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Characteristic equation and asymptotic behavior of 2-dimensional delay-differential equations

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ABSTRACT. Consider 2-dimensional delay-differential equations

$$\dot{x}(t) = A\int_{-r}^{0} x(t+s)d\eta(s),$$

where A is a 2×2 constant matrix, r is a positive constant, and $\eta\colon [-r,0] \to \mathbf{R}$ is monotone on $[-r,0]$ and continuous to the left on $(-r,0)$. The purpose of this work is to show that a necessary and sufficient condition under which the zero solution of (AL) is uniformly asymptotically stable can be obtained, if we impose a restriction on η as follows:

$$\eta(s)+\eta(-r-s)=\eta(0)+\eta(-r) \qquad \text{for a.e.} \quad s\in[-r,0].$$

The proof will be give by using the characteristic equation.

1. Main Results

Consider 2-dimensional equations

$$\dot{x}(t) = A\int_{-r}^{0} x(t+s)d\eta(s), \tag{AL}$$

where A is a 2×2 constant matrix, r is a positive constant, and $\eta\colon [-r,0] \to \mathbf{R}$ is monotone on $[-r,0]$ and continuous to the left on $(-r,0)$. Moreover, we assume that

$$\eta(s)+\eta(-r-s)=\eta(0)+\eta(-r) \qquad \text{for a.e.} \quad s\in[-r,0]. \tag{H1}$$

In [5] we have discussed 1-dimensional case:

Theorem A. *Suppose* (H1) *holds. The zero solution of a scalar equation*

$$\dot{x}(t) = -\int_{-r}^{0} x(t+s)d\eta(s) \tag{1.1}$$

is uniformly asymptoticaly stable if and only if $\eta(0) > \eta(-r)$ *and*

$$\int_{-r}^{0} \sin\left(-\frac{s}{r}\pi\right) d\eta(s) < \frac{\pi}{r}.$$

In [1] we also have shown the following:

Theorem B. *The zero solution of 2-dimensional equations*

$$\dot{x}(t) = -a \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \sum_{k=1}^{N} x(t-\tau_k),$$

where τ_k *is an arithmetic sequence, that is,* $\tau_k = \tau + (k-1)l$ *with* $\tau \geq 0$ *and* $l > 0$ *for* $k = 1, 2, \ldots, N$, $\tau_N > 0$ *and* $|\theta| < \frac{\pi}{2}$, *is uniformly asymptoticaly stable if and only if*

$$a > 0$$

and

$$\frac{a(\tau_1+\tau_N)}{2} \frac{\sin\left(\dfrac{Nl}{\tau_1+\tau_N}\left(\dfrac{\pi}{2}-|\theta|\right)\right)}{\sin\left(\dfrac{l}{\tau_1+\tau_N}\left(\dfrac{\pi}{2}-|\theta|\right)\right)} < \frac{\pi}{2} - |\theta|.$$

By using the ideas of the proofs of Theorems A and B, we will give more extended results (Theorems 1.1 and 1.2.)

By the transfomation $x(t) = Py(t)$ with an appropriate regular matrix P, we can rewrite (AL) as

$$\dot{y}(t) = P^{-1}AP\int_{-r}^{0} y(t+s)d\eta(s).$$

Consequently, we only have to consider the equations (AL) where the matrix A is either of the following two matrices:

(I) in the case matrix A has real eigenvalues,

$$A = -\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}$$

where a_1, a_2 and b are real numbers;

(II) in the case matrix A has complex eigenvalues,

$$A = -R(\theta) = -\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $|\theta| \leq \frac{\pi}{2}$.

