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Multiobjective Programming of Set Functions[1]

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Abstract

Pareto optimality conditions in multiobjective programming with subdifferentiable set functions are established. We define a generalized $(\Im^*, \rho, \theta)$-convex and prove that an $(\Im^*, \rho, \theta)$-convex set functions is a convex set function. We discuss the Wolfe-type and Mond-Weir-type duality, and establish the weak-duality and strong-duality theorems for the two types of duality models.

1. INTRODUCTION

There are many types of functions. For instance functions of point to point; point to set; point to vector or set to point; set to vector; set to set

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etc. In this talk we will discuss about set functions in some programming problems. Throughout the paper, we consider an atomless finite measure space (X, Γ, μ) with seperable $L^1(X, \Gamma, \mu)$ space. For each measurable set $\Omega \in \Gamma$, it corresponds a characteristic function $\chi_\Omega \in L^\infty(X, \Gamma, \mu) = L^1(X, \Gamma, \mu)^*$, and so for any $f \in L^1(X, \Gamma, \mu)$, the dual pair is represented by

$$\langle f, \chi_\Omega \rangle = \int_X f(x)\chi_\Omega(x) d\mu(x) = \int_\Omega f(x) d\mu(x).$$

Since $\mu(x) < \infty$,

$$L^\infty(X, \gamma, \mu) \subset L^1(X, \Gamma, \mu).$$

Like functions defined on linear space, we will define both the convex family of measurable sets and convex set functions, and investigate the optimality conditions of the multiobjective programming with set functions. Formally, we give the programming problem with set functions as follows:

$$\begin{array}{lll} (P) & \text{Minimize} & F(\Omega) \\ & \text{subject to} & \Omega \in \mathcal{S} \subset \Gamma \quad \text{and} \\ & & G(\Omega) \leq \theta \end{array}$$

where $F : \Gamma \mapsto \mathbb{R}^n$ and $G : \Gamma \mapsto \mathbb{R}^m$ are convex set functions and $\mathcal{S}$ is a convex family of measurable subsets of X. Then under suitable conditions, Lai and Lin [5, Theorem 12] established the necessary optimality condition for problem (P). In this paper, we define a generalized $(\Im, \rho, \theta)$-convex and $(\Im^*, \rho, \theta)$-convex, and proved that every $(\Im^*, \rho, \theta)$-convex set function is a convex set function. This is a key theorem to establish a sufficient optimality conditions for (P). We are also state Wolfe-type duality model and Mond-Weir type duality model, and establish the weak and strong duality theorems for the above two duality models.

2. Preliminaries and Definitions

Let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable. Then we can find a countable L^1-dense subset of elements in $L^\infty(X, \Gamma, \mu)$. It follows that for every $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0,1]$, there is a Morris sequence $\{V_n\} = \{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ with properties as follows:

$$\chi_{\Omega_n} \longrightarrow^{w^*} \lambda \chi_{\Omega \backslash \Lambda} \quad \text{and} \quad \chi_{\Lambda_n} \longrightarrow^{w^*} (1-\lambda)\chi_{\Lambda \backslash \Omega} \tag{2.1}$$

imply

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \longrightarrow^{w^*} \lambda \chi_\Omega + (1-\lambda)\chi_\Lambda, \tag{2.2}$$

where $\longrightarrow^{w^*}$ denotes the weak* convergence of elements in $L_\infty(X, \Gamma, \mu)$.

We need the following definitions like the concept of functions defined in linear space.

Definition 2.1. *[5]. A* **subfamily** $\mathcal{S}$ *of* Γ *is called* **convex** *if for any* $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0,1]$ *associated with a Morris sequence* $\{V_n\}$ *in* Γ, *there exists a subsequence* $\{V_{n_k}\}$ *such that*

$$V_{n_k} = \Omega_{n_k} \cup \Lambda_{n_k} \cup \{\Omega \cap \Lambda\} \in \mathcal{S} \qquad \textit{for all} \quad k. \tag{2.3}$$

