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Continuous Selectors of Fixed Point Sets of Multifunctions with Decomposable Values and Their Applications

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Abstract. The existence theorem of continuous selectors that values are fixed points of multivalued contractions are proved. As an application, an existence of continuous selectors on mild solution sets of nonconvex differential inclusions of evolution type, depending on a parameter, is presented.

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1. Introduction

The results of [1] appeared to be basic to study an existence problem for continuous selectors of multifunctions with nonconvex decomposable values [3, 12, 14, 23, 24]. Recently, by using Filippov's successive approximation process [25] and selection theorems [3, 12] there were proved the existence of continuous selectors, that values are solutions of Lipschitzean differential inclusions [7, 21]. It should be mentioned that in this case we have to do with multifunctions, that values are non-decomposable and non-convex sets. Remark finally that the existence of continuous selectors passing through the fixed points of multivalued contractions depending on a parameter with non-convex values were obtained in [18].

Our results contain as a special case the selection theorem [15] and supplement the results of [18]. The contents of the present paper can be represented by the following results.

Let $(X, \|\cdot\|)$ be a separable Banach space, M be a separable metric space, T be a locally compact σ-compact metric space with a positive, finite, nonatomic Radon measure μ_0, $L_1(T,X)$ be the Banach space of Bochner-integrable functions $x : T \mapsto X$ with the norm $\| v \|_L = \int_T \| v(t) \| \, d\mu_0$.

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Consider a function $P : M \times L_1(T, X) \to [0, +\infty)$,

$$P(\xi, x) = \int_T p(t, \xi, x(t)) d\mu_0, \tag{1.1}$$

where $p : T \times M \times X \to [0, +\infty)$ is a function with the following properties:

i) $p(t, \xi, x)$ is measurable for every $(\xi, x) \in M \times X$ and continuous with respect to (ξ, x) a.e. on T;

ii) for every $\xi \in M$ the function $p(t, \xi, \cdot)$ is a semi-norm on X a.e. on T;

iii) $p(t, \xi, x) \leq c\|x\|$, $c > 0$ a.e. on T for every $(\xi, x) \in M \times X$.

Assume that for every $\xi \in M$, $x \in L_1(T, X)$

$$m(\xi)\|x\|_L \leq P(\xi, x),$$

where $m : M \to (0, +\infty)$ is continuous. ¿From the latter and iii) one has

$$m(\xi)\|x\|_L \leq P(\xi, x) \leq c\|x\|_L. \tag{1.2}$$

This implies, thanks to ii), that for every $\xi \in M$ the function $P(\xi, \cdot)$ is a norm in $L_1(T, X)$, equivalent to the usual one.

For every pair of nonempty closed sets $A, B \in L_1(T, X)$ and $\xi \in M$ we denote by $d_H^L(\xi)(A, B)$ the Hausdorff distance between A and B, generated by the norm $P(\xi, \cdot)$.

Let $\Gamma : M \times L_1(T, X) \mapsto L_1(T, X)$ be a multifunction with non-empty, closed, decomposable values and $\mathrm{Fix}\Gamma(\xi)$ be the set of all fixed points of $\Gamma(\xi, x)$ for every $\xi \in M$.

THEOREM 1.1. *Let* $\Gamma : M \times L_1(T, X) \mapsto L_1(T, X)$ *be a multifunction with non-empty, closed, decomposable values. Assume that:*

i) *the multifunction* $\xi \to \Gamma(\xi, x)$ *is lower semicontinuous for every* $x \in L_1(T, X)$;

ii) *there exists an upper semicontinuous function* $k : M \to [0, 1)$ *such that for every* $\xi \in M$, $x, y \in L_1(T, X)$ *one has*

$$d_H^L(\xi)\left(\Gamma(\xi, x), \Gamma(\xi, y)\right) \leq k(\xi) P(\xi, x - y). \tag{1.3}$$

Then the following assertions are true:

a) $\mathrm{Fix}\Gamma(\xi) \neq \emptyset$ *for every* $\xi \in M$ *and there exists a continuous function* $u : M \to L_1(T, X)$ *such that*

$$u(\xi) \in \mathrm{Fix}\Gamma(\xi), \quad \forall \xi \in M; \tag{1.4}$$

b) *if a set* $D \subset M$ *is closed and* $u_D : D \to L_1(T, X)$ *is a continuous function,* $u_D(\xi) \in \mathrm{Fix}\Gamma(\xi)$, $\xi \in D$, *then there exists a continuous function* $u : M \to L_1(T, X)$ *such that* (1.4) *holds and* $u(\xi) = u_D(\xi)$, $\forall \xi \in D$.

COROLLARY 1.2. *Suppose that all the assumptions of Theorem* 1.1 *are valid. Then:*

a) *the multifunction* $\xi \to \mathrm{Fix}\Gamma(\xi)$ *is closed-valued and lower semicontinuous;*

b) *for every* $\xi \in M$ *the set* $\mathrm{Fix}\Gamma(\xi)$ *is an absolute retract and, consequently, it is arcwise connected;*

c) *if the multifunction* $\xi \to \Gamma(\xi, x)$ *has the closed graph, a retraction can be chosen which depends continuously on* ξ*, namely, there exists a continuous map* $g : M \times L_1(T, X) \to L_1(T, X)$ *such that*

$$g(\xi, x) \in \mathrm{Fix}\Gamma(\xi), \quad \forall x \in L_1(T, X),$$
$$g(\xi, x) = x, \quad \forall x \in \mathrm{Fix}\Gamma(\xi).$$

This paper is organized as follows.

Section 2 contains notations and terminology.

The main results are proved in section 3.

As an application of the obtained results, the continuous selectors of mild solution sets of nonconvex differential inclusions of evolution type, depending on a parameter, are studied in section 4. In particular, we establish some topological properties of these solution sets.

In section 5, comments are given.

2. Notations and Definitions

Let
- $(X, \|\cdot\|)$ be a separable Banach space,
- M be a separable metric space,
- T be a locally compact σ-compact metric space with a positive, finite, nonatomic Radon measure μ_0 and a σ-algebra Σ of μ_0-measurable subsets of T,
- $L_1(T, X)$ be the Banach space of eqiuvalence classes of Bochner-integrable functions $x : T \mapsto X$ with the usual norm.

For a normed space Y let
- cY be the family of all nonempty, closed subsets of Y,
- $d(x, K)$ be the distance of a point $x \in X$ to a subset $K \subset X$,
- $d_L(v, Q)$ be the distance of a point $v \in L_1(T, X)$ to a subset $Q \subset L_1(T, X)$.

If A and D are subsets of X, then $d(A, D) = \sup\{d(a, D); a \in A\}$ is the excess of A over D, and $d_H(A, D) = \max\{d(A, D), d(D, A)\}$ is the Hausdorff distance between A and D.

If $A, D \subset L_1(T, X)$, then $d_L(A, D) = \sup\{d_L(a, D); a \in A\}$ is the excess of A over D and $d_H^L(A, D) = \max\{d_L(A, D), d_L(D, A)\}$ is the Hausdorff distance between A and D.

For a function $P(\xi, x)$, that is a norm in $L_1(T, X)$ for every $\xi \in M$, we denote by $d_L(\xi)(\cdot, \cdot)$ a metric in $L_1(T, X)$, induced by the norm $P(\xi, \cdot)$.