For the case (I), we have

Theorem 1.1. *Suppose* (H1) *holds. The zero solution of* (AL) *is uniformly asymptotically stable if and only if* $\eta(0) > \eta(-r)$,

$$a_i > 0 \qquad \textit{and} \qquad a_i \int_{-r}^{0} \sin(-\frac{s}{r}\pi) d\eta(s) < \frac{\pi}{r} \qquad i = 1, 2.$$

For the case (II), we have

Theorem 1.2. *Suppose* (H1) *holds. The zero solution of* (AL) *is uniformly asymptotically stable if and only if* $\eta(0) > \eta(-r)$ *and*

$$\int_{-r}^{0} \cos\left\{\frac{r+2s}{r}\left(\frac{\pi}{2} - |\theta|\right)\right\} d\eta(s) < \frac{\pi - 2|\theta|}{r}. \tag{1.2}$$

Remark 1.1. If θ=0, Theorem 1.2 coincides with Theorem A. We state in the following section that Theorem B is included by Theorem 1.2.

Remark 1.2. In the case A is an n×n matrix, we can obtain the necessary and sufficient condition for the uniform asymptotic stability of the zero solution of (AL) by applying Theorems 1.1, 1.2 and theory of Jordan canonical form. (*cf.* [2, Theorem 3.4]).

The proofs of Theorems 1.1 and 1.2 are very similar, so that we give only the proof of Theorem 1.2.

We prepare a lemma to prove the theorem.

Lemma 1.1. *For any integer n, if $|\theta| \le \frac{\pi}{2}$ and $|\alpha| \le 1$, then the following inequality holds:*

$$\left|\cos\left(\frac{(2n+1)\pi+2\theta}{2}\alpha\right)\right| < |2n+1|\cos\left\{\left(\frac{\pi}{2}-|\theta|\right)|\alpha|\right\}$$

Proof of Lemma 1.1. First of all we note that $|\sin k\phi| \le |k||\sin\theta|$ for any integer k and $\phi \in \mathbf{R}$. Then we have

$$\begin{aligned}|\cos\{(2k+1)\phi\}| &= \left|\sin\left\{(2k+1)\left(\frac{\pi}{2}-\phi\right)\right\}\right| \\ &\le |2k+1|\left|\sin\left(\frac{\pi}{2}-\phi\right)\right| \\ &= |2k+1||\cos\phi|.\end{aligned}$$

Using these relations, we have

$$\begin{aligned}&\left|\cos\left(\frac{(2n+1)\pi+2\theta}{2}\alpha\right)\right| \\ =&\left|\cos\left\{(2n+1)\frac{\pi}{2}\alpha\right\}\cos(\theta\alpha)-\sin\left\{(2n+1)\frac{\pi}{2}\alpha\right\}\sin(\theta\alpha)\right| \\ \le&|2n+1|\left|\cos\left(\frac{\pi}{2}\alpha\right)\right||\cos(\theta\alpha)|+|2n+1|\left|\sin\left(\frac{\pi}{2}\alpha\right)\right||\sin(\theta\alpha)| \\ =&|2n+1|\left\{\cos\left(\frac{\pi}{2}|\alpha|\right)\cos(|\theta||\alpha|)+\sin\left(\frac{\pi}{2}|\alpha|\right)\sin(|\theta||\alpha||)\right\} \\ =&|2n+1|\cos\left\{\left(\frac{\pi}{2}-|\theta|\right)|\alpha|\right\}.\end{aligned}$$

This completes the proof. □

Lemma 1.2. *Suppose* (H1) *holds. If $f\colon [-r,0] \to \mathbf{R}$ is continuous, then*

$$\int_{-r}^{0} f(s)d\eta(s) = -\int_{-r}^{0} f(s)d\eta(-r-s).$$

Proof of Lemma 1.2. Define $\tilde{\eta}\colon [-r,0] \to \mathbf{R}$ by $\tilde{\eta}(s) = \eta(-r-s)$. Then $\tilde{\eta}$ is monotone on $[-r,0]$ and f is Riemann-Stieltjes integrable with respect to $\tilde{\eta}$. By the assumption (H1), for any positive integer n there exist $t_k \in (-r, 0)$ such that $-r(1-\frac{k-1}{n}) < t_k < -r(1-\frac{k}{n})$ and $\eta(t_k) = -\tilde{\eta}(t_k) + \eta(0) + \eta(-r)$ for $k = 1, 2, \cdots, n$. For a partition D_n: $-r = t_0 < t_1 < \cdots < t_n < t_{n+1} = 0$ of $[-r, 0]$ and any choice of $\xi_k \in [t_{k-1}, t_k]$, we consider the Riemann sum $S(f; \eta; D_n; \xi)$. Then we have

