brief discussion is included in the last section.

Since epidemics will increase the death rates of infectives and removed, it is biologically natural to assume that

$$\mu_1 \le \min\{\mu_2,\ \mu_3\}.$$

2 Disease Free Equilibrium

It is known that the disease free equilibrium E_0 is globally asymptotically stable whenever $b/\mu_1 < S^* = (\mu_2+\lambda)/\beta$ [5]. In this section, we shall further show that the following result is still true.

Theorem 1. *The disease free equilibrium E_0 is globally attractive whenever $b/\mu_1 = S^* = (\mu_2 + \lambda)/\beta$.*

Proof. For any solution $(S(t), I(t), R(t))$ of (1), let us first consider the case (a): $S(t) > S^*$ for all $t \ge t_0$. In this case, from (1), we see that for all $t \ge t_0$,

$$\begin{aligned} \dot{S}(t) - \dot{S}^* + \dot{I}(t) + \dot{R}(t) &= -\mu_1 (S(t) - S^*) - \mu_2 I(t) - \mu_3 R(t) \\ &\le -\mu_1 (S(t) - S^* + I(t) + R(t)). \end{aligned}$$

Thus,

$$\lim_{t\to+\infty} S(t) = S^*, \quad \lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} R(t) = 0.$$

Now let us consider the case (b): $\varphi_1 < S^*$ and $S(t) < S^*$ for all $t \geq t_0$. Set

$$G = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C \mid 0 \leq \varphi_1 \leq S^*, \varphi_2 \geq 0, \varphi_3 \geq 0\}.$$

We define

$$V(\varphi) = \varphi_2(0) + \beta S^* \int_0^h f(s) \int_{-s}^0 \varphi_2(u) du ds.$$

Then,

$$\dot{V}(\varphi) \mid_{(1)} = -\beta \left(S^* - \varphi_1(0)\right) \int_0^h f(s) \varphi_2(-s) ds \leq 0 \tag{4}$$

for $\varphi \in G$. Thus, $V(\varphi)$ is a Liapunov function on the subset G in C . Let

$$Q = \left\{\varphi \in G \mid \dot{V}(\varphi) \mid_{(1)} = 0\right\}$$

and M be the largest set in Q which is invariant with respect to (1). Clearly, M is not empty since $(S^*, 0, 0) \in M$.

From (4) we see that $\dot{V}(\varphi) \mid_{(1)} = 0$ if and only if $S^* - \varphi_1(0) = 0$ or $\varphi_2 = 0$. Note that $S^* - \varphi_1(0) = 0$ implies that $\varphi_2 = 0$ by (1). Thus, we always have $\varphi_2 = 0$ if $\dot{V}(\varphi) \mid_{(1)} = 0$. Observe that any solution of (1) is bounded by the following inequality

$$\dot{S}(t) + \dot{I}(t) + \dot{R}(t) \leq -\mu_1 \left(S(t) + I(t) + R(t)\right) + b. \tag{5}$$

Thus, it follows from the Liapunov-LaSalle invariance principle that $\lim_{t \to +\infty} I(t) = 0$ (see [7] or [9]). Hence, $\lim_{t \to +\infty} R(t) = 0$ by $\lim_{t \to +\infty} I(t) = 0$ and the last equation of (1). Furthermore, note that boundedness of $S(t)$ and $\int_0^h f(s) I(t-s) ds \to 0$ as $t \to +\infty$ by $\lim_{t \to +\infty} I(t) = 0$, we can also easily have that $\lim_{t \to +\infty} S(t) = S^*$ by the first equation of (1).

In the rest, let us consider the case (c): there is some $\hat{s}$ with $0 \leq \hat{s} < h$ such that $\varphi_1(-\hat{s}) \geq S^*$ and $S(t) < S^*$ for all $t \geq t_0$, or there is some $\hat{t}_0 \geq t_0$ such that $S(\hat{t}_0) = S^*$.