Similarly, $d_L(\xi)(A, D)$, $d_H^L(\xi)(A, D)$ are the excess of A over D and the Hausdorff distance between $A, D \subset L_1(T, X)$, generated by the metric $d_L(\xi)(\cdot, \cdot)$.

A set $A \subset L_1(T, X)$ is called decomposable if for any $u, v \in A$ and $E \in \Sigma$ the element $\chi(E)u + \chi(T \setminus E)v$ belongs to A, where $\chi(E)$ is the characteristic function of E.

We denote by $\mathrm{dc}L_1(T, X)$ the set of nonempty, decomposable, closed subsets of $L_1(T, X)$.

A sequence $x_n \in L_1(T, X)$, $n \geq 1$, is called uniformly integrable if, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\int_E \|x_n(t)\| d\mu_0 < \epsilon$$

for every subset $E \in \Sigma$ with $\mu_0(E) \leq \delta$ and for all $n \geq 1$.

A multifunction $F : T \mapsto c(X)$ is called measurable if the set $\{t \in T;\ F(t) \cap U \neq \emptyset\}$ is measurable for any closed subset $U \subset X$.

A multifunction F from a topological space Y into a topological space Z is called lower semicontinuous (l.s.c.) at a point $y_0 \in Y$ if, for any open set $V \subset Z$, $F(y_0) \cap V \neq \emptyset$, there exists a neighbourhood $U(y_0)$ of y_0 such that $F(y) \cap V \neq \emptyset$ for every $y \in U(y_0)$.

If Y, Z are metric spaces then a multifunction $F : Y \to Z$ is l.s.c. at a point $y_0 \in Y$ if and only if for any $z_0 \in F(y_0)$ and every sequence $y_n \in Z$, $y_n \to y_0$, $n \to \infty$

$$\lim_{n \to \infty} d_Z(z_0, F(y_n)) = 0,$$

where $d_Z(z_0, F(y_n))$ is the distance of the point z_0 to the set $F(y_n)$.

A subset A of Y is called a retract of Y if there is a continuous map $g : Y \to A$ satisfying $g(x) = x$ for every $x \in A$. Any such map g is called a retraction of Y onto A.

3. The Proof of main result

We give some results that are applied in the proof of Theorem 1.1.

Let $p : T \times M \times X \to [0, +\infty)$ be a function, having the following properties:

P1) $p(t,\xi,x)$ is measurable for every $(\xi,x) \in M \times X$ and continuous with respect to (ξ,x) a.e. on T;

P2) for every $\xi \in M$ the function $p(t,\xi,\cdot)$ is a seminorm on X a.e. on T;

P3) $p(t,\xi,x) \le c\|x\|$, $c > 0$ a.e. on T for every $(\xi,x) \in M \times X$.

Thanks to P1), P3) for every $x(\cdot) \in L_1(T,X)$, $\xi \in M$ the function $p(t,\xi,x(t))$ is integrable. Then there is defined the function $P : M \times L_1(T,X) \to [0,+\infty)$

$$P(\xi,x) = \int_T p(t,\xi,x(t))d\mu_0.$$

PROPOSITION 3.1. *Suppose Properties* P1)-P3) *of the function* $p(t,\xi,x)$ *hold. Then the function* $P(\xi,x)$ *is continuous with respect to* (ξ,x) *and for every* $\xi \in M$ *the function* $P(\xi,\cdot)$ *is a continuous seminorm in* $L_1(T,X)$.

Properties of the function $P(\xi,x)$ arise directly from Properties P1)-P3) of the function $p(t,\xi,x)$.

Let the function $p(t,\xi,x)$ has Properties P1)-P3). Denote by $B(\xi)$ the open unit ball, generated by the seminorm $P(\xi,\cdot)$

$$B(\xi) = \{x \in L_1(T,X);\ P(\xi,x) < 1\},\ \xi \in M.$$

THEOREM 3.2. *Let* $\Gamma : M \mapsto dcL_1(T,X)$ *be a l.s.c. multifunction,* $\phi : M \to (0,+\infty)$ *be a l.s.c. function and* $g : M \to L_1(T,X)$ *be a continuous function. If for every* $\xi \in M$

$$\Phi(\xi) = \Gamma(\xi) \cap (g(\xi) + \phi(\xi)B(\xi)) \neq \emptyset,$$

then the multifunction $\xi \to \Phi(\xi)$ *has a continuous selector* $f : M \to L_1(T,X)$. *Furthemore, if* $D \subset M$ *is a closed set,* $f_D : D \to L_1(T,X)$ *is a continuous function and* $f_D(\xi) \in \Phi(\xi)$, $\xi \in D$, *then* f *can be chosen so that* $f(\xi) = f_D(\xi)$, $\xi \in D$.

Theorem 3.2 can be proved analogously to theorem 3.1 in [14].

Proof of Theorem 1.1.

At first we prove that for any continuous function $u : M \to L_1(T,X)$ the multifunction $\Gamma(\xi,u(\xi))$ is l.s.c. Fix $\xi_0 \in M$. Taking into account (1.2) one has

$$d_H^L(\Gamma(\xi,u(\xi_0)),\Gamma(\xi,u(\xi)) \le \frac{c\,k(\xi)}{m(\xi)}\|u(\xi) - u(\xi_0)\|_L.$$

Fix $v_0 \in \Gamma(\xi_0, u(\xi_0))$ and $\xi_n \to \xi_0$, $n \to \infty$. Upper semicontinuity of $k(\xi)$ and continuity of $m : M \to (0, +\infty)$ imply that

$$d_H^L(\Gamma(\xi_n, u(\xi_n)), \Gamma(\xi_n, u(\xi_0)) \leq a \, \|u(\xi_n) - u(\xi_0)\|_L \tag{3.1}$$

for some $a > 0$ and all $n \geq 1$. Thanks to (3.1) one has

$$\begin{aligned} d_L(v_0, \Gamma(\xi_n, u(\xi_n)) &\leq d_L(v_0, \Gamma(\xi_n, u(\xi_0)) + d_H^L(\Gamma(\xi_n, u(\xi_n)), \Gamma(\xi_n, u(\xi_0)) \\ &\leq d_L(v_0, \Gamma(\xi_n, u(\xi_0)) + a \, \|u(\xi_n) - u(\xi_0)\|_L. \end{aligned}$$

¿From this, taking into account lower semicontinuity of the multifunction $\xi \to \Gamma(\xi, u(\xi_0))$, it follows that the multifunction $\Gamma(\xi, u(\xi))$ is l.s.c.at point ξ_0.