Definition 2.2. *[5]. A* **set function** $F : \mathcal{S} \mapsto \mathbb{R}$ *is called* **convex** *on a convex subfamily* $\mathcal{S} \subset \Gamma$ *if for any* $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0,1]$, *there exists a Morris sequence* $\{V_n\}$ *in* $\mathcal{S}$ *such that*

$$\lim_{n \to \infty} \sup F(V_n) \leqq \lambda F(\Omega) + (1-\lambda) F(\Lambda). \tag{2.4}$$

Definition 2.3. *[3]. An element $f \in L_1(X,\Gamma,\mu)$ is called a* **subgradient** *of a set function $F : \Gamma \mapsto \mathbb{R}$ at Ω_0 if it satisfies the inequality*

$$F(\Omega) \geqq F(\Omega_0) + \langle \chi_\Omega - \chi_{\Omega_0}, f\rangle \qquad \text{for all} \quad \Omega \in \Gamma. \tag{2.5}$$

The set of all subgradients f of a set function F at Ω_0 is denoted by $\partial F(\Omega_0)$ and is called the **subdifferential** *of F at Ω_0. If $\partial F(\Omega_0) \neq \emptyset$, F is called* **subdifferentiable** *at Ω_0.*

Remark 2.1. *Every convex real-valued set function is subdifferentiable but the converse is not true.*

Definition 2.4. *[5]. A set function $F : \Gamma \mapsto \mathbb{R} \cup \{\infty\}$ with*

$$DomF = \{\Omega \in \Gamma | F(\Omega) \text{ is finite}\} = \mathcal{S}, \tag{2.6}$$

is called **w^*-lower (-upper) semicontinuous** *at $\Omega \in \mathcal{S}$ if*

$$-\infty < F(\Omega) \leqq \lim_{n\to\infty} \inf F(\Omega_n) \tag{2.7}$$

$$(\lim_{n\to\infty} \sup F(\Omega_n) \leqq F(\Omega) < \infty)$$

for any sequence $\{\Omega_n\} \subset \mathcal{S}$ with $\chi_{\Omega_n} \to^{w^*} \chi_\Omega$.

The function F is said to be **w^*-continuous** *at Ω if*

$$F(\Omega) = \lim_{n\to\infty} F(\Omega_n) \tag{2.8}$$

for any sequence $\{\Omega_n\} \subset \mathcal{S}$ with $\chi_{\Omega_n} \to^{w^*} \chi_\Omega$.

We will use the convention that $F(\emptyset) = 0$ and denote the weak*-closure of $\mathcal{S}$ by $\bar{\mathcal{S}}$ throughout. A set function $F : \Gamma \mapsto \mathbb{R} \cup \{\infty\}$ is said to be **proper** if $F \neq \infty$ on Γ.

Definition 2.5. *A functional* $\Im$ *on* $\Gamma \times \Gamma \times L_1(X, \Gamma, \mu)$ *is said to be* **sublinear** *with respect to its third argument if for any* $\Omega, \Omega_0 \in \Gamma$,

$$\Im(\Omega, \Omega_0; f_1 + f_2) \leqq \Im(\Omega, \Omega_0; f_1) + \Im(\Omega, \Omega_0; f_2) \tag{2.9}$$

for any $f_1, f_2 \in L_1(X, \Gamma, \mu)$, *and*

$$\Im(\Omega, \Omega_0; \alpha f) = \alpha \Im(\Omega, \Omega_0; f) \tag{2.10}$$

for any $\alpha \in \mathbb{R}, \alpha \geqq 0$, *and* $f \in L_1(X, \Gamma, \mu)$.

Now, we consider the notion of generalized $(\Im, \rho, \theta)$-convexity, an extension of generalized $(\Im, \rho)$-convexity defined by Preda [10], for nondifferentiable set functions. Let us consider a sublinear functional $\Im$: $\Gamma \times \Gamma \times L_1(X, \Gamma, \mu) \mapsto \mathbb{R}$ and a set function $F : \Gamma \mapsto \mathbb{R}$. Let $\rho \in \mathbb{R}$ and $\theta : \Gamma \times \Gamma \mapsto \mathbb{R}_+ \equiv [0, \infty)$ such that $\theta(\Omega, \Omega_0) \neq 0$ if $\Omega \neq \Omega_0$. Throughout the paper we assume that the set functions are subdifferentiable. The following definitions are essential in the paper.