If $S(t) < S^*$ for all $t \geq t_0$, observe that system (1) is autonomous and the solution of (1) with any initial function $\varphi \in C$ is unique, by the same argument as used in case (b) with $t_0 = t_0 + 2h$, we can show that $\lim_{t \to +\infty} S(t) = S^*$ and $\lim_{t \to +\infty} I(t) = \lim_{t \to +\infty} R(t) = 0$.

If there is some $\hat{t}_0 \geq t_0$ such that $S(\hat{t}_0) = S^*$, by (1) we see that

$$\begin{aligned}\dot{S}(\hat{t}_0) - \dot{S}^* &= -\beta S(\hat{t}_0)\int_0^h f(s)I(\hat{t}_0 - s)ds - \mu_1 S(\hat{t}_0) + b \\ &= -\beta S^* \int_0^h f(s)I(\hat{t}_0 - s)ds < 0.\end{aligned}$$

Thus, for all $t > \hat{t}_0$, $S(t) - S^* < 0$, i.e. $S(t) < S^*$. Again by the same argument as used in case (b) with $t_0 = \hat{t}_0 + 2h$, we can show that $\lim_{t\to+\infty} S(t) = S^*$ and $\lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} R(t) = 0$.

This completes the proof of Theorem 1.

3 Endemic Equilibrium

Throughout this section, we always assume that the endemic equilibrium E_+ exists for (1), this is, we assume that (3) is true. Let us define

$$T \equiv \int_0^h sf(s)ds.$$

The following Theorem 2 is actually main result of this paper.

Theorem 2. *If there is some $\tilde{S}$ satisfying $S^* < \tilde{S} < b/(\mu_2+\lambda)$ such that the following conditions hold true:*

(i) $h < \min\left\{(2\beta\tilde{S})^{-1},\ (\tilde{S} - S^*)/(b - \mu_1 S^*)\right\}$;

(ii) $b < \tilde{S}\left[\beta\left(b/(\mu_2 + \lambda) - \tilde{S}\right) + \mu_1\right]$,

then the endemic equilibrium E_+ is globally asymptotically stable.

Proof. From (5), we see that, for any sufficiently small $\epsilon > 0$, there is a $t_0^* \geq t_0$ such that for $t \geq t_0^*$,

$$S(t) + I(t) + R(t) \leq \frac{b}{\mu_1} + \epsilon \equiv K_\epsilon.$$

For any positive constant $\tilde{S}$ satisfying $S^* < \tilde{S} < b/(\mu_2 + \lambda)$, define

$$\Omega_\varepsilon \equiv \left\{(S, I, R) \in R^3 \mid S + I + R \le K_\varepsilon, S > 0, I > 0, R > 0\right\},$$

$$\Omega_{\varepsilon,\tilde{S}} \equiv \left\{(S, I, R) \in \Omega_\varepsilon \mid S \le \tilde{S}\right\}.$$

Let us first show that the following *Assertion* A is true.

Assertion A : *For any positive constant $\tilde{S}$ satisfying $S^* < \tilde{S} < b/(\mu_2 + \lambda)$, if $h < (2\beta\tilde{S})^{-1}$, then any solution (1) will enter in $\Omega_{\varepsilon,\tilde{S}}$ in a finite time.*

In fact, if not, there are some $\tilde{S}$ satisfying $S^* < \tilde{S} < b/(\mu_2 + \lambda)$ and some solution $(S(t), I(t), R(t))$ of (1) such that $S(t) > \tilde{S}$ for all $t \ge t_0$ and $h < (2\beta\tilde{S})^{-1}$. Define a function

$$V(t) = I(t) + \beta\tilde{S}\int_0^h f(s)\int_{t-s}^t I(u)duds. \tag{6}$$

Thus,

$$\begin{aligned} I(t) &\le V(t) + \beta\tilde{S}\int_0^h f(s)\int_{t-s}^t I(u)duds \\ &\le V(t) + \beta\tilde{S}T \max_{-h\le\theta\le 0} I(t+\theta) \end{aligned} \tag{7}$$