Lemma 3.6 [9] implies that there exists a continuous function $k_1 : M \to R$ such that $k(\xi) < k_1(\xi) < 1$, $\xi \in M$. Now, from (1.3) it follows that for every $x, y \in L_1(T, X)$, $x \neq y$

$$d_H^L(\xi)(\Gamma(\xi, x), \Gamma(\xi, y)) < k_1(\xi) \, P(\xi, x - y). \tag{3.2}$$

Fix any continuous function $u_0 : M \to L_1(T, X)$. We shall construct a Cauchy sequence of successive continuous approximations $u_n : M \to L_1(T, X)$ such that for every $\xi \in M$, $n \geq 1$,

$$u_n(\xi) \in \Gamma(\xi, u_{n-1}(\xi)), \tag{3.3}$$

$$P(\xi, u_{n+1}(\xi) - u_n(\xi)) \leq k_1(\xi) \, P(\xi, u_n(\xi) - u_{n-1}(\xi)). \tag{3.4}$$

Suppose we have defined the functions $u_1(\xi), ..., u_n(\xi)$ satisfying (3.3), (3.4). If $u_n(\xi) \neq u_{n-1}(\xi)$ for every $\xi \in M$, then

$$\begin{aligned} &d_L(\xi)(u_n(\xi), \Gamma(\xi, u_n(\xi)) \leq d_H^L(\xi)(\Gamma(\xi, u_{n-1}(\xi), \Gamma(\xi, u_n(\xi)) \\ &\leq k(\xi) \, P(\xi, u_n(\xi) - u_{n-1}(\xi)) < k_1(\xi) \, P(\xi, u_n(\xi) - u_{n-1}(\xi)). \end{aligned} \tag{3.5}$$

Proposition 3.1 implies that the function $k_1(\xi) \, P(\xi, u_n(\xi) - u_{n-1}(\xi))$ is continuous. Thanks to (3.5) and Theorem 3.2 one get a continuous selector $u_{n+1}(\xi)$ of the multifunction $\Gamma(\xi, u_n(\xi))$ such that

$$P(\xi, u_{n+1}(\xi) - u_n(\xi)) \leq k_1(\xi) \, P(\xi, u_n(\xi) - u_{n-1}(\xi)), \; \xi \in M. \tag{3.6}$$

If $u_n(\xi) = u_{n-1}(\xi)$ for some $\xi \in M$, then the set

$$D = \{\xi \in M; \; P(\xi, u_n(\xi) - u_{n-1}(\xi)) = 0\}$$

is closed and for every $\xi \in D$,

$$u_n(\xi) \in \Gamma(\xi, u_{n-1}(\xi)) = \Gamma(\xi, u_n(\xi)).$$

Consider on the open set $M \setminus D$ the restriction of the multifunction $\Gamma(\xi, u_n(\xi))$, that is l.s.c. in $M \setminus D$ and for every $\xi \in M \setminus D$ inequality (3.5) holds. Repeating above-mentioned arguments we get a continuous selector $u^*_{n+1} : M \setminus D \to L_1(T, X)$ of the restriction of $\Gamma(\xi, u_n(\xi))$ on $M \setminus D$ such that for every $\xi \in M \setminus D$

$$P(\xi, u^*_{n+1}(\xi) - u_n(\xi)) \leq k_1(\xi)\, P(\xi, u_n(\xi) - u_{n-1}(\xi)),\ \xi \in M. \tag{3.7}$$

Denote by $u^*_{n+1}(\xi)$ the function, defined in the following way:

$$u_{n+1}(\xi) = u^*_{n+1}(\xi),\ \xi \in M \setminus D,$$
$$u_{n+1}(\xi) = u_n(\xi),\ \xi \in D.$$

This function is a selector of the multifunction $\Gamma(\xi, u_n(\xi))$ and satisfies inequality (3.4). Since the set $M \setminus D$ is open, then inequality (3.7) implies that the selector $u_{n+1}(\xi)$ is continuous in M. Clearly, $u_{n+1}(\xi)$ satisfies (3.3), (3.4). From (3.4) we obtain that for every $\xi \in M,\ l > n$

$$\begin{aligned} P(\xi, u_n(\xi) - u_l(\xi)) &\leq (1 + k_1(\xi) + \dots + k_1^{l-n-1}(\xi))\, k_1^n(\xi) P(\xi, u_1(\xi) - u_0(\xi)) \\ &\leq \frac{k_1^n(\xi)}{1 - k_1(\xi)}\, P(\xi, u_1(\xi) - u_0(\xi)). \end{aligned} \tag{3.8}$$

According to (3.8), (1.2) one has that

$$\|u_n(\xi) - u_l(\xi)\|_L \leq k_1^n(\xi) \frac{c}{m(\xi)(1 - k_1(\xi))} \|u_1(\xi) - u_0(\xi)\|_L. \tag{3.9}$$

Therefore, (3.9) implies that the sequence $\{u_n(\cdot)\}$ converges uniformly on every compact subsets of M to a continuous function $u(\cdot)$ and

$$\|u(\xi) - u_0(\xi)\|_L \leq \frac{c}{m(\xi)(1 - k_1(\xi))} \|u_1(\xi) - u_0(\xi)\|_L. \tag{3.10}$$

To see that $u(\xi) \in \Gamma(\xi, u(\xi))$, $\xi \in M$, it is enough to take advantage of (3.3) and (1.3). This ends the proof of statement a) of Theorem 1.1.

Let a set $D \subset M$ be closed and $u_D : D \to L_1(T, X)$ be a continuous function such that $u_D(\xi) \in \Gamma(\xi, u_D(\xi))$, $\xi \in D$. Take any continuous function $u_0 : M \to L_1(T, X)$, $u_0(\xi) = u_D(\xi)$, $\xi \in D$. Then there exists a continuous function $u : M \to L_1(T, X)$ such that $u(\xi) \in \Gamma(\xi, u(\xi))$, $\xi \in M$ and inequality (3.10) holds.

Consider the multifunction $\Gamma^*(\xi, u_0(\xi))$,

$$\Gamma^*(\xi, u_0(\xi)) = \Gamma(\xi, u_0(\xi)),\ \xi \in M \setminus D,$$
$$\Gamma^*(\xi, u_0(\xi)) = u_D(\xi),\ \xi \in D.$$

This multifunction is l.s.c. Then there exists a continuous selector $u_1(\xi)$ of the multifunction $\Gamma^*(\xi, u_0(\xi))$ [3]. Clearly that $u_1(\xi) \in \Gamma(\xi, u_0(\xi))$, $\xi \in M$ and $u_1(\xi) = u_0(\xi)) = u_D(\xi)$, $\xi \in M \setminus D$. Now from (3.10) it follows that $u(\xi) = u_0(\xi)) = u_D(\xi)$, $\xi \in D$. This completes the proof of Theorem 1.1.

Proof of Corollary 1.2.

The statement a) of Corollary 1.2 directly follows from the statement b) of Theorem 1.1 and (1.3).

Let us prove the statement b). According to the definition of an absolute retract [16] we have to prove that for fixed $\xi_0 \in M$, for any separable metric space Y, for any closed set $D \subset Y$ and for any continuous function $v_D : D \to \mathrm{Fix}\Gamma(\xi_0)$ there exists a continuous function $v : Y \to \mathrm{Fix}\Gamma(\xi_0)$ such that $v(y) = v_D(y)$, $y \in D$.

Consider the multifunction $\Gamma^* : Y \times L_1(T, X) \to \mathrm{dc}L_1(T, X)$ defined in the following way:

$$\Gamma^*(y, x) = \Gamma(\xi_0, x), \ y \in Y, \ x \in L_1(T, X).$$

It is clear that the multifunction $y \to \Gamma^*(y, x)$ is l.s.c. for every $x \in L_1(T, X)$ and

$$d_H^L(\xi_0)(\Gamma^*(y, x), \Gamma^*(y, z)) \le k(\xi_0)P(\xi_0, x - z), \ \ x, z \in L_1(T, X), \ y \in Y.$$

Therefore, for the multifunction $\Gamma^*(y, x)$ all assumptions of Theorem 1.1 hold. Since $v_D(y) \in \mathrm{Fix}\Gamma(\xi_0)$, $y \in D$, then $v_D(y) \in \Gamma(\xi_0, v_D(y)) = \Gamma^*(y, v_D(y))$, $y \in D$. Taking into account the statement a) of Theorem 1.1 for the multifunction $\Gamma^* : Y \times L_1(T, X) \to \mathrm{dc}L_1(T, X)$ we obtain that there exists a continuous function $v : Y \to L_1(T, X)$ such that $v(y) \in \Gamma^*(y, v(y)) = \Gamma(\xi_0, v(y))$, $y \in Y$ and $v(y) = v_D(y)$, $y \in D$. It means that $v(y) \in \mathrm{Fix}\Gamma(\xi_0)$, $y \in Y$. Hence the statement b) of Corollary 1.2 is proved.

