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Some Criteria in Set-Valued Optimization*

黒岩 大史 (Daishi KUROIWA)†‡

Department of Mathematics and Computer Science,

Faculty of Science and Engineering,

Shimane University, Matsue 690, Japan.

Abstract: In this paper, we define several relations of two sets with respect to an ordering convex cone. By using the relations, we define some kinds of criteria of solutions in set-valued optimization. Moreover, we investigate conditions which guarantee any solution of local optimal is a solution of global optimal when a set-valued map is convex.

Key words: Set-valued analysis, convexity of set-valued maps, minimal solutions, minimum solutions, vector-valued analysis, optimization.

1. Introduction

How should be criteria of set-valued optimization problems defined? In optimization theory, we find a lot of set-valued optimization problems, for example, a duality problem of a vector-valued optimization problem, or a minimax problem of a vector-valued map, see [2, 3, 13, 11, 14]. However, the ordinary criteria of solutions, which is considered in such set-valued problems, is based on comparisons of two vectors, and we suspect, it may be a defect and it should be based on comparisons of two sets.

The end of this paper is to answer the question. We define some criteria of solutions of a set-valued minimization problem with comparisons of two images of the set-valued map in this paper. The organization of the paper is the following: In Section 2, we consider some concepts of comparisons of two sets with respect to a vector-ordering, and we give and observe six kinds of relations between two sets. Using the relations, in Section 3, we introduce some criteria of solutions, which are based on comparisons of two sets with a vector-ordering; [8, 9], in a set-valued optimization problem. In this reason, they are natural and simple criteria. Finally, we investigate some conditions which guarantee any local solution is a global solution if a set-valued map is convex in Section 4.

2. Relations between Two Sets with a Vector-Ordering

In this section, we discuss and construct concepts of comparisons of two nonempty sets with respect to a vector ordering. Throughout this paper, let Y be an ordered vector space with

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‡email: kuroiwa@riko.shimane-u.ac.jp

the vector ordering $\leq_K$ induced by a convex cone K: for $x, y \in Y$,

$$x \leq_K y \text{ if } y - x \in K. \tag{2.1}$$

First, for each vectors $a, b \in Y$, we have one of the following:

(i) $a \in b + K$ (equivalently $b \in a - K$); (iii) $b \in a + K$ (equivalently $a \in b - K$);

(ii) $a \notin b + K$ (equivalently $b \notin a - K$); (iv) $b \notin a + K$ (equivalently $a \notin b - K$).

These relationships are summarized as $b \leq_K a$, $b \not\leq_K a$ or $a \leq_K b$, $a \not\leq_K b$, that is, one vector is dominated by the other vector or otherwise. In the case of relationship between a nonempty set $A \subset Y$ and a vector $b \in Y$, a different situation is observed; we have two domination structure

(i) for all $a \in A$, $a \leq_K b$;

(ii) there exists $a \in A$ such that $a \leq_K b$.

The first relation means the vector b dominates the whole set A from above with respect to the vector ordering $\leq_K$. The second relation means the vector b is dominated from below by an element of the set A. If the set A is singleton, they are coincident with each other. These relationships are denoted by $b \in A \cap^{+} K$ and $b \in A \cup^{+} K$, respectively, where

$$A \cap^{+} K := \bigcap_{a \in A} (a + K) \quad \text{and} \quad A \cup^{+} K := \bigcup_{a \in A} (a + K). \tag{2.2}$$

Analogously, we use the following notations for a nonempty set $B \subset Y$:

$$B \cap^{-} K := \bigcap_{b \in B} (b - K) = B \cap^{+} (-K) \quad \text{and} \quad B \cup^{-} K := \bigcup_{b \in B} (b - K) = B \cup^{+} (-K). \tag{2.3}$$

It is easy to see that $A \cap^{+} K \subset A \cup^{+} K$ and $B \cap^{-} K \subset B \cup^{-} K$, and also that $A \cup^{+} B = A + B$ and $A \cup^{-} B = A - B$.

Secondly, we consider the relationship between two nonempty sets in Y, which is strongly concerned with intersection and inclusion in set theory. Given nonempty sets $A, B \subset Y$, exactly one of following conditions holds: (i) $A \cap B = \emptyset$; (ii) $A \cap B \neq \emptyset$. The latter case includes its special cases $A \subset B$ and $A \supset B$.

By using above two ideas, we classify the relationship between two nonempty sets $A, B \subset Y$ in the sense that A is (partially) dominated from above by B or A (partially) dominates B from below:

(i) $A \subset B \cap^{-} K$; (v) $A \cap^{+} K \supset B$;

(ii) $A \cap (B \cap^{-} K) \neq \emptyset$; (vi) $(A \cap^{+} K) \cap B \neq \emptyset$;

(iii) $A \cup^{+} K \supset B$; (vii) $A \subset B \cup^{-} K$;

(iv) $(A \cup^{+} K) \cap B \neq \emptyset$; (viii) $A \cap (B \cup^{-} K) \neq \emptyset$.

Since conditions (i) and (v) coincide and conditions (iv) and (viii) coincide, we define six kinds of classification for set-relationship.