Definition 2.6.

(1) *The function* F *is said to be* $(\Im, \rho, \theta)$-**convex** *at* Ω_0 *if for each* $\Omega \in \Gamma$ *and* $f \in \partial F(\Omega_0)$, *we have*

$$F(\Omega) - F(\Omega_0) \geqq \Im(\Omega, \Omega_0; f) + \rho\theta(\Omega, \Omega_0). \tag{2.11}$$

(2) *The function* F *is said to be* $(\Im, \rho, \theta)$-**quasiconvex** *at* Ω_0 *if for each* $\Omega \in \Gamma$ *and* $f \in \partial F(\Omega_0)$,

$$F(\Omega) \leqq F(\Omega_0) \quad \textit{implies} \quad \Im(\Omega, \Omega_0; f) \leqq -\rho\theta(\Omega, \Omega_0). \tag{2.12}$$

(3) *The function* F *is said to be* **Ponstein** $(\Im, \rho, \theta)$-**quasiconvex** *at* Ω_0 *(cf. [12]) if for each* $\Omega \in \Gamma$ *and* $f \in \partial F(\Omega_0)$,

$$F(\Omega) < F(\Omega_0) \quad \textit{implies} \quad \Im(\Omega, \Omega_0; f) \leqq -\rho\theta(\Omega, \Omega_0). \tag{2.13}$$

(4) *The function F is said to be* $(\Im, \rho, \theta)$**-pseudoconvex** *at Ω_0 if for each $\Omega \in \Gamma$ and $f \in \partial F(\Omega_0)$,*

$$\Im(\Omega, \Omega_0; f) \geqq -\rho\theta(\Omega, \Omega_0) \quad \textit{implies} \quad F(\Omega) \geqq F(\Omega_0). \tag{2.14}$$

(5) *The function F is said to be* **strictly** $(\Im, \rho, \theta)$**-pseudoconvex** *at Ω_0 if for each $\Omega \in \Gamma$ and $f \in \partial F(\Omega_0)$,*

$$\Im(\Omega, \Omega_0; f) \geqq -\rho\theta(\Omega, \Omega_0) \quad \textit{implies} \quad F(\Omega) > F(\Omega_0). \tag{2.15}$$

Definition 2.7. *In Definition 2.6, if $\rho \geq 0$ and the functional $\Im : \Gamma \times \Gamma \times L_1(X, \Gamma, \mu) \mapsto \mathbb{R}$ is taken by a special case:*

$$\Im(\Omega, \Omega_0; f) = \langle \chi_\Omega - \chi_{\Omega_0}, f \rangle,$$

then $(\Im, \rho, \theta)$-convex is called $(\Im^*, \rho, \theta)$-convex.

Remark 2.2. *From the Definition 2.6, it is easy to see that the following implications hold:*

(a) *(1) $\Rightarrow$ (2) $\Rightarrow$ (3),*
(b) *(1) $\Rightarrow$ (4),*
(c) *(5) $\Rightarrow$ (4).*

Remark 2.3. *If a set function F is differentiable and $(\Im^*, \rho, \theta)$-convex at Ω_0 with $\rho = 0$, then F becomes a convex set function at Ω_0 (cf. [1, Theorem 4.6]) .*

3. SUFFICIENT CONDITIONS

In this section, we derive sufficient conditions for optima of (P) under the assumption of a particular form of $(\Im, \rho, \theta)$-convexity. Let $\mathbb{R}^n$ be the

n-dimensional Euclidean space. Throughout the paper, the following convention for vectors in $\mathbb{R}^n$ will be adopted:

$$x > y \Longleftrightarrow x_i > y_i \quad \text{for all} \quad i = 1, \cdots, n;$$

$$x \geqq y \Longleftrightarrow x_i \geqq y_i \quad \text{for all} \quad i = 1, \cdots, n;$$

$$x \geq y \Longleftrightarrow x_i \geq y_i \quad \text{for all} \quad i = 1, \cdots, n, \ \ \textit{but} \ \ x \neq y;$$