Now we prove the statement c). Let us show that the multifunction $\xi \to \mathrm{Fix}\Gamma(\xi)$ has a closed graph $\mathrm{gr}\,\mathrm{Fix}\Gamma$. Take sequences $\xi_n \to \xi_0$, $x_n \to x_0$, $x_n \in \mathrm{Fix}\Gamma(\xi_n)$, and $\epsilon_n \to 0$. Then there exists $y_n \in \Gamma(\xi_n, x_0)$ such that

$$\begin{aligned} P(\xi_n, x_n - y_n)) &\le d_L(\xi_n)(x_n, \Gamma(\xi_n, x_0)) + \epsilon_n \\ \le d_L(\xi_n)(x_n, \Gamma(\xi_n, x_n)) &+ d_H^L(\xi_n)(\Gamma(\xi_n, x_n), \Gamma(\xi_n, x_0)) + \epsilon_n \\ &\le k(\xi_n)P(\xi_n, x_n - x_0) + \epsilon_n. \end{aligned}$$

Thanks to (1.2) the latter implies

$$\|x_n - y_n\|_L \le \frac{k(\xi_n) \cdot c}{m(\xi)} \cdot \|x_n - x_0\|_L + \epsilon_n.$$

¿From this it follows that the sequence y_n, $y_n \in \Gamma(\xi_n, x_0)$ converges to x_0. As the multifunction $\xi \to \Gamma(\xi, x_0)$ has the closed graph one get that $x_0 \in \Gamma(\xi_0, x_0)$. Hence the multifunction $\xi \to \mathrm{Fix}\Gamma(\xi)$ has a closed graph $D = \mathrm{gr}\,\mathrm{Fix}\Gamma$.

Consider the multifunction $\Gamma^* : M \times L_1(T,X) \times L_1(T,X) \to \mathrm{dc}L_1(T,X)$, $\Gamma^*(\xi, u, x) = \Gamma(\xi, x)$, $\xi \in M$, $u \in L_1(T,X)$, $x \in L_1(T,X)$. Considering (ξ, u) as a new parameter one has that for the multifunction $\Gamma^*(\xi, u, x)$ all assumptions of Theorem 1.1 hold.

Consider a function $g_D : D \to L_1(T,X)$, $g_D(\xi, u) = u$, $(\xi, u) \in D$. The function $g_D(\xi, u)$ is continuous on the closed set D and $g_D(\xi, u) \in \Gamma(\xi, g_D(\xi, u)) = \Gamma^*(\xi, u, g_D(\xi, u))$, $(\xi, u) \in D$. Taking into account the statement a) of Theorem 1.1 for the multifunction $\Gamma^*(\xi, u, x)$ we obtain that there exists a continuous function $g : M \times L_1(T,X) \to L_1(T,X)$ such that $g(\xi, u) \in \Gamma^*(\xi, u, g(\xi, u)) = \Gamma(\xi, g(\xi, u))$, $(\xi, u) \in M \times L_1(T,X)$ and $g(\xi, u) = g_D(\xi, u)$, $(\xi, u) \in D$. It means that $g(\xi, u) \in \mathrm{Fix}\Gamma(\xi)$, $(\xi, u) \in M \times L_1(T,X)$ and $g(\xi, u) = u$ for all $u \in \mathrm{Fix}\Gamma(\xi)$. The statement c) of Corollary 1.2 is proved.

4. Application

Throughout $T = [0, a]$ is a segment with the Lebesgue σ-algebra; $(X, \|\cdot\|)$ is a separable Banach space whose null element is denoted by Θ_X; Λ is a separable metric space; F is a multifunction from $T \times X \times \Lambda$ into X with non-empty closed values.

Consider the evolution inclusion

$$\dot{x}(t) \in A(t)x(t) + F(t, x(t), \lambda), \qquad \text{(H)}$$

$$x(0) = \xi,$$

where $\{A(t);\ t \in T\}$ is a family of densely defined, closed, linear operators, that generates an evolution operator

$$S : \Delta = \{(t,s) \in T \times T :\ 0 \le s \le t \le a\} \to \mathcal{L}(X).$$

Recall if $S(t,s)$ is an evolution operator (or fundamental solutions), then $S : \Delta \to \mathcal{L}(X)$ is strongly continuous, $S(t,\tau)S(\tau,s) = S(t,s)$ for $0 \le s \le \tau \le t \le a$ (semigroup property), and $S(t,t) = I$ for all $t \in T$.

By a solution of (H) for fixed $\xi \in X$, $\lambda \in \Lambda$ we understand a mild solution $x(\cdot) \in C(T,X)$ of the form

$$x(t) = S(t,0)\xi + \int_0^t S(t,s)f(s)ds,\ t \in T,$$

where $f(\cdot) \in L_1(T, X)$ and a.e. on T $f(t) \in F(t, x(t), \lambda)$.

Let $\mathcal{H}(\xi, \lambda)$ be the set of all mild solutions of (H). Our purpose is to study the existence of continuous selectors of the multifunction $(\xi, \lambda) \to \mathcal{H}(\xi, \lambda)$ and properties of this multifunction.

We denote by $C^*(T, X)$ the space of all continuous functions $x : T \to X$ represented in the following way:

$$x(t) = S(t, 0)\xi + \int_0^t S(t, s)f(s)ds, \ t \in T, \ \xi \in X, \ f(\cdot) \in L_1(T, X). \tag{4.1}$$

Consider on $C^*(T, X)$ a function

$$\|x\|_{C^*(T,X)} = \|x\|_{C(T,X)} + \|f\|_{L_1(T,X)}. \tag{4.2}$$

PROPOSITION 4.1. *Function* (4.2) *is a norm and* $C^*(T, X)$ *with the norm, given by* (4.2) *is a separable Banach space. The map* $\mathcal{T} : X \times L_1(T, X) \to C^*(T, X)$ *defined by (4.1), is a one-to-one linear operator. Moreover,* $\mathcal{T}$ *is a topological isomorphism.*