for $t \ge t_0$. The time derivative of $V(t)$ along solution $(S(t), I(t), R(t))$ satisfies

$$\begin{aligned} \dot{V}(t) &= \dot{I}(t) + \beta\tilde{S}(I(t) - \int_0^h f(s)I(t-s)ds) \\ &= \beta S(t)\int_0^h f(s)I(t-s)ds - \beta S^* I(t) + \beta\tilde{S}\left(I(t) - \int_0^h f(s)I(t-s)ds\right) \\ &\ge \beta\left(\tilde{S} - S^*\right)I(t) > 0 \end{aligned} \tag{8}$$

for $t \ge t_0$. Set

$$\omega(t) = \begin{cases} I(t) - V(t)/(1-\beta\tilde{S}T), & \text{if } I(t) \ge V(t)/(1-\beta\tilde{S}T), \\ 0, & \text{if } I(t) < V(t)/(1-\beta\tilde{S}T). \end{cases} \tag{9}$$

Here $1 - \beta\tilde{S}T > 1 - 2\beta\tilde{S}T > 1 - 2\beta\tilde{S}h > 0$ by $T < h$. Clearly, $\omega(t)$ is nonnegative and continuous for $t \ge t_0$, and for $I(t) < V(t)/(1-\beta\tilde{S}T)$ and $t \ge t_0 + h$,

$$\omega(t) = 0 \le \beta\tilde{S}T \max_{-h\le\theta\le 0} \omega(t+\theta).$$

For $I(t) \geq V(t)/(1-\beta\tilde{S}T)$ and $t \geq t_0 + h$, by (7), (8) and (9), we have

$$\begin{aligned}
\omega(t) &= I(t) - \frac{1}{1-\beta\tilde{S}T}V(t) \\
&\leq V(t) + \beta\tilde{S}T \max_{-h\leq\theta\leq 0} I(t+\theta) - \frac{1}{1-\beta\tilde{S}T}V(t) \\
&= V(t) + \beta\tilde{S}T \max_{-h\leq\theta\leq 0} \max\left\{\omega(t+\theta) + \frac{1}{1-\beta\tilde{S}T}V(t+\theta), \frac{1}{1-\beta\tilde{S}T}V(t+\theta)\right\} \\
&\quad -\frac{1}{1-\beta\tilde{S}T}V(t) \\
&= \left(1 - \frac{1}{1-\beta\tilde{S}T}\right)V(t) + \beta\tilde{S}T \max_{-h\leq\theta\leq 0}\left\{\omega(t+\theta) + \frac{1}{1-\beta\tilde{S}T}V(t+\theta)\right\} \\
&\leq \left(1 - \frac{1}{1-\beta\tilde{S}T} + \frac{\beta\tilde{S}T}{1-\beta\tilde{S}T}\right)V(t) + \beta\tilde{S}T \max_{-h\leq\theta\leq 0}\omega(t+\theta) \\
&= \beta\tilde{S}T \max_{-h\leq\theta\leq 0}\omega(t+\theta).
\end{aligned}$$

Thus, for all $t \geq t_0 + h$, we have

$$\omega(t) \leq \beta\tilde{S}T \max_{-h\leq\theta\leq 0}\omega(t+\theta). \tag{10}$$

By $h < (2\beta\tilde{S})^{-1}$, we can choose a positive constant α which is only dependent on $\beta, \tilde{S}$ and h such that

$$\beta\tilde{S}he^{\alpha h} < 1.$$

We next show that, for any constant $k > 0$ and all $t \geq t_0 + h$, the following inequality holds (also see [10]):

$$\omega(t) < \left(k + \max_{-h\leq\theta\leq 0}\omega(t_0+h+\theta)\right)e^{-\alpha(t-t_0-h)} \equiv g_k(t). \tag{11}$$