$$x \not\geq y \quad \text{is the negation of} \quad x \geq y.$$

We now consider the following nondifferentiable multiobjective programming problem as the primal problem:

$$\begin{aligned} (P) \qquad & \text{Minimize} && F(\Omega) = (F_1(\Omega), \cdots, F_n(\Omega)) \\ & \text{subject to} && G_j(\Omega) \leqq 0, \quad j = 1, 2, \cdots, m, \\ & && \Omega \in \mathcal{S}, \end{aligned} \tag{3.1}$$

where $\mathcal{S}$ is a subfamily of Γ, $F_i : \mathcal{S} \mapsto \mathbb{R}, i = 1, 2, \cdots, n$, and $G_j : \mathcal{S} \mapsto \mathbb{R}, j = 1, 2, \cdots, m$.

Let H denote the set of all feasible solutions of (P). We say that a measurable set $\Omega^* \in H$ is a Pareto optimal solution of (P) if there is no $\Omega \in H$ to satisfy $F(\Omega) \leq F(\Omega^*)$.

In [5], Lai and Lin proved the necessary optimality conditions of (P). For convenience, we write $\alpha^\top F = \sum_{i=1}^n \alpha_i F_i = \langle F, \alpha \rangle_n$, for $\alpha \in \mathbb{R}^n$.

Theorem 3.1. *[5, Theorem 12]. In problem (P), let $\mathcal{S}$ be a convex subfamily of Γ and $F_i, i = 1, \cdots, n, G_j, j = 1, \cdots, m$, be proper convex set functions on Γ. Let Ω^* be a Pareto optimal solution of problem (P). Suppose that for each $i \in \{1, 2, \cdots, n\}$, there corresponds a $\Omega_i \in \mathcal{S}$ such that*

$$\begin{aligned} & G_k(\Omega_i) < 0, && k = 1, 2, \cdots, m \\ & F_j(\Omega_i) < F_j(\Omega^*), && \text{for} \quad j = 1, \cdots, n, j \neq i \end{aligned}$$

and assume that all functions $F_1, \cdots, F_n, G_1, \cdots, G_m$, except possibly one, are w^*- continuous on $\mathcal{S}$ and that $\bar{\mathcal{S}}$ contains a relative interior point. Then there exist $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*)$ with $\alpha_i^* \geqq 1, i = 1, 2, \cdots, n$, and $\lambda^* = (\lambda_1^*, \cdots, \lambda_m^*)$ in $\mathbb{R}_+^m$ such that

$$\langle \lambda^*, G(\Omega^*) \rangle_m = 0 \tag{3.2}$$

$$\lambda^* \geq 0 \tag{3.3}$$

$$\alpha^* \geqq e, \tag{3.4}$$

$$0 \in \langle \alpha^*, \partial F(\Omega^*) \rangle_n + \langle \lambda^*, \partial G(\Omega^*) \rangle_m + N_{\mathcal{S}}(\Omega^*) \tag{3.5}$$

where $e = (1, 1, \cdots, 1)$ *in* $\mathbb{R}^n$ *and*

$$N_{\mathcal{S}}(\Omega^*) = \{ f \in L_1(X, \Gamma, \mu) | \langle \chi_\Omega - \chi_{\Omega^*}, f \rangle \leqq 0 \quad \text{for all} \quad \Omega \in \mathcal{S} \}. \tag{3.6}$$

□

In order to establish a theorem on sufficient conditions for a feasible solution to be a Pareto optimal solution of (P) under the assumption of $(\Im, \rho, \theta)$-convexity of set functions, the following theorem is essential to key such problem and strong duality theorem.