Proof. Clearly $\mathcal{T}$ is linear. To prove that $\mathcal{T}$ is one-to-one suppose that $\mathcal{T}(\xi_1, f_1) = \mathcal{T}(\xi_2, f_2)$ for some $(\xi_1, f_1), (\xi_2, f_2) \in X \times L_1(T, X)$. This implies that $\xi_1 = \xi_2$ and setting $f(s) = f_1(s) - f_2(s)$

$$\int_0^t S(t, s)f(s)ds = \theta_x, \ \text{for } t \in T. \tag{4.3}$$

Since for $h > 0$

$$\int_0^t S(t + h, s)f(s)ds = S(t + h, t) \int_0^t S(t, s)f(s)ds = \theta_x,$$

then for each $h > 0$

$$\int_0^t S(t + h, s)f(s)ds = \theta_x, \ \text{for } t \in T. \tag{4.4}$$

Let $0 < t < a$ be arbitrary. By virtue of (4.3), (4.4), for $h > 0$ sufficiently small, we have

$$\int_0^{t+h} S(t + h, s)f(s)ds = \int_0^t S(t + h, s)f(s)ds + \int_t^{t+h} S(t + h, s)f(s)ds = \theta_x,$$

from which, dividing by h,

$$\frac{1}{h} \int_t^{t+h} S(t, s)f(s)ds = \theta_x. \tag{4.5}$$

Let J be the set of all $t \in (0, a)$ such that

$$\lim_{h\to 0} \frac{1}{2h} \int_{t-h}^{t+h} \|f(t) - f(s)\| ds = 0 \tag{4.6}$$

and observe that $T \setminus J$ has zero Lebesque measure ([8], p. 217). Let $t \in J$ be arbitrary. We have

$$\frac{1}{h} \int_t^{t+h} S(t,s) f(s) ds$$
$$= \frac{1}{h} \int_t^{t+h} S(t,s)[f(s) - f(t)]ds + \frac{1}{h} \int_t^{t+h} S(t,s) f(t) ds.$$

Whence, in view of (4.5)

$$\frac{1}{h} \int_t^{t+h} S(t,s) f(t) ds = \frac{1}{h} \int_t^{t+h} S(t,s)[f(t) - f(s)]ds.$$

Since for fixed t, $h > 0$, the function $s \to S(t+h, s) f(t)$, $0 \le s \le t+h$ is continuous, there exists $t < c(h) < t+h$ such that

$$S(t+h, c(h)) f(t) = \frac{1}{h} \int_t^{t+h} S(t,s)[f(t) - f(s)]ds. \tag{4.7}$$

Let $M > 0$ be such that

$$\|S(t,s)\|_{\mathcal{L}} \le M, \quad 0 \le s \le t \le a.$$

Taking into account (4.7) one has

$$\|S(t+h, c(h)) f(t)\| \le 2M \cdot \frac{1}{2h} \int_{t-h}^{t+h} \|f(t) - f(s)\| ds.$$

¿From this and (4.6) it follows that

$$\|f(t)\| = \lim_{h\to 0} \|S(t+h, c(h)) f(t)\| = 0.$$

As $t \in J$ is arbitrary and $T \setminus J$ has zero Lebesgue measure, we have $f_1 = f_2$. Hence $(\xi_1, f_1) = (\xi_2, f_2)$, and $\mathcal{T}$ is one-to-one. This means that the function $\|x\|_{C^*(T,X)}$, defined by (4.2), is a norm. ¿From the estimate

$$\|\xi\| + \|f\|_{L_1(T,X)} \le \|x\|_{C^*(T,X)} \le M(\|\xi\| + \|f\|_{L_1(T,X)})$$

it follows that $\mathcal{T} : X \times L_1(T,X) \to C^*(T,X)$ is a topological isomorphism, and $C^*(T,X)$ is the separable Banach space. This completes the proof.

To study the problem (H) we introduce the following hypothesis:

(H1) the multifunction $t \to F(t,x,\lambda)$ is measurable for every $x \in X$, $\lambda \in \Lambda$;

(H2) there exists a continuous function $L : \Lambda \to L_1(T,R^+)$ such that for every $x,y \in X$, $\lambda \in \Lambda$, and for almost every $t \in T$ one has

$$d_H\left(F(t,x,\lambda), F(t,y,\lambda)\right) \leq L(\lambda)(t)\|x-y\|;$$

(H3) the multifunction $\lambda \to F(t,x,\lambda)$ is lower semicontinuous for every $x \in X$ and for almost every $t \in T$;

(H4) for each convergent sequence $\{\lambda_n\}$ in Λ the sequence $\{d(\theta_x, F(t,\theta_x,\lambda_n))\}$ is uniformly integrable.

Remark 4.2. Since for every $\lambda \in \Lambda$ the function $d(\theta_x, F(t,\theta_x,\lambda)$ is measurable, condition (H4) holds, if there exists a continuous function $\beta : \Lambda \to L_1(T,R^+)$ such that for every $\lambda \in \Lambda$

$$d(\theta_x, F(t,\theta_x,\lambda) \leq \beta(\lambda)(t) \text{ a.e. in } T.$$

THEOREM 4.3. *Let* (H1)-(H4) *be satisfied. Then*

a) *there exists a continuous function* $u : X \times \Lambda \to C^*(T,X)$ *such that for every* $(\xi,\lambda) \in X \times \Lambda$

$$u(\xi,\lambda) \in \mathcal{H}(\xi,\lambda); \tag{4.8}$$

b) *if* $D \in X \times \Lambda$ *is a closed set,* $u_D : D \to C^*(T,X)$ *is a continuous function such that for every* $(\xi,\lambda) \in D$,

$$u_D(\xi,\lambda) \in \mathcal{H}(\xi,\lambda),$$

then there exists a continuous function $u : X \times \Lambda \to C^*(T,X)$ *such that for every* $(\xi,\lambda) \in X \times \Lambda$ *inclusions* (4.8) *is true and*

$$u(\xi,\lambda) = u_D(\xi,\lambda), \quad (\xi,\lambda) \in D;$$

c) *for every* $(\xi,\lambda) \in X \times \Lambda$ *the set* $\mathcal{H}(\xi,\lambda)$ *is a closed absolute retract;*

d) *if the multifunction* $\lambda \to F(t,x,\lambda)$ *has a closed graph a.e. on* T, *a retraction can be chosen which depends continuously on* (ξ,λ), *namely, there exists a continuous map* $h : X \times \Lambda \times C^*(T,X) \to C^*(T,X)$ *such that*

$$h(\xi,\lambda,u) \in \mathcal{H}(\xi,\lambda), \quad \forall u \in C^*(T,X),$$
$$h(\xi,\lambda,u) = u, \quad \forall u \in \mathcal{H}(\xi,\lambda).$$

COROLLARY 4.4. *Suppose that* (H1)-(H4) *hold. Then for every* $(\xi,\lambda) \in X \times \Lambda$ *the set* $\mathcal{H}(\xi,\lambda)$ *is the closed, arcwise connected subset of* $C^*(T,X)$ *and the multifunction* $(\xi,\lambda) \to \mathcal{H}(\xi,\lambda)$ *is lower semicontinuous.*

Proof of Theorem 4.2. For every $(\xi, \lambda) \in X \times \Lambda$, $\psi \in L_1(T, X)$ put

$$\Phi(\xi, \lambda, \psi) = \{\varphi \in L_1(T, X);\ \varphi(t) \in F(t, \eta(\xi, \psi)(t), \lambda)\}$$

a.e. in T, where

$$\eta(\xi, \psi)(t) = S(t, 0)\xi + \int_0^t S(t, s)\psi(s)ds.$$

Using well-known arguments we obtain that $\Phi(\xi, \lambda, \psi) \in \mathrm{dc}L_1(T, X)$.

Now, we prove that for every fixed $\lambda \in \Lambda$ the multifunction $\Phi(\xi, \lambda, \psi)$ is Lipschitzian from $X \times L_1(T, X)$ into $\mathrm{dc}L_1(T, X)$ with Lipschitz constant $r(\lambda)$, continuously depending on λ.