$$\begin{aligned} S(f;\eta;D_n;\xi) &= \sum_{k=1}^{n+1} f(\xi_k)[\eta(t_k) - \eta(t_{k-1})] \\ &= -\sum_{k=1}^{n+1} f(\xi_k)[\tilde{\eta}(t_k) - \tilde{\eta}(t_{k-1})] = -S(f;\tilde{\eta};D_n;\xi). \end{aligned}$$

Noticing that $d(D_n) = \max_{1\le k\le n+1}(t_k - t_{k-1}) < \frac{2r}{n}$, we have

$$\int_{-r}^{0} f(s)d\eta(s) = -\int_{-r}^{0} f(s)d\tilde{\eta}(s)$$

as $n \to \infty$. This completes the proof. □

Proof of Theorem 1.2. The characteristic equation of (AL) is the following form;

$$\Delta(\lambda) = \det\left[\lambda I + R(\theta)\int_{-r}^{0} e^{\lambda s} d\eta(s)\right] = 0, \tag{1.3}$$

where I is the 2×2 identity matrix. We use the fact that the zero solution of (AL) is uniformly asymptotically stable if and only if all the roots of the characteristic equation (1.3) lie in the left half of the complex plane, that is, the real part of every characteristic root λ of (1.3) is negative.

Let

$$p^{\pm}(\lambda;\mu) = \lambda + \rho e^{\pm i\theta}\int_{-r}^{0} e^{\mu\lambda s} d\eta(s)$$

for a parameter $\mu \in [0, 1]$. Then the characteristic equation of (1.3) can be expressed in the following form:

$$\begin{aligned} \Delta(\lambda) &= \det\begin{bmatrix} \lambda + \cos\theta\int_{-r}^{0} e^{\lambda s}d\eta(s) & -\sin\theta\int_{-r}^{0} e^{\lambda s}d\eta(s) \\ \sin\theta\int_{-r}^{0} e^{\lambda s}d\eta(s) & \lambda + \cos\theta\int_{-r}^{0} e^{\lambda s}d\eta(s) \end{bmatrix} \\ &= \left(\lambda + \cos\theta\int_{-r}^{0} e^{\lambda s}d\eta(s)\right)^2 + \left(\sin\theta\int_{-r}^{0} e^{\lambda s}d\eta(s)\right)^2 \\ &= \left(\lambda + e^{i\theta}\int_{-r}^{0} e^{\lambda s}d\eta(s)\right)\left(\lambda + e^{-i\theta}\int_{-r}^{0} e^{\lambda s}d\eta(s)\right) \end{aligned}$$

$$= p^+(\lambda; 1)p^-(\lambda; 1) = 0.$$

Now we consider the distribution of the zeros of $p^+(\lambda;\mu)$ and $p^-(\lambda;\mu)$ in the complex plane.