Clearly, for $t_0 \leq t \leq t_0 + h$, $\omega(t) < g_k(t)$. If (11) is not true, by the continuity of $\omega(t)$ and $g_k(t)$, there are some constant $k_0 > 0$ and $\bar{t}_0 > t_0 + h$ such that

$$\omega(t) < g_{k_0}(t), \qquad t_0 \leq t < \bar{t}_0, \tag{12}$$

$$\omega(\bar{t}_0) = g_{k_0}(\bar{t}_0). \tag{13}$$

On the other hand, from (10), (11) and (12), we have

$$\begin{aligned}
\omega(\bar{t}_0) &\le \beta\tilde{S}T \max_{-h\le\theta\le 0}\omega(\bar{t}_0+\theta) \\
&\le \beta\tilde{S}T \max_{-h\le\theta\le 0} g_{k_0}(\bar{t}_0+\theta) \\
&= \beta\tilde{S}T \max_{-h\le\theta\le 0}\left\{\left(k_0+\max_{-h\le\theta\le 0}\omega(t_0+h+\theta)\right)e^{-\alpha\left(\bar{t}_0+\theta-t_0-h\right)}\right\} \\
&= \beta\tilde{S}Te^{\alpha h}\left(k_0+\max_{-h\le\theta\le 0}\omega(t_0+h+\theta)\right)e^{-\alpha\left(\bar{t}_0-t_0-h\right)} \\
&= \beta\tilde{S}Te^{\alpha h}g_{k_0}(\bar{t}_0) \\
&< g_{k_0}(\bar{t}_0),
\end{aligned}$$

which contradicts to (13). This proves (11).

In (11), letting $k\to 0^+$, we have that for $t\ge t_0+h$,

$$\omega(t)\le \max_{-h\le\theta\le 0}\omega(t_0+h+\theta)e^{-\alpha(t-t_0-h)}\equiv Me^{-\alpha(t-t_0-h)}, \tag{14}$$

where $M\equiv \max_{-h\le\theta\le 0}\omega(t_0+h+\theta)$. Therefore, it follows from (9) and (14) that for $t\ge t_0+h$,

$$I(t)\le Me^{-\alpha(t-t_0-h)}+\frac{1}{1-\beta\tilde{S}T}V(t). \tag{15}$$

Thus, it follows from (6), (8) and (15) that for $t\ge t_0+2h$,

$$\begin{aligned}
\dot{V}(t) &\ge \beta\left(\tilde{S}-S^*\right)I(t) \\
&= \beta\left(\tilde{S}-S^*\right)\left[V(t)-\beta\tilde{S}\int_0^h f(s)\int_{t-s}^t I(u)duds\right] \\
&\ge \beta\left(\tilde{S}-S^*\right)\left[V(t)-\beta\tilde{S}\int_0^h f(s)\int_{t-s}^t\left(Me^{-\alpha(u-t_0-h)}+\frac{1}{1-\beta\tilde{S}T}V(u)\right)duds\right] \\
&\ge \beta\left(\tilde{S}-S^*\right)\left[\frac{1-2\beta\tilde{S}T}{1-\beta\tilde{S}T}V(t)-\beta\tilde{S}M\int_0^h f(s)\int_{t-s}^t e^{-\alpha(u-t_0-h)}duds\right].
\end{aligned} \tag{16}$$

We have used that $V(t)$ is nondecreasing and $\tilde{S}>S^*$ in (16).

Note that $\int_0^h f(s)\int_{t-s}^t e^{-\alpha(u-t_0-h)}duds\to 0$ as $t\to+\infty$, $\tilde{S}>S^*$ and $1-2\beta\tilde{S}T>0$, we easily have that $V(t)\to+\infty$ as $t\to+\infty$ by (16), which contradicts to that $I(t)$ is bounded for $t\ge t_0$. This proves *Assertion* A.

Next, let us further show that the following *Assertion* B is also true.

Assertion B : *If the conditions* (i) *and* (ii) *hold, then any solution of* (1) *will eventually stay in* $\Omega_{\varepsilon,\tilde{S}}$.