Theorem 3.2. *Let F be a $(\Im^*, \rho, \theta)$-convex real-valued set function at Ω_0. Then F is convex at Ω_0.*

Proof. For any $\Omega, \Omega_0 \in \Gamma$, there is a Morris sequence $\{V_n\} = \{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Omega_0)\}$ with $\Omega_n \subset \Omega \setminus \Omega_0$ and $\Lambda_n \subset \Omega_0 \setminus \Omega$ such that

$$\chi_{\Omega_n} \longrightarrow^{w^*} \lambda \chi_{\Omega \setminus \Omega_0} \quad \text{and} \quad \chi_{\Lambda_n} \longrightarrow^{w^*} (1 - \lambda) \chi_{\Omega_0 \setminus \Omega}$$

imply

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Omega_0)} \longrightarrow^{w^*} \lambda \chi_\Omega + (1 - \lambda) \chi_{\Omega_0}, \quad \text{for any} \quad \lambda \in [0, 1]. \tag{3.7}$$

By assumption, we have

$$F(\Omega) - F(\Omega_0) \geqq \langle \chi_\Omega - \chi_{\Omega_0}, f\rangle + \rho\theta(\Omega, \Omega_0) \tag{3.8}$$

and

$$F(\Omega_0) \geqq \langle \chi_{\Omega_0}, f\rangle + \rho\theta(\Omega_0, \emptyset). \tag{3.9}$$

Then, multiplying (3.8) by $\lambda(> 0)$ and adding the resulting inequality to (3.9), we have

$$\begin{aligned} F(\Omega_0) + \lambda[F(\Omega) - F(\Omega_0)] \geqq & \lambda\langle \chi_\Omega - \chi_{\Omega_0}, f\rangle + \langle \chi_{\Omega_0}, f\rangle \\ & + \rho[\lambda\theta(\Omega, \Omega_0) + \theta(\Omega_0, \emptyset)]. \end{aligned} \tag{3.10}$$

Now, for $(\Omega, \Omega_0, \lambda) \in \Gamma \times \Gamma \times [0, 1]$, there is a Morris sequence: $\{V_n\} = \{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Omega_0)\}, n = 1, 2, \ldots$ as before, and for each n, we let $0 < \lambda_n < \lambda < 1$ and satisfy

$$\rho\lambda_n\theta(V_n, \Omega_0) \leqq \rho\lambda\theta(\Omega, \Omega_0) \tag{3.11}$$

and

$$\limsup_n \rho\lambda_n\theta(V_n, \Omega_0) = \rho\lambda\theta(\Omega, \Omega_0). \tag{3.12}$$

From (3.8), (3.9), and (3.11), we have

$$\begin{aligned} F(V_n) &= F(\Omega_n \cup \Lambda_n \cup (\Omega \cap \Omega_0)) \\ &= F(\Omega_n \cup \Lambda_n \cup (\Omega \cap \Omega_0)) - F(\Omega_0) + F(\Omega_0) - F(\emptyset) \\ &\geqq \langle \chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Omega_0)} - \chi_{\Omega_0}, f\rangle + \langle \chi_{\Omega_0}, f\rangle \\ &\quad + \rho[\lambda\theta(V_n, \Omega_0) + \theta(\Omega_0, \emptyset)] \\ &\geqq \langle \chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Omega_0)} - \chi_{\Omega_0}, f\rangle + \langle \chi_{\Omega_0}, f\rangle \\ &\quad + \rho[\lambda_n\theta(V_n, \Omega_0) + \theta(\Omega_0, \emptyset)]. \end{aligned}$$

We let $\epsilon_n > 0$ be such that

$$F(V_n) = \langle \chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Omega_0)}, f \rangle + \rho[\lambda_n \theta(V_n, \Omega_0) + \theta(\Omega_0, \emptyset)] + \epsilon_n$$

where $\epsilon_n \to 0$ as $n \to \infty$. It follows from (3.7), (3.12), and (3.10), the limit superior of the above expression gives

$$\begin{aligned}
\lim_{n\to\infty} \sup F(V_n) &= \langle \lambda\chi_\Omega + (1-\lambda)\chi_{\Omega_0}, f\rangle + \rho[\lambda\theta(\Omega,\Omega_0) + \theta(\Omega_0,\emptyset)] \\
&= \lambda\langle \chi_\Omega - \chi_{\Omega_0}, f\rangle + \langle \chi_{\Omega_0}, f\rangle + \rho[\lambda\theta(\Omega,\Omega_0) + \theta(\Omega_0,\emptyset)] \\
&\leqq \lambda[F(\Omega) - F(\Omega_0)] + F(\Omega_0) \\
&= \lambda F(\Omega) + (1-\lambda)F(\Omega_0)
\end{aligned}$$

since F is $(\Im, \rho, \theta)$-convex at Ω_0. This shows that F is also convex. □

Now, we come to one of our main theorems on sufficient criteria for problem (P) under generalized convexity of set functions.