(Sufficiency) If $\mu = 0$, then $\lambda = -e^{\pm i\theta}\{\eta(0) - \eta(-r)\}$. By (1.2) and $\eta(0) > \eta(-r)$, $|\theta| < \frac{\pi}{2}$. Then we have $\mathrm{Re}\lambda = -\cos(\pm\theta)\{\eta(0) - \eta(-r)\} < 0$. Noticing that each branch of λ is continuous in the parameter μ, we only show that there is no zeros on the imaginary axis for any $\mu \in (0, 1]$. If $\Delta(i\omega) = 0$ for an $\omega \in \mathbf{R}$, we have $p^+(i\omega;\mu) = 0$ or $p^-(i\omega;\mu) = 0$. When $\bar{\lambda}$ is the complex conjugate of any complex λ, the relation

$$p^+(\lambda;\mu) = \overline{p^-(\bar{\lambda};\mu)} \tag{1.4}$$

stands and $p^-(i\omega;\mu) = 0$ implies $p^+(-i\omega;\mu) = 0$. Thus, we only have to consider the case $p^+(i\omega;\mu) = 0$. Calculating $p^+(i\omega : \mu)$, we have

$$\begin{aligned} p^+(i\omega;\mu) &= i\omega + e^{i\theta}\int_{-r}^{0} e^{i\mu\omega s}d\eta(s) \\ &= \int_{-r}^{0} \cos(\theta + \mu\omega s)d\eta(s) + i\left(\omega + \int_{-r}^{0} \sin(\theta + \mu\omega s)d\eta(s)\right). \end{aligned}$$

Therefore we have

$$\int_{-r}^{0} \cos(\theta + \mu\omega s)d\eta(s) = 0 \qquad \text{and} \qquad \omega = -\int_{-r}^{0} \sin(\theta + \mu\omega s)d\eta(s).$$

Using the assumption (H1) and Lemma 1.2, we have

$$\begin{aligned} \int_{-r}^{0} \cos(\theta + \mu\omega s)d\eta(s) &= \frac{1}{2}\left[\int_{-r}^{0} \cos(\theta + \mu\omega s)d\eta(s) + \int_{0}^{-r} \cos\{\theta + \mu\omega(-r - s)\}\, d\eta(-r - s)\right] \\ &= \int_{-r}^{0} \frac{\cos(\theta + \mu\omega s) + \cos\{\theta + \mu\omega(-r - s)\}}{2} d\eta(s) \\ &= \cos\frac{2\theta - \mu\omega r}{2}\int_{-r}^{0} \cos\frac{\mu\omega(r + 2s)}{2} d\eta(s) \end{aligned}$$

and

$$\begin{aligned} -\int_{-r}^{0} \sin(\theta + \mu\omega s)d\eta(s) &= -\int_{-r}^{0} \frac{\sin(\theta + \mu\omega s) + \sin\{\theta + \mu\omega(-r - s)\}}{2} d\eta(s) \\ &= -\sin\frac{2\theta - \mu\omega r}{2}\int_{-r}^{0} \cos\frac{\mu\omega(r + 2s)}{2} d\eta(s). \end{aligned}$$

Therefore we have

$$\cos\frac{2\theta-\mu\omega r}{2}\int_{-r}^{0}\cos\frac{\mu\omega(r+2s)}{2}d\eta(s)=0 \tag{1.5}$$

and

$$\omega=-\sin\frac{2\theta-\mu\omega r}{2}\int_{-r}^{0}\cos\frac{\mu\omega(r+2s)}{2}d\eta(s). \tag{1.6}$$

By (1.5) and $|\theta|<\frac{\pi}{2}$, we get $\int_{-r}^{0}d\eta(s)=0$ for $\omega=0$. This contradicts the inequality $\eta(0)>\eta(-r)$. Then $\omega\neq 0$, and hence $\int_{-r}^{0}\cos\frac{\mu\omega(r+2s)}{2}d\eta(s)\neq 0$ by (1.6). Therefore we obtain $\cos\frac{2\theta-\mu\omega r}{2}=0$ from (1.5), that is,

$$\omega=\frac{(2n+1)\pi+2\theta}{\mu r}\qquad\text{for some integer } n.$$

Substituting the above ω in (1.6), we have

$$\omega=\frac{(2n+1)\pi+2\theta}{\mu r}=(-1)^n\int_{-r}^{0}\cos\left\{\frac{(2n+1)\pi+2\theta}{2}\frac{r+2s}{r}\right\}d\eta(s). \tag{1.7}$$