In fact, if not, by *Assertion* A, there is some solution $(S(t), I(t), R(t))$ of (1) such that, for any positive constant $\tilde{S}_1$ satisfying $S^* < \tilde{S}_1 < \tilde{S} < b/(\mu_2 + \lambda)$, there are two time sequences $\{t_n\}$ and $\{t_n'\}$ with $t_n < t_n' < t_{n+1} < t_{n+1}'$, $t_n \to +\infty$ and $t_n' \to +\infty$, such that

$$S(t_n) = \tilde{S}_1, \quad S(t_n') = \tilde{S}, \quad \tilde{S}_1 \le S(t) \le \tilde{S} \quad \text{for } t_n \le t \le t_n', \tag{17}$$

and $\dot{S}(t_n') \ge 0$.

From (1), we have

$$\begin{aligned}\tilde{S} - \tilde{S}_1 &= S(t_n') - S(t_n) \\ &= -\beta \int_{t_n}^{t_n'} S(v) \int_0^h f(s) I(v-s)\,dv\,ds - \mu_1 \int_{t_n}^{t_n'} S(v)\,dv + b(t_n' - t_n),\end{aligned}$$

which , together with (17), yields

$$\begin{aligned}b(t_n' - t_n) &= \tilde{S} - \tilde{S}_1 + \beta \int_{t_n}^{t_n'} S(v) \int_0^h f(s) I(v-s)\,dv\,ds + \mu_1 \int_{t_n}^{t_n'} S(v)\,dv \\ &\ge \tilde{S} - \tilde{S}_1 + \mu_1 \tilde{S}_1 (t_n' - t_n).\end{aligned}$$

Thus,

$$t_n' - t_n \ge \frac{\tilde{S} - \tilde{S}_1}{b - \mu_1 \tilde{S}_1} \tag{18}$$

and

$$\frac{\tilde{S} - \tilde{S}_1}{b - \mu_1 \tilde{S}_1} \to \frac{\tilde{S} - S^*}{b - \mu_1 S^*} > h \quad \text{as} \quad \tilde{S}_1 \to S^* \tag{19}$$

by condition (i). From (1), we also have that for $t \ge t_0$,

$$\begin{aligned}\dot{S}(t) + \dot{I}(t) &= -\mu_1 S(t) - (\mu_2 + \lambda) I(t) + b \\ &\ge -(\mu_2 + \lambda)\left(S(t) + I(t)\right) + b,\end{aligned}$$

which, together with $\tilde{S} < b/(\mu_2 + \lambda)$, implies that, for any sufficiently small positive constant η, there is a large $t_1^* \geq t_0^*$ such that for $t \geq t_1^*$,

$$S(t) + I(t) \geq \frac{b}{\mu_2 + \lambda} - \eta \equiv N(\eta) > \tilde{S} \tag{20}$$

and

$$S^* + I^* = \frac{(\mu_2 + \lambda)(\mu_2 + \lambda - \mu_1) + b\beta}{\beta(\mu_2 + \lambda)} > N(\eta) > I^* = \frac{b\beta - \mu_1(\mu_2 + \lambda)}{\beta(\mu_2 + \lambda)}. \tag{21}$$

(20) and (21) show that the points $(\tilde{S}, 0, 0)$ and $(0, I^*, 0)$ are in the lower left hand side of plane $S + I = N(\eta)$, and the positive equilibrum (S^*, I^*, R^*) is in the upper right hand side of plane $S + I = N(\eta)$. We see that the planes $S = \tilde{S}$ and $S + I = N(\eta)$ intersect at $(\tilde{S}, N(\eta) - \tilde{S}, R)$ for any $R > 0$ (see Fig. 1).

Thus, it follows from (17), (18), (19) and (20) that, for large $t_n' \geq t_1^*$ and $\tilde{S}_1$ which is sufficiently close to S^*,

$$I(t_n' - s) \geq N(\eta) - \tilde{S} > 0, \qquad 0 \leq s \leq h. \tag{22}$$

(also see Fig.1). (22) and condition (ii) enable us to show that $\dot{S}(t_n') < 0$ which is a contradiction to that $\dot{S}(t_n') \geq 0$.