In the following theorems, we state here without proofs. The complete paper will appear elsewhere.

Theorem 3.3 (Sufficient Optimality Conditions). *Let $\Omega^* \in H$ and assume that Ω^*, α^*, and λ^* satisfy (3.2)-(3.6), and that $\Im(\Omega, \Omega^*; -h) \geqq 0$, for each $h \in N_S(\Omega^*), \Omega \in H$. Assume furthermore any one of the following conditions holds:*

(1) *F_i is $(\Im, \rho_{1i}, \theta)$-convex at Ω^*, $i = 1, \cdots, n, G_j$ is $(\Im, \rho_{2j}, \theta)$-convex at Ω^*, $j = 1, \cdots, m$, and $\langle \alpha^*, \rho_1\rangle_n + \langle \lambda^*, \rho_2\rangle_m \geqq 0$,*
(2) *${\alpha^*}^\top F + {\lambda^*}^\top G$ is $(\Im, \rho, \theta)$-convex at Ω^* and $\rho \geqq 0$,*
(3) *${\alpha^*}^\top F + {\lambda^*}^\top G$ is Ponstein $(\Im, \rho, \theta)$-quasiconvex at Ω^* and $\rho > 0$,*
(4) *${\alpha^*}^\top F$ is $(\Im, \rho_1, \theta)$-pseudoconvex at Ω^* , ${\lambda^*}^\top G$ is $(\Im, \rho_2, \theta)$-quasiconvex at Ω^* , and $\rho_1 + \rho_2 \geqq 0$,*

(5) $\alpha^{*\top}F$ *is* $(\Im,\rho_1,\theta)$*-quasiconvex at* Ω^*, $\lambda^{*\top}G$ *is strictly* $(\Im,\rho_2,\theta)$*-pseudoconvex at* Ω^* *, and* $\rho_1+\rho_2 \geqq 0$,

(6) $\alpha^{*\top}F$ *is Ponstein* $(\Im,\rho_1,\theta)$*-quasiconvex at* Ω^* *,* $\lambda^{*\top}G$ *is* $(\Im,\rho_2,\theta)$*-quasiconvex at* Ω^* *, and* $\rho_1+\rho_2 > 0$.

Then Ω^* *is a Pareto optimal solution of (P).*

4. DUALITY THEOREMS

The result of Theorem 3.2 is used to formulate two dual problems of both the Wolfe-type (D_1) under convexity and Mond-Weir-type (D_2) under generalized convexity for (P) as follows:

(D_1) Maximize $F(U)+\langle\lambda,G(U)\rangle_m e$

$$= (F_1(U)+\langle\lambda,G(U)\rangle_m,\cdots,F_n(U)+\langle\lambda,G(U)\rangle_m)$$

subject to

$$\lambda_j \geqq 0, j=1,\cdots,m, \quad U\in\mathcal{S}, \tag{4.1}$$

$$\alpha_i > 0, i=1,\cdots,n, \sum_{i=1}^{n}\alpha_i = 1, \tag{4.2}$$

$$0\in\langle\alpha,\partial F(U)\rangle_n+\langle\lambda,\partial G(U)\rangle_m+N_{\mathcal{S}}(U), \tag{4.3}$$

(D_2) Maximize $F(U)=(F_1(U),\cdots,F_n(U))$

subject to

$$\langle\lambda,G(U)\rangle_m \geqq 0, \tag{4.4}$$

$$\lambda_j \geqq 0, j=1,\cdots,m, \quad U\in\mathcal{S}, \tag{4.5}$$

$$\alpha_i > 0, i=1,\cdots,n, \sum_{i=1}^{n}\alpha_i = 1, \tag{4.6}$$

$$0\in\langle\alpha,\partial F(U)\rangle_n+\langle\lambda,\partial G(U)\rangle_m+N_{\mathcal{S}}(U). \tag{4.7}$$

We denote, respectively by K_1 and K_2, the sets of feasible solutions of problems (D_1) and (D_2). Then for the dual problem (D_1), we have both weak duality and strong duality as follows.