From Lemma 1.1 and $0<\mu\leq 1$, we obtain

$$\begin{aligned}\left|\frac{(2n+1)\pi+2\theta}{r}\right| &\leq \mu\int_{-r}^{0}\left|\cos\left\{\frac{(2n+1)\pi+2\theta}{2}\frac{r+2s}{r}\right\}\right|d\eta(s)\\ &\leq |2n+1|\int_{-r}^{0}\cos\left\{\left(\frac{\pi}{2}-|\theta|\right)\frac{r+2s}{r}\right\}d\eta(s)\\ &<|2n+1|\frac{\pi-2|\theta|}{r}<\frac{|2n+1|\pi-2|\theta|}{r},\end{aligned}$$

where we used (1.2). This is a contradiction. Therefore there is no characteristic root on the imaginary axis when $\mu\in(0,1]$.

(Necessity) If $\eta(0)=\eta(-r)$, then $\eta(s)\equiv 0$ for $s\in[-r,0]$. In this case, for any solution $x(t_0,\phi)$ of (AL), $x(t_0,\phi)(t)\equiv\phi(0)$ for all $t\geq t_0$. This is a contradiction and we obtain $\eta(0)\neq\eta(-r)$.

Suppose that $\eta(0)<\eta(-r)$ or

$$\int_{-r}^{0}\cos\left\{\frac{r+2s}{r}\left(\frac{\pi}{2}-|\theta|\right)\right\}d\eta(s)\geq\frac{\pi-2|\theta|}{r}. \tag{1.8}$$

Claim 1. There exist a $\mu_0\in[0,1]$ and a $\lambda_0\in\mathbf{C}$ such that $p^{+}(\lambda_0;\mu_0)=0$ and $\mathrm{Re}\lambda_0\geq 0$.

Proof of Claim 1. If $|\theta| = \frac{\pi}{2}$, then $\mu_0 = 0$ and $\lambda_0 = -i(\sin\theta)\{\eta(0) - \eta(-r)\}$. If $|\theta| < \frac{\pi}{2}$ and $\eta(0) < \eta(-r)$, then $\mu_0 = 0$ and $\lambda_0 = -e^{i\theta}\{\eta(0) - \eta(-r)\}$. If $|\theta| < \frac{\pi}{2}$ and $\eta(0) > \eta(-r)$, then there exists a $\mu_0 \in (0,1]$ such that

$$\mu_0 \int_{-r}^{0} \cos\left\{\frac{r+2s}{r}\left(\frac{\pi}{2} - |\theta|\right)\right\} d\eta(s) = \frac{\pi - 2|\theta|}{r}$$

by (1.8). This yeilds that $\lambda_0 = i\frac{\pi - 2|\theta|}{\mu_0 r}$.

Claim 2. Assume that there exist an $\omega \in \mathbf{R}$ and a $\tilde{\mu} \in [0,1]$ such that $p^+(i\omega;\tilde{\mu}) = 0$. Consider the zero of $p^+(\lambda;\mu)$ with parameter $\mu \in [0,1]$ and let λ be a branch of the zero through $\lambda = i\omega$ at $\mu = \tilde{\mu}$. Then $\mathrm{Re}(\frac{\partial\lambda}{\partial\mu}) > 0$ when $\lambda = i\omega$ and $\mu = \tilde{\mu}$.