In fact, from (1) and (22), we have that

$$\begin{aligned} \dot{S}(t_n') &= -\beta S(t_n') \int_0^h f(s) I(t_n' - s) ds - \mu_1 S(t_n') + b \\ &= -\beta \tilde{S} \int_0^h f(s) I(t_n' - s) ds - \mu_1 \tilde{S} + b \\ &\leq -\beta \tilde{S} \left(N(\eta) - \tilde{S} \right) - \mu_1 \tilde{S} + b \\ &= -\tilde{S} \left[\beta \left(N(\eta) - \tilde{S} \right) + \mu_1 \right] + b \\ &\equiv G(\tilde{S}, \eta). \end{aligned} \tag{23}$$

By condition (ii), we see that

$$G(\tilde{S}, 0) = -\tilde{S} \left[\beta \left(N(0) - \tilde{S} \right) + \mu_1 \right] + b < 0. \tag{24}$$

Thus, it follows from (23), (24) and the continuity of $G(\tilde{S}, \eta)$ with respect to η that $\dot{S}(t_n') \leq G(\tilde{S}, \eta) < 0$ for sufficiently small $\eta > 0$. This proves our second assertion.

Now, by *Assertions A* and *B,* we can complete the proof of Theorem 2 by using the following Liapunov functional

$$\begin{aligned} V(t,S,I_t) &= S - S^* \ln \frac{S}{S^*} + \frac{\omega_1}{2}(S - S^* + I - I^*)^2 \\ &\quad + \omega_2 \int_0^h f(s) \int_{t-s}^t (I(u) - I^*)^2 \, du ds, \end{aligned}$$

where ω_1 and ω_2 are some positive constants chosen later and $(S(t), I(t), R(t))$ is any solution of (1).

By *Assertion B,* for $\tilde{S}$ satisfying $S^* < \tilde{S} < b/(\mu_2 + \lambda)$, there is a sufficiently large time $\hat{t} > t_0$ such that for $t \geq \hat{t}$,

$$S(t) \leq \tilde{S}. \tag{25}$$

The derivative $\dot{V}(t,S,I_t)$ of $V(t,S,I_t)$ along the solution of (1) satisfies

$$\begin{aligned} \dot{V}(t,S,I_t) &= -\delta \left[(S - S^*)^2 + (I - I^*)^2 \right] \\ &\quad - \frac{1}{2} \int_0^h f(s) \left[W(t,s) B(S(t)) W^T(t,s) \right] ds, \end{aligned} \tag{26}$$

for all $t \geq \hat{t}$, where δ is some positive constant chosen later,

$$B(S(t)) = \begin{bmatrix} 2\left(\omega_1\mu_1 - \delta + (\mu_1 + \beta I^*)/S(t)\right) & \omega_1(\mu_1 + \beta S^*) & \beta \\ \omega_1(\mu_1 + \beta S^*) & 2(\omega_1 \beta S^* - \omega_2 - \delta) & 0 \\ \beta & 0 & 2\omega_2 \end{bmatrix},$$

$$W(t,s) = (S(t) - S^*, I(t) - I^*, I(t-s) - I^*).$$

We can easily see that the symmetric matrix $B(S(t))$ is positive dominant diagonal for every $t \geq \hat{t}$, if

$$\frac{2(\mu_1 + \beta I^*)}{S(t)} - 4\delta - \beta > \omega_1(\beta S^* - \mu_1) - 2\delta > 2\omega_2 > \beta. \tag{27}$$

Let us choose δ small enough such that

$$0 < \delta < \frac{\beta}{2\tilde{S}} \left(\frac{b}{\mu_2 + \lambda} - \tilde{S} \right).$$