Fix $\lambda \in \Lambda$, $\psi_1, \psi_2 \in L_1(T, X)$, $(f_1, g_1), (f_2, g_2) \in X$, $\alpha \in \Phi(\xi_1, \lambda, \psi_1)$ and $\epsilon > 0$. Using properties of measurable multifunctions we get that there exists a measurable function $\beta : T \to X$ such that

$$\beta(t) \in F(t, \eta(\xi_2, \psi_2)(t), \lambda)$$

and

$$\|\alpha(t) - \beta(t)\| \leq d\left(\alpha(t), F(t, \eta(\xi_2, \psi_2)(t), \lambda)\right) + \epsilon$$

a.e. in T. Thanks to (H2) one has

$$\begin{aligned} &\|\alpha(t) - \beta(t)\| \\ &\leq L(\lambda)(t)\left(\|S(t, 0)\xi_1 - S(t, 0)\xi_2\| + \int_0^t \|S(t, s)\|\|\psi_1(s) - \psi_2(s)\|ds\right) + \epsilon \\ &\leq L(\lambda)(t) \cdot M\left(\|\xi_1 - \xi_2\| + \|\psi_1 - \psi_2\|_L\right) + \epsilon \end{aligned} \tag{4.9}$$

¿From (4.9) it follows

$$\begin{aligned} &\|\alpha - \beta\|_L \\ &\leq r(\lambda)\left(\|\xi_1 - \xi_2\| + \|\psi_1 - \psi_2\|_L\right) + \epsilon \cdot a. \end{aligned} \tag{4.10}$$

Therefore (4.10) implies that

$$\begin{aligned} &d_L\left(\Phi(\xi_1, \lambda, \psi_1), \Phi(\xi_2, \lambda, \psi_2)\right) \\ &\leq r(\lambda)\left(\|\xi_1 - \xi_2\| + \|\psi_1 - \psi_2\|_L\right) \end{aligned} \tag{4.11}$$

and, interchanging the roles of ξ_1, ψ_1 and ξ_2, ψ_2,

$$\begin{aligned} &d_L\left(\Phi(\xi_2, \lambda, \psi_2), \Phi(\xi_1, \lambda, \psi_1)\right) \\ &\leq r(\lambda)\left(\|\xi_1 - \xi_2\| + \|\psi_1 - \psi_2\|_L\right). \end{aligned} \tag{4.12}$$

¿From (4.11), (4.12) it follows that

$$d_H^L\left(\Phi(\xi_1,\lambda,\psi_1),\Phi(\xi_2,\lambda,\psi_2)\right) \\ \le r(\lambda)\left(\|\xi_1-\xi_2\|+\|\psi_1-\psi_2\|_L\right). \tag{4.13}$$

So, our claim is proved.

Let us show that for every fixed $\xi \in X$, $\psi \in L_1(T,X)$ the multifunction $\lambda \to \Phi(\xi,\lambda,\psi)$ is l.s.c. Fix $\lambda^* \in \Lambda$, $\varphi^* \in \Phi(\xi,\lambda^*,\psi)$ and a sequence$\{\lambda_n\}$ in Λ converging to λ^*.

Put $\alpha_n(t) = d(\varphi^*(t), F(t,\eta(\xi,\psi)(t),\lambda_n))$ for all $t \in T$, $n \ge 1$. From properties of measurable multifunctions it follows that each $\alpha_n(t)$ is measurable. Moreover, thanks to (H3), one has $\lim_{n\to\infty}\alpha_n(t) = 0$ a.e.in T. Now, observe that for almost every $t \in T$ one has

$$\alpha_n(t) \le d(\varphi^*(t), F(t,\theta_x,\lambda_n)) + d_H(F(t,\theta_x,\lambda_n), F(t,\eta(\xi,\psi)(t),\lambda_n)) \\ \le \|\varphi^*(t)\| + d(\theta_x, F(t,\theta_x,\lambda_n)) + L(\lambda_n)(t)\cdot M\left(\|\xi\|+\|\psi\|_L\right).$$

¿From this, taking into account (H4), it follows that the sequence $\{\alpha_n\}$ is uniformly integrable. Hence $\lim_{n\to\infty}\alpha_n = 0$ in $L_1(T,R)$. For every $n \ge 1$ there exists a measurable function $\varphi_n : T \to X$ such that $\varphi_n(t) \in F(t,\eta(\xi,\psi)(t),\lambda_n)$ and $\|\varphi_n(t)-\varphi^*(t)\| \le \alpha_n(t) + 1/n$ a.e.in T. So $\varphi_n \in \Phi(\xi,\lambda_n,\psi)$ for all $n \ge 1$ and $\{\varphi_n\}$ converges to φ^* in $L_1(T,X)$, as desired.

Fix $\lambda^* \in \Lambda$, $\xi^* \in X$, $\varphi^* \in \Phi(\xi^*,\lambda^*,\psi)$ and a sequence $\{\xi_n,\lambda_n\}$ in $X \times \Lambda$ converging to (ξ^*,λ^*). Thanks to (4.13) one has

$$d_L(\varphi^*,\Phi(\xi_n,\lambda_n,\psi)) \le d_L(\varphi^*,\Phi(\xi^*,\lambda_n,\psi)) + d_H^L(\Phi(\xi_n,\lambda_n,\psi),\Phi(\xi^*,\lambda_n,\psi)) \\ \le d_L(\varphi^*,\Phi(\xi^*,\lambda_n,\psi)) + r(\lambda_n)\|\xi^*-\xi_n\|.$$

Hence $\lim_{n\to\infty} d_L(\varphi^*,\Phi(\xi_n,\lambda_n,\psi)) = 0$. It means that for every $\psi \in L_1(T,X)$ the multifunction $(\xi,\lambda) \to \Phi(\xi,\lambda,\psi)$ is lower semicontinuous.

Consider for every $\lambda \in \Lambda$ the function

$$P(\lambda,\psi) = \int_T e^{-\int_0^t 2M\cdot L(\lambda)(s)ds}\|\psi(t)\|dt. \tag{4.14}$$

It is clear that the function $p(\lambda,t,x) = \exp(-\int_0^t 2M\cdot L(\lambda)(s)ds)\|x\|$ has the properties, indicated in Introduction, and function $P(\lambda,\psi)$ (4.14) satisfies an inequality similar to (1.2). Of course, the function $P(\lambda,\cdot)$ for every $\lambda \in \Lambda$ is a norm in $L_1(T,X)$, equivalent to the usual one. For every λ, $A, B \subset L_1(T,X)$ we denote by $d_L(\lambda)(x,A)$, $d_L(\lambda)(A,B)$, $d_H^L(\lambda)(A,B)$ the distance of a point x to the set A, the excess of A over B and the Hausdorff distance between A and B, where $d_L(\cdot)(\cdot,\cdot)$ is a metric, induced by norm (4.14).

Now, we prove that

$$d_H^L(\lambda)\left(\Phi(\xi,\lambda,\psi),\Phi(\xi,\lambda,\omega)\right)\leq \frac{1}{2}P(\lambda,\psi-\omega) \tag{4.15}$$

for every $\lambda\in\Lambda$, $\xi\in X$.