DEFINITION 2.1. For nonempty sets $A, B \subset Y$, we denote

- $A \mathbin{\cap\!\!\!\!+} K \supset B$ by $A \leq_K^{(\mathrm{i})} B$;
- $(A \mathbin{\cap\!\!\!\!+} K) \cap B \neq \emptyset$ by $A \leq_K^{(\mathrm{iv})} B$;
- $A \cap (B \mathbin{\cap\!\!\!\!-} K) \neq \emptyset$ by $A \leq_K^{(\mathrm{ii})} B$;
- $A \subset B \mathbin{\cup\!\!\!\!-} K$ by $A \leq_K^{(\mathrm{v})} B$;
- $A \uplus K \supset B$ by $A \leq_K^{(\mathrm{iii})} B$;
- $(A \uplus K) \cap B \neq \emptyset$ by $A \leq_K^{(\mathrm{vi})} B$.

PROPOSITION 2.1. For nonempty sets $A, B \subset Y$,

$$A \leq_K^{(p)} B \text{ if and only if } -B \leq_K^{(q)} -A$$

for (p, q)=(i,i), (ii,iv), (iii, v), (vi, vi).

PROPOSITION 2.2. For nonempty sets $A, B \subset Y$, we denote the following statements hold:

- $A \leq_K^{(\mathrm{i})} B$ implies $A \leq_K^{(\mathrm{ii})} B$;
- $A \leq_K^{(\mathrm{i})} B$ implies $A \leq_K^{(\mathrm{iv})} B$;
- $A \leq_K^{(\mathrm{ii})} B$ implies $A \leq_K^{(\mathrm{iii})} B$;
- $A \leq_K^{(\mathrm{iv})} B$ implies $A \leq_K^{(\mathrm{v})} B$;
- $A \leq_K^{(\mathrm{iii})} B$ implies $A \leq_K^{(\mathrm{vi})} B$;
- $A \leq_K^{(\mathrm{v})} B$ implies $A \leq_K^{(\mathrm{vi})} B$.

We investigate some properties about set-relations.

PROPOSITION 2.3. For nonempty sets $A, B \subset Y$, the following statements hold:

- $A \leq_K^{(\mathrm{i})} B$ and $B \leq_K^{(\mathrm{i})} A$ implies $A = B = \{x\}$ for some x;
- $A \leq_K^{(\mathrm{ii})} B$ and $B \leq_K^{(\mathrm{ii})} A$ implies $\mathrm{Sup}_K A = \mathrm{Sup}_K B$;
- $A \leq_K^{(\mathrm{iii})} B$ and $B \leq_K^{(\mathrm{iii})} A$ implies $\mathrm{Min}_K A = \mathrm{Min}_K B$;
- $A \leq_K^{(\mathrm{iv})} B$ and $B \leq_K^{(\mathrm{iv})} A$ implies $\mathrm{Inf}_K A = \mathrm{Inf}_K B$;
- $A \leq_K^{(\mathrm{v})} B$ and $B \leq_K^{(\mathrm{v})} A$ implies $\mathrm{Max}_K A = \mathrm{Max}_K B$.

3. Criteria of a Set-Valued Minimization Problem

Now, we consider some criteria of solutions of a set-valued minimization problem. We set our set-valued minimization problem as follows:

$$\text{(P)} \quad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & x \in S, \end{array}$$

where F is a set-valued map from a nonempty set X to Y and S is a nonempty subset of X satisfying $F(x) \neq \emptyset$ for each $x \in S$.

First, to consider criteria of the solutions, we recall some concept, which are well known and play important roles in vector-valued optimization, as follows:

- Minimal or Maximal;
  Let A be a nonempty subset of Y. An element a of A is said to be **minimal** in A (with ordering cone K) if any element x of A with $x \leq_K a$ satisfies $x = a$. The set of all minimal elements in A is called minimal set of A, which is written by $\mathrm{Min}_K A$. Concept of maximal is considered as minimal with ordering cone $-K$, and the maximal set of A is written by $\mathrm{Max}_K A$.

- Minimum or Maximum;
  Let A be a nonempty subset of Y. An element a of A is said to be **minimum** in A (with ordering cone K) if $a \leq_K x$ for each element x of A. Such an element does not always exist, however, if it exists, it is unique and it is the only $\mathrm{Min}_K A$ element. Concept of maximum is considered as minimum with ordering cone $-K$.

- Infimum or Supremum (see, [4, 9]);
  Let A be a nonempty subset of Y. An element a of A is said to be **infimum** in A (with ordering cone K) when a is a maximum element of $\cap_{a \in A}(a - K)(= A \mathbin{\cap\!\!\!\!-} K)$, and the set of all infimum element in A is written by $\mathrm{Inf}_K A$. Such an element of $\mathrm{Inf}_K A$ does exist under some weak conditions, see [9], and if it exists, it is also unique. An element a of A is said to be **supremum** in A (with ordering cone K), if a is a minimum element of $\cap_{a \in A}(a + K)(= A \mathbin{\cap\!\!\!\!+} K)$, and the set of all supremum element in A is written by $\mathrm{Sup}_K A$.