Then, for all $t \geq \hat{t}$,

$$\frac{2(\mu_1 + \beta I^*)}{S(t)} - 4\delta - \beta > \beta.$$

Thus, note that $\beta S^* - \mu_1 = \mu_2 + \lambda - \mu_1 > 0$, we can easily choose the positive constants ω_1, ω_2 and δ satisfying (27). Hence, it follows from (26) that for all $t \geq \hat{t}$,

$$\dot{V}(t, S, I_t) \leq -\delta \left[(S - S^*)^2 + (I - I^*)^2 \right],$$

from which we have that for all $t \geq \hat{t}$,

$$V(t, S, I_t) \leq V(\hat{t}, S(\hat{t}), I_{\hat{t}}) - \delta \int_{\hat{t}}^{t} \left[(S(u) - S^*)^2 + (I(u) - I^*)^2 \right] du.$$

Thus,

$$\int_{t_0}^{+\infty} (S(u) - S^*)^2 \, du < +\infty, \quad \int_{t_0}^{+\infty} (I(u) - I^*)^2 \, du < +\infty.$$

By (1), we see that $\frac{d}{dt}(S(t) - S^*)^2$ and $\frac{d}{dt}(I(t) - I^*)^2$ are also uniformly bounded for $t \geq t_0$. Thus, the well-known Barbălat's lemma (see [2]) shows that

$$(S(t) - S^*)^2 + (I(t) - I^*)^2 \to 0 \qquad \text{as } t \to +\infty. \tag{28}$$

That $R(t) - R^* \to 0$ as $t \to +\infty$ is an immediate result of (28) and the third equation of (1) (see Lemma 4 in [5]).

The proof of Theorem 2 is completed.

In the following, let us give a simpler and more practical criterion by Theorem 2.

By (23),

$$G(\tilde{S}, 0) \equiv b - \tilde{S} \left[\beta \left(\frac{b}{\mu_2 + \lambda} - \tilde{S} \right) + \mu_1 \right] = \beta \tilde{S}^2 - \left(\frac{b\beta}{\mu_2 + \lambda} + \mu_1 \right) \tilde{S} + b.$$

It is easy to see that equation $G(\tilde{S}, 0) = 0$ has two different positive real roots g_1 and g_2 ($g_1 < g_2$),

$$g_{1,2} = \frac{1}{2\beta} \left[\frac{b\beta}{\mu_2 + \lambda} + \mu_1 \mp \sqrt{\left(\frac{b\beta}{\mu_2 + \lambda} + \mu_1 \right)^2 - 4\beta b} \right],$$

if

(iii) $b\beta > (\lambda+\mu_2)^2\left(2-\frac{\mu_1}{\lambda+\mu_2}+2\sqrt{1-\frac{\mu_1}{\lambda+\mu_2}}\right).$

It is not difficult to see that condition (iii) is more restrictive than the necessary condition (3) for the existence of the endemic equilibrium E_+. Also note that (iii) ensures that $S^* < b/(\mu_2+\lambda)$.

Theorem 3. *Assume that condition* (iii) *and*

(iv) $h < \min\left\{(2\beta g_2)^{-1},\ (g_2 - S^*)/(b-\mu_1 S^*)\right\}$

are satisfied, then the endemic equilibrium E_+ is globally asymptotically stable.

Proof. By $G(S^*,0) = (\mu_2+\lambda)(\mu_2+\lambda-\mu_1)/\beta > 0$ and (iii), we see that $S^* < g_1$. We can also easily check that $g_2 < b/(\mu_2+\lambda)$. Thus, $S^* < g_1 < g_2 < b/(\mu_2+\lambda)$. Choose $\tilde{S}$ such that $S^* < g_1 < \tilde{S} < g_2$. Then, $G(\tilde{S},0) < G(g_2,0) = 0$, which together with condition (iv) of Theorem 3 shows that, while $\tilde{S}$ is sufficiently close to g_2, conditions (i) and (ii) of Theorem 2 can also be satisfied. This proves Theorem 3.