Fix $\lambda\in\Lambda$, $\xi\in X$, $\psi,\omega\in L_1(T,X)$, $\alpha\in\Phi(\xi,\lambda,\psi)$ and $\epsilon>0$. Then there exists a measurable function $\beta:T\to X$ such that

$$\beta(t)\in F(t,\eta(\xi,\omega)(t),\lambda)$$

and

$$\|\alpha(t)-\beta(t)\|\leq d\left(\alpha(t),F(t,\eta(\xi,\omega)(t),\lambda)\right)+\epsilon$$

a.e.in T. Thanks to (H2) one has

$$\|\alpha(t)-\beta(t)\|\leq L(\lambda)(t)\cdot M\cdot\int_0^t\|\psi(s)-\omega(s)\|ds+\epsilon$$

a.e. in T. Then

$$\begin{aligned}&\int_T e^{-\int_0^t 2M\cdot L(\lambda)(s)ds}\|\alpha(t)-\beta(t)\|dt\\ \leq&\int_T\left(e^{-\int_0^t 2M\cdot L(\lambda)(s)ds}\cdot L(\lambda)(t)\cdot M\cdot\int_0^t\|\psi(s)-\omega(s)\|ds\right)dt+\epsilon\cdot a.\end{aligned}\tag{4.16}$$

Integrating by parts one can estimate the right-hand side of (4.16) in the following way:

$$\begin{aligned}&\int_T\left(e^{-\int_0^t 2M\cdot L(\lambda)(s)ds}\cdot L(\lambda)(t)\cdot M\cdot\int_0^t\|\psi(s)-\omega(s)\|ds\right)dt\\ &\leq -\frac{1}{2}e^{-\int_0^t 2M\cdot L(\lambda)(s)ds}\cdot\int_0^t\|\psi(s)-\omega(s)\|ds\,\Big|_0^a\\ &\quad+\frac{1}{2}\int_T e^{-\int_0^t 2M\cdot L(\lambda)(s)ds}\cdot\|\psi(t)-\omega(t)\|dt.\end{aligned}\tag{4.17}$$

By adding (4.16), (4.17) we obtain that

$$P(\lambda,\alpha-\beta)\leq\frac{1}{2}P(\lambda,\psi-\omega)+\epsilon\cdot a.$$

¿From this it follows that

$$d_L(\lambda)\left(\Phi(\xi,\psi,\lambda),\Phi(\xi,\omega,\lambda)\right)\leq\frac{1}{2}P(\lambda,\psi-\omega)$$

and, interchanging the roles of ψ and ω

$$d_L(\lambda)\left(\Phi(\xi,\omega,\lambda)\Phi(\xi,\psi,\lambda)\right) \leq \frac{1}{2}P(\lambda,\psi-\omega).$$

So, inequality (4.15) is proved.

If the multifunction $\lambda \to F(t,x,\lambda)$ has a closed graph a.e. on T, then it is easy to prove that the multifunction $\lambda \to \Phi(\xi,\lambda,\psi)$ has a closed graph.

Let us show that the multifunction $(\xi,\lambda) \to \Phi(\xi,\lambda,\psi)$ has a closed graph. To this end, fix sequences $\{\xi_n,\lambda_n\}$ in $X\times\Lambda$ converging to (ξ^*,λ^*), a sequence $\{\alpha_n\}$, $\alpha_n \in \Phi(\xi_n,\lambda_n,\psi)$, converging to α^* a sequence $\{\epsilon_n\}$, converging to 0. Then there exist a sequence $\{\beta_n\}$, $\beta_n \in \Phi(\xi^*,\lambda_n,\psi)$ such that

$$\begin{aligned}\|\alpha_n - \beta_n\|_L &\leq d_L(\alpha_n, \Phi(\xi^*,\lambda_n,\psi)) + \epsilon_n \\ &\leq d_H^L(\Phi(\xi_n,\lambda_n,\psi),\Phi(\xi^*,\lambda_n,\psi)) + \epsilon_n.\end{aligned}$$

The latter and (4.13) imply

$$\|\alpha_n - \beta_n\|_L \leq r(\lambda_n)\|\xi_n - \xi^*\|_L + \epsilon_n.$$

¿From this it follows that $\lim_{n\to\infty}\beta_n = \alpha^*$. Since the multifunction $\lambda \to \Phi(\xi^*,\lambda,\psi)$ has a closed graph, we obtain that $\alpha^* \in \Phi(\xi^*,\lambda^*,\psi)$. Hence the multifunction $(\xi,\lambda) \to \Phi(\xi,\lambda,\psi)$ has a closed graph.

As the final result we obtain that the mutifunction $\Phi : X\times\Lambda\times L_1(T,X) \to L_1(T,X)$ satisfies all assumptions of Theorem 1.1 and Corollary 1.2.

For all $(\xi,\lambda) \in \Xi\times\Lambda$ we put

$$\mathrm{Fix}\Phi(\xi,\lambda) = \{\varphi \in L_1(T,X);\ \varphi \in \Phi(\xi,\lambda,\varphi)\}.$$

Taking into account Theorem 1.1 and Corollary 1.2 we have:

a') For all $(\xi,\lambda) \in \Xi\times\Lambda$ the set $\mathrm{Fix}\Phi(\xi,\lambda)$ is non-empty and there exists a continuous function $v : X\times\Lambda \to L_1(T,X)$ such that

$$v(\xi,\lambda) \in \mathrm{Fix}\Phi(\xi,\lambda); \tag{4.18}$$

b') if $D \subset X\times\Lambda$ is a closed subset of $X\times\Lambda$ and $v_D : D \to L_1(T,X)$ is a continuous function, $v_D(\xi,\lambda) \in \mathrm{Fix}\Phi(\xi,\lambda)$, $(\xi,\lambda) \in D$, then there exists a continuous function $v : X\times\Lambda \to L_1(T,X)$ such that (4.18) is true and $v(\xi,\lambda) = v_D(\xi,\lambda)$, $(\xi,\lambda) \in D$;

c') $(\xi,\lambda) \to \mathrm{Fix}\Phi(\xi,\lambda)$ is the closed-valued lower semicontinuous multifunction and for every $(\xi,\lambda) \in X\times\Lambda$ the set $\mathrm{Fix}\Phi(\xi,\lambda)$ is an absolute retract in space $L_1(T,X)$ and, consequently, arcwise connected.

d') if the multifunction $\lambda \to F(t,x,\lambda)$ has a closed graph a.e. on T, then a retraction can be chosen which depends continuously on (ξ,λ), namely, there exists a continuous map $v : X \times \Lambda \times L_1(T,X) \to L_1(T,X)$ such that

$$v(\xi,\lambda,z) \in \mathrm{Fix}\Phi(\xi,\lambda), \quad \forall z \in L_1(T,X),$$
$$v(\xi,\lambda,z) = z, \quad \forall z \in \mathrm{Fix}\Phi(\xi,\lambda).$$

Now, consider the operator $\mathcal{T} : X \times L_1(T,X) \to C^*(T,X)$ defined by putting

$$\mathcal{T}(\xi,\varphi)(t) = S(t,0)\xi + \int_0^t S(t,s)\varphi(s)ds, \ t \in T$$

for all $\varphi \in L_1(T,X)$ and all $\xi \in X$. Observe that

$$\mathcal{H}(\xi,\lambda) = \mathcal{T}\left(\xi, \mathrm{Fix}\Phi(\xi,\lambda)\right). \tag{4.19}$$

Thanks to Proposition 4.1 the operator $\mathcal{T} : X \times L_1(T,X) \to C^*(T,X)$ is a topological isomorphism. Now, Theorem 4.2 and Corollary 4.3 follows directly from (4.19) and a')-d'). This ends the proof of Theorem 4.2 and Corollary 4.3.