Proof of Claim 2. Taking the partial derivative of λ with μ on $p^+(\lambda : \mu) = 0$, we obtain

$$\frac{\partial\lambda}{\partial\mu} = \frac{-\lambda e^{i\theta}\displaystyle\int_{-r}^{0} s e^{\mu\lambda s} d\eta(s)}{1 + \mu e^{i\theta}\displaystyle\int_{-r}^{0} s e^{\mu\lambda s} d\eta(s)}$$

If $\tilde{\mu} = 0$ and $|\theta| < \frac{\pi}{2}$, there is no $\omega \in \mathbf{R}$ such that $p^+(i\omega;0) = 0$. If $\tilde{\mu} = 0$ and $|\theta| = \frac{\pi}{2}$, then $\omega = -\sin\theta\{\eta(0) - \eta(-r)\}$ and

$$\begin{aligned}\mathrm{Re}\left(\frac{\partial\lambda}{\partial\mu}\right)\bigg|_{\lambda=i\omega} &= \mathrm{Re}\left\{-i\omega e^{i\theta}\int_{-r}^{0} s e^{\mu\lambda s} d\eta(s)\right\} \\ &= \omega\sin\theta\int_{-r}^{0} s\,d\eta(s) = \frac{r\omega^2}{2} > 0\end{aligned}$$

If $\tilde{\mu} \in (0,1]$, we have $\omega = \frac{(2n+1)\pi + 2\theta}{\tilde{\mu} r}$ for some integer n from (1.7). Letting $L_c = \int_{-r}^{0} s\cos(\theta + \tilde{\mu}\omega s)\,d\eta(s)$ and $L_s = \int_{-r}^{0} s\sin(\theta + \tilde{\mu}\omega s)\,d\eta(s)$. Noticing that $\sin\{\theta + \tilde{\mu}\omega(-r - s)\} = \sin(\theta + \tilde{\mu}\omega s)$ and the assumption (H1), we have

$$\begin{aligned}L_s &= \int_{-r}^{0} s\sin(\theta + \tilde{\mu}\omega s)\,d\eta(s) + \int_{0}^{-r} (-r - s)\sin\{\theta + \tilde{\mu}\omega(-r - s)\}\,d\eta(-r - s) \\ &= -\frac{r}{2}\int_{-r}^{0} \sin(\theta + \tilde{\mu}\omega s)\,d\eta(s) = \frac{r\omega}{2},\end{aligned}$$

where we used the equality $\omega = -\int_{-r}^{0} \sin(\theta + \tilde{\mu}\omega s) d\eta(s)$ obtained from $\mathrm{Im}p^+(i\omega;\tilde{\mu}) = 0$. Therefore we obtain

$$\mathrm{Re}\left(\frac{\partial\lambda}{\partial\mu}\right)\bigg|_{\lambda=i\omega} = \mathrm{Re}\left\{\frac{-i\omega(L_c + iL_s)}{1 + \tilde{\mu}(L_c + iL_s)}\right\}$$

$$= \mathrm{Re}\left[\frac{-i\omega\left\{L_c(1+\tilde{\mu}L_c)+\tilde{\mu}L_s^2\right\}+\omega L_s}{(1+\tilde{\mu}L_c)^2+(\tilde{\mu}L_s)^2}\right]$$

$$= \frac{\frac{\omega^2 r}{2}}{(1+\tilde{\mu}L_c)^2+(\tilde{\mu}L_s)^2} > 0.$$

Claims 1 and 2 yield that the branch of the zero of $p^+(\lambda;\mu)$ through the point $\lambda = \lambda_0$ at $\mu = \mu_0$ continues to lie in the right half of the complex plane for $\mu \in (\mu_0, 1]$. So the zero solution of (AL) is not uniformly asymptotically stable. This is a contradiction and the proof is completed. □

2. Applications

We will give some applications of Theorems 1.1 and 1.2.