In a vector-valued optimization problem, a criterion of solutions is presented by one of the above six concepts and to find such solutions is the end of the problem. For example, $x_0 \in S$ is a minimal solution on S of a map from X to Y if $f(x_0)$ is an element of $\mathrm{Min}_K f(S)$. In almost literatures concerned with set-valued optimization, a criterion of solutions is defined the following: $x_0 \in S$ is a (minimal) solution of (P) if there exists $y_0 \in F(x_0)$ such that y_0 is an element of $\mathrm{Min}_K f(S)$, or $y_0 \leq_K y$ for each $y \in F(x)$ with $y \leq_K y_0$, and $x \in \mathbf{Dom}(F)$. From this, we can see the criterion of solutions is based on comparison of a vector y_0 and a set $f(S)$, or comparison of two vectors y_0 and y. In set-valued optimization, criterion of solutions should be based on comparisons of two sets!

Now, we introduce some criteria of solutions in set-valued optimization. These criteria of solutions are based on comparisons of two images of F with the ordering cone K. Though we can define various types of such concepts, we define only four as follows:

DEFINITION 3.2. Let $k =$ i, ii, ..., iv. A vector $x_0 \in S$ is said to be

- a type (k) **minimum** solution of (P) if for each $x \in S$ which satisfies $F(x_0) \neq F(x)$, $F(x) \leq_K^{(k)} F(x_0)$;

- a type (k) **minimal** solution of (P) if for each $x \in S$ which satisfies $F(x) \leq_K^{(k)} F(x_0)$, $F(x_0) \leq_K^{(k)} F(x)$;

- a type (k) **maximum** solution of (P) if for each $x \in S$ which satisfies $F(x) \neq F(x_0)$, $F(x_0) \leq_K^{(k)} F(x)$;

- a type (k) **maximal** solution of (P) if for each $x \in S$ which satisfies $F(x_0) \leq_K^{(k)} F(x)$, $F(x) \leq_K^{(k)} F(x_0)$.

In this paper, concepts of infimum and supremum in set-valued version are omitted since they are more complicated.

REMARK 3.1. When F is a single-valued map, that is $F(x)$ is a singleton for all $x \in \mathbf{Dom}(F)$, these concepts in Definition 3.2. are equivalent to vector-valued version each other.

PROPOSITION 3.4. For each k =, i, ii, ..., vi, if x_0 is a type (k) minimum solution of (P) then x_0 is a type (k) minimal solution of (P).

4. Convexity of Set-Valued Maps and Optimality

In this section, we investigate some conditions which guarantee each local solution is a global solution when a set-valued map is convex. We begin to define concepts of convexity of set-valued maps. The rest of this paper, let F be a set-valued maps from a topological vector space X to the ordered space Y.

DEFINITION 4.3. Let k =i, ii, ..., iv. F is said to be

- type (k) **convex** if for each x_1, $x_2 \in \mathrm{Dom}(F)$ and $\lambda \in (0,1)$,

$$F((1-\lambda)x_1 + \lambda x_2) \leq_K^{(k)} (1-\lambda)F(x_1) + \lambda F(x_2);$$

- type (k) **concave** if for each x_1, $x_2 \in \mathrm{Dom}(F)$ and $\lambda \in (0,1)$,

$$(1-\lambda)F(x_1) + \lambda F(x_2) \leq_K^{(k)} F((1-\lambda)x_1 + \lambda x_2),$$

where $\mathrm{Dom}(F) = \{x \in X | F(x) \neq \emptyset\}$.

PROPOSITION 4.5. For (p,q)=(i,i), (ii,iv), (iii, v), (vi, vi), F is type (p) convex and $-F$ is type (q) concave are equivalent.

DEFINITION 4.4. An element x_0 is a type (k) local minimal (resp. minimum, maximal, maximum) solution of (P) if there exists a neighborhood of x_0 such that x_0 is a type (k) minimal (resp. minimum, maximal, maximum) solution on the neighborhood.

From the definitions, we have the following results.

THEOREM 4.1. The following statements hold:

(i) if F is a type (k) convex set-valued map then each solution of type (k) local minimum becomes a solution of type (k) global minimum, for k=i, ii;

(ii) if F is a type (iii) convex set-valued map and $F(x) + K$ is closed convex for each $x \in X$ then each solution of type (iii) local minimum becomes a solution of type (iii) global minimum.

THEOREM 4.2. The following statements hold:

(i) if F is a type (k) convex set-valued map then each solution of type (k) local minimal becomes a solution of type (k) global minimal, for k=i, ii;

(ii) if F is a type (iii) convex set-valued map and $F(x) + K$ is a closed convex for each $x \in X$ then each solution of type (iii) local minimal become a solution of type (iii) global minimal.

COROLLARY 4.1. The following statements hold:

(i) if F is a type (k) concave set-valued map then each solution of type (k) local maximum (resp. maximal) become a solution of type (k) global maximum (resp. maximal) for k=i, iv;

(ii) if F is a type (v) concave set-valued map and $F(x) - K$ is a closed convex for each $x \in X$ then each solution of type (v) local maximum (resp. maximal) become a solution of type (v) global maximum (resp. maximal).

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