COROLLARY 4.5. *Let* (H1)-(H4) *be satisfied and the multifunction* $\lambda \to F(t,x,\lambda)$ *has a closed graph a.e. on* T. *For* $i = 1,2$ *let* $u_i : X \times \Lambda \to C^*(T,X)$ *be a continuous map such that* $u_i(\xi,\lambda) \in \mathcal{H}(\xi,\lambda)$ *for every* $(\xi,\lambda) \in X \times \Lambda$. *Then there exists a continuous map* $g : X \times \Lambda \times [0,1] \to C^*(T,X)$ *satisfying*

$$g(\xi,\lambda,0) = u_1(\xi,\lambda), \ g(\xi,\lambda,1) = u_2(\xi,\lambda), \quad \textit{for every } (\xi,\lambda) \in X \times \Lambda,$$
$$g(\xi,\lambda,\tau) \in \mathcal{H}(\xi,\lambda), \quad \textit{for every } (\xi,\lambda,\tau) \in X \times \Lambda \times [0,1].$$

Proof. Thanks to Theorem 4.2 d) there exists a continuous map $h : X \times \Lambda \times C^*(T,X) \to C^*(T,X)$ such that

$$h(\xi,\lambda,u) \in \mathcal{H}(\xi,\lambda), \quad \forall u \in C^*(T,X), \tag{4.20}$$
$$h(\xi,\lambda,u) = u, \quad \forall u \in \mathcal{H}(\xi,\lambda).$$

Define $g : X \times \Lambda \times [0,1] \to C^*(T,X)$ by

$$g(\xi,\lambda,\tau) = h(\xi,\lambda,(1-\tau)u_1(\xi,\lambda) + \tau u_2(\xi,\lambda)). \tag{4.21}$$

Clearly g is well defined, since $(1-\tau)u_1(\xi,\lambda) + \tau u_2(\xi,\lambda) \in C^*(T,X)$ for every $(\xi,\lambda,\tau) \in X \times \Lambda \times [0,1]$ and continuous as composition of continuous functions. Furthermore, from (4.19), (4.20) it follows that g is desired. This completes the proof.

Corollary 4.4 means that any two continuous selections of the multifunction $(\xi, \lambda) \to \mathcal{H}(\xi, \lambda)$ can be joined by a homotopy with values in $\mathcal{H}(\xi, \lambda)$.

5. Comments

(1) Theorem 3.2 is proved by using some ideas of [14].

(2) Theorem 1.1 is obtained in the standart way by using Theorem 3.2.

(3) Continuous selectors of fixed point sets of multifunctions with nonconvex values were studied in [15, 18]. Our result contains as a special case the selection theorem in [15] and supplements the result in [18].

(4) Proposition 4.1 is proved analogously to Proposition 2.1 in [10].

(5) The absolute retractness of fixed point sets for multivalued contraction with closed decomposable values were proved in [2] under severe constraints then ours.

(6) The problem (H) under severe constraints were studied in [10]. Theorem 4.2 contains more information about properties of the multifunction $(\xi, \lambda) \to \mathcal{H}(\xi, \lambda)$.

(7) It should be mentioned that the continuous selections and properties of solution sets for different classes of Lipschitzean differential inclusions were studied in [4, 5, 6, 7, 10, 11, 13, 19, 20, 21, 22]. The majority of these results can be obtained easily by using Theorem 1.1 and Corollary 1.2.

References

1. Antosievicz, A. and Cellina, A.: Continuous selectors and differential relations, J. Differ. Equat. 19 (1975), 386-398.
2. Bressan, A., Cellina, A. and Fryszkowski, A.: A class of absolute retracts in spaces of integrable functions, Proc. Amer. Math. Soc. 112 (1991), 413-418.
3. Bressan, A. and Colombo, G.: Extensions and selections of maps with decomposable values, Studia Math. 90 (1988), 69-86.
4. Cellina, A.: On the set of solutions to Lipschitzean differential inclusions, Diff. and Int. Equat. 1 (1988), 495-500.
5. Cellina, A. and Ornelas, A.: Representation of the attainable set for Lipschitzean differential inclusions, Rocky Mountain J. Math. 22 (1992), 117-124.
6. Cellina, A. and Staicu, V.: On evolution equations having monotonicities of opposite sign, J. Diff. Equat. 90 (1991), 71-80.
7. Colombo, R.M., Fryszkowski, A., Rzezuchowski, T. and Staicu, V.: Continuous selections of solution sets of Lipschitzean differential inclusions, Funkcial Ekvac. 34 (1991), 321-330.
8. Dunford, N. and Schwartz, T.: Linear operators, part 1, Interscience, New York, 1957.
9. De Blasi, F.S. and Piangiani, G.: Remarks on Hausdorff continuous multifunctions and selections, Comment. Math. Univ. Carolinae, 24 (1983), 553-561.
10. De Blasi, F.S., Piangiani, G. and Staicu, V.: Topological properties of nonconvex differential inclusions of evolution type, Centro mathematica V. Volterra Universita di Roma II, Preprint 116 (1992), 1-14.

11. De Blasi, F.S., Piangiani, G. and Staicu, V.: On the solution sets of some nonconvex hyperbolic differential inclusions, Centro mathematica V. Volterra Universita di Roma II, Preprint 117 (1992), 1-12.
12. Fryszkowski, A.: Continuous selections for a class of nonconvex multivalued maps, Studia Math. 76 (1983), 163-174.
13. Goncharov, V.V.: Some properties of viability problems depending on a parameter, NoDEA 2 (1995), 1-19.
14. Goncharov, V.V and Tolstonogov, A.A.: Mutial continuous selections of multifunctions with nonconvex values and its applications, Math. USSR Sbornik 182 (1991), 946-969.
15. Kannai, Z.: Contraction-selection theorem and perturbed Lipschitz inclusions, Pure Math. and Appl. 4 (1993), 479-491.
16. Kuratowski, K.: Topology, v.II, Academic Press, New-York and London. Panstwowe Wydawnictwo Naukove, Warszawa, 1966.
17. Marano, S.A.: Generalized solutions of partial differential inclusions depending on a parameter, Rend. Accad. Naz. Sci. XL Mem. Mat. 107 (1989), 281-285.
18. Marano, S.A.: Fixed points of multivalued contractions with nonclosed, nonconvex values, Rend. Mat. Acc. Lincei 5 (1994), 203-212.
19. Ornelas, A.: A continuous version of the Fillipov-Gronwall inequality for differential inclusions, Atti. Acad. Naz. Lincei Rend. Cl. Sei. Fis. Mat. Natur. 1 (1990), 105-110.
20. Staicu, V.: Continuous selections of solution sets to evolution equations, Proc. Amer. Math. Soc. 113 (1991), 403-413.
21. Staicu, V.: On a non-convex hyperbolic differential inclusion, Proc. Edinburgh Math. Soc. 35 (1992), 375-382.
22. Staicu, V. and He Wu: Arcwise connectedness of solution sets to Lipschitzean differential inclusions, Bolletino U.M.I. 5A (1991), 253-256.
23. Tolstonogov, A.A. and Tolstonogov, D.A.: L_p-continuous extreme selectors of multifunctions with decomposable values: Existence theorems, Set-Valued Analysis 4 (1996), 173-203.
24. Tolstonogov, A.A. and Tolstonogov, D.A.: L_p-continuous extreme selectors of multifunctions with decomposable values: Relaxation theorems, Set-Valued Analysis 4 (1996), 237-269.
25. Filippov, A.F.: Classical solutions of differential equations with multivalued right-hand side, Vestnik Moskow Univ. Matematika, Mechanika 7 (1967), 16-26.