Example 2.1. Consider 2-dimensional delaly differential equations with N delays

$$\dot{x}(t) = -R(\theta)\sum_{k=1}^{N} a_k x(t-\tau_k), \tag{2.1}$$

where $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $|\theta| \le \frac{\pi}{2}$. Suppose $\tau_k = \tau + (k-1)l$ and $a_{N-k+1} = a_k$ for $k = 1, 2, \cdots, N$. Here $\tau \ge 0$ and $l > 0$ are constants. We also suppose that $a_i a_j \ge 0$ for $i, j = 1, 2, \cdots, N$ and $\tau_N > 0$. Let $r = \tau_N + \tau_1$ and

$$\eta(s) = \sum_{k=1}^{N} e_k(s), \qquad e_k(s) = \begin{cases} 0 & s \in [-r, -\tau_k], \\ a_k & s \in (-\tau_k, 0]. \end{cases}$$

Then the assumption (H1) holds, and Theorem 1.2 is applicable. Let us compute the formula in the condition (1.2).

$$\int_{-r}^{0} \cos\left\{\frac{r+2s}{r}\left(\frac{\pi}{2}-|\theta|\right)\right\} d\eta(s) = \sum_{k=1}^{N} a_k \cos\left\{\frac{r-2\tau_k}{r}\left(\frac{\pi}{2}-|\theta|\right)\right\}$$

Then we have the following:

Corollary 2.1. *The zero solution of* (2.1) *is uniformly asymptotically stable if and only if*

$$\sum_{k=1}^{N} a_k > 0 \quad \textit{and} \quad \sum_{k=1}^{N} a_k \cos\left\{\frac{\tau_{N-k+1} - \tau_k}{\tau_1 + \tau_N}\left(\frac{\pi}{2} - |\theta|\right)\right\} < \frac{\pi - 2|\theta|}{\tau_1 + \tau_N}.$$

Remark 2.1. Thorem B is given by putting $a_k = a$ in this corollary. Stépán [6, p. 65] and Kuang [4, p. 87] proved that the zero solution of the scalar delay differential equation with two delays

$$x'(t) = -ax(t - \tau_1) - ax(t - \tau_2),$$

where $a > 0,\ \tau_1, \tau_2 \geq 0,\ \tau_1 + \tau_2 > 0$, is uniformly asymptotically stable if and only if

$$2a(\tau_1 + \tau_2)\cos\left(\frac{\tau_1 - \tau_2}{\tau_1 + \tau_2}\frac{\pi}{2}\right) < \pi. \tag{2.2}$$

If $\theta = 0, a_k = a$ and $N = 2$, the condition in Corollary 2.1 coincides with (2.2). The proof of Theorem 1.2 is given by generalizing the proof given by Kuang.

Example 2.2. Consider a scalar integro-differential equation

$$\dot{x}(t) = -\int_{t-r}^{t} c(t - s)x(s)ds, \tag{2.3}$$

where $c\colon [0, r] \to [0, \infty)$ is continuous satisfying $c(s) = c(r - s)$ and r is a positive constant. If we choose $\eta(s) = \int_0^s c(-\xi)d\xi$ for $s \in [-r, 0]$, then the assumption (H1) holds. Applying Theorem 1.2 for $\theta = 0$, we have

Corollary 2.2. *The zero solution of* (2.3) *is uniformly asymptotically stable if and only if*

$$0 < \int_0^r c(s)\sin\left(\frac{s}{r}\pi\right)ds < \frac{\pi}{r}. \tag{2.4}$$

Remark 2.2. Krisztin [3] gives the following excellent sufficient condition as far as the author knows: If

$$0 < \int_{-r}^{0} |s| d\eta(s) < \frac{\pi}{2}, \tag{2.5}$$

then the zero solution of (1.1) is asymptotically stable. In fact, it becomes a necessary and sufficient condition in case $N = 1$ and $\theta = 0$ in Example 2.1. However, let $c(s) = 1$ for $s \in [0, r]$ in (2.3), then conditions (2.4) and (2.5) are reduced to $0 < r < \pi/\sqrt{2} = 2.221\ldots$ and $0 < r < \sqrt{\pi} = 1.772\ldots$, respectively. This gap suggests us it should be difficult to obtain an explicit necessary and sufficient condition ensuring the uniform asymptotic stability of the zero solution of (AL) without some restriction on η.

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