Theorem 4.1 (Weak Duality). *Let $\Omega \in H, (\alpha, \lambda, U) \in K_1$, and $\Im(\Omega, U, -h) \geqq 0$. If any one of the following conditions hold:*

(a) *F_i is $(\Im, \rho_{1i}, \theta)$-convex, $i = 1, \cdots, n, G_j$ is $(\Im, \rho_{2j}, \theta)$-convex, $j = 1, \cdots, m$, and $\langle \alpha, \rho_1 \rangle_n + \langle \lambda, \rho_2 \rangle_m \geqq 0$,*

(b) *$\alpha^\top F + \lambda^\top G$ is $(\Im, \rho, \theta)$-convex and $\rho \geqq 0$,*

(c) *$\alpha^\top F + \lambda^\top G$ is Ponstein $(\Im, \rho, \theta)$-quasiconvex and $\rho > 0$,*

then

$$F(\Omega) \nleq F(U) + \langle \lambda, G(U) \rangle_m e.$$

Theorem 4.2 (Strong Duality). *In Theorems 3.1 and 4.1, we let the functions $F_i, i = 1, 2, \cdots, n$, and $G_j, j = 1, 2, \cdots, m$, be $(\Im^*, \rho, \theta)$-convex. Assume furtheremore these functions satisfy the other conditions in Theorems 3.1 and 4.1. Suppose that Ω^* is a Pareto optimal solution for (P). Then there exist $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*)$ with $\alpha_i^* > 0, i = 1, \cdots, n$,and $\lambda^* = (\lambda_1^*, \cdots, \lambda_m^*)$ with $\lambda_j^* \geqq 0, j = 1, \cdots, m$, such that $(\alpha^*, \lambda^*, \Omega^*)$ is a Pareto optimal solution for (D_1) and the optimal values of (P) and (D_1) are equal.*

Theorem 4.3 (Weak Duality). *Let $\Omega \in H, (\alpha, \lambda, U) \in K_2$, and $\Im(\Omega, U, -h) \geqq 0$. If any one of the following conditions hold:*

(a) *$\alpha^\top F$ is $(\Im, \rho_1, \theta)$-pseudoconvex , $\lambda^\top G$ is $(\Im, \rho_2, \theta)$-quasiconvex , and $\rho_1 + \rho_2 \geqq 0$,*

(b) *$\alpha^\top F$ is $(\Im, \rho_1, \theta)$-quasioconvex , $\lambda^\top G$ is strictly $(\Im, \rho_2, \theta)$-pseudoconvex , and $\rho_1 + \rho_2 \geqq 0$*

(c) $\alpha^{\top} F$ *is Ponstein* $(\Im, \rho_1, \theta)$*-quasioconvex* , $\lambda^{\top} G$ *is* $(\Im, \rho_2, \theta)$*-quasiconvex* , *and* $\rho_1 + \rho_2 > 0$.

then,

$$F(\Omega) \nleq F(U).$$

Theorem 4.4 (Strong Duality). *In Theorems 3.1 and 4.3, let the functions* $F_i, i = 1, 2, \cdots, n$, *and* $G_j, j = 1, 2, \cdots, m$, *be* $(\Im^*, \rho, \theta)$*-convex. Assume furtheremore these functions satisfy the other conditions in Theorems 3.1 and 4.3. Suppose that* Ω^* *is a Pareto optimal solution for (P). Then there exist* $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*)$ *with* $\alpha_i^* > 0, i = 1, \cdots, n$, *and* $\lambda^* = (\lambda_1^*, \cdots, \lambda_m^*)$ *with* $\lambda_j^* \geqq 0, j = 1, \cdots, m$, *such that* $(\alpha^*, \lambda^*, \Omega^*)$ *is a Pareto optimal solution for* (D_2) *and the optimal values of (P) and* (D_2) *are equal.*

The complete proof of Theorems 4.1 - 4.4 will appear elsewhere.

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