4 Conclusion

In this paper, we have considered the global asymptotic properties of the disease free equilibrium and the endemic equilibrium of the delay SIR epidemic model and obtained Theorems 1, 2 and 3. Theorem 1 shows that the disease free equilibrium is still globally attractive whenever $b/\mu_1 = S^* = (\mu_2+\lambda)/\beta$, i.e. the disease will eventually disappear. Theorems 2 and 3 give sufficient conditions to ensure the global asymptotic stability of the endemic equilibrium whenever it exists, i.e. the conditions that the disease always remains endemic. Based on Hethcote's analysis for the SIR epidemic model without delay and general properties for delay differential equations, it is natural to conjecture that *for sufficiently small delay h, condition* (3) *implies the global asymptotic stability of the endemic equilibrium, i.e. condition* (3) *should be the threshold of* (1) *for an epidemic* *to occur.* Unfortunately, we need more restrictive conditions (ii) and (iii) in Theorems 2 and 3 in order to ensure the global asymptotic stability of the endemic state. Our proofs suggest that, to complete the analysis on the above problem, we need to construct new Liapunov functionals and to give better estimate on the lower bound of $I(t)$ than one given by (22).

Figure legend

Fig. 1: The solution satisfies (22). Here $A = (K_\varepsilon, 0, 0)$, $B = (0, K_\varepsilon, 0)$, $C = (0, 0, K_\varepsilon)$, $P = (N(\eta), 0, 0)$, $P1 = (\tilde{S}, 0, 0)$, $W = (\tilde{S}, N(\eta) - \tilde{S}, 0)$, $Q = (S^*, 0, 0)$, $Q1 = (\tilde{S}_1, 0, 0)$, $E_+ = (S^*, I^*, R^*)$.

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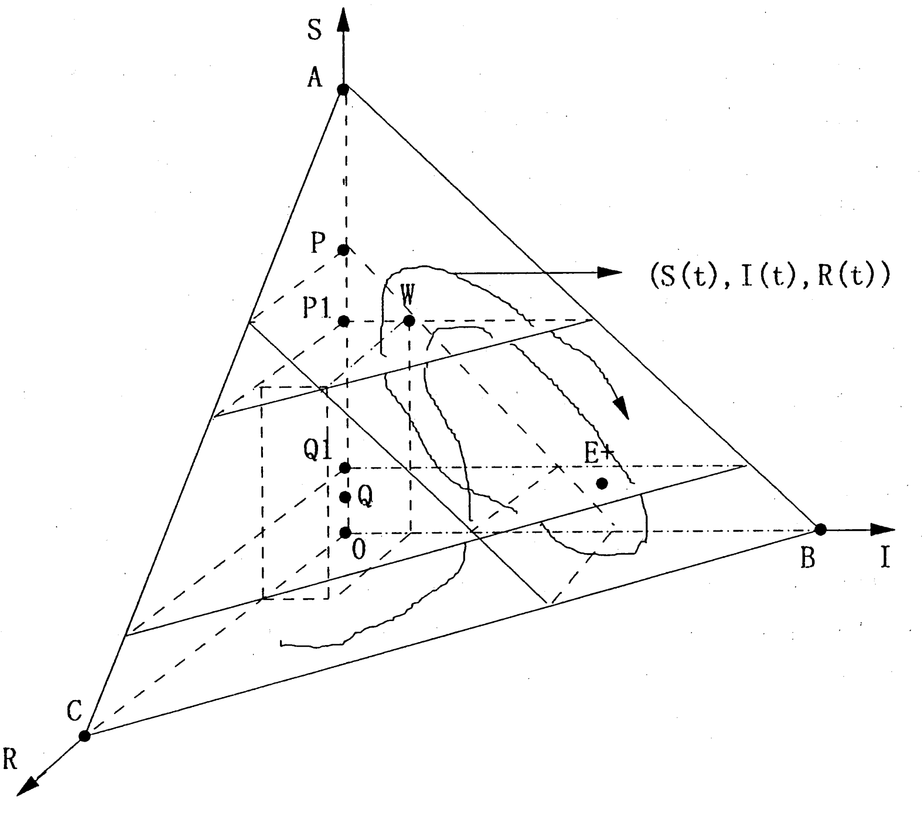

(Fig. 1)