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Continuity of ε-Approximate Solution Set

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Abstract

In this note, we present the continuity of ε-approximate solutions set for the nonlinear progrmming problems. In [1], the similar continuity for the unconstrained problem was shown. We show another result. The ocntinuity of the approximate solution set is estimated by using the ρ-distance.

1 Preliminaries

In this note, we consider the following nonlinear programming problem:

(P) minimize $f(x)$

subject to $g(x)\leq 0$

where $g=(g_{1},\ldots,g_{m})$, f and $g_{i}(i=1,\ldots,m):\boldsymbol{R}^{n}\rightarrow\boldsymbol{R}$.

We denote the feasible set $\{x\in\boldsymbol{R}^{n}\mid g(x)\leq 0\}$ by K.

We suppose that the following assumption is satisfied.

Assumption. Let f and $g_{i}(i=1,\ldots,m)$ be convex and f be bounded from below. Let $K\neq\emptyset$. The parameter ε is positive.

For the problem (P), the ε -approximate sulution is well known as follows.

Definition 1.1. An element $\bar{x}\in K$ is said to be an ε-approximate solution for (P) if and only if $\bar{x}$ satisfies that $f(x)+\varepsilon\geq f(\bar{x})$ for any $x\in K$.

We set $inf_{K}f=inf\{f(x)\mid x\in K\}$ and denote theε-approximate solution set $\{\bar{x}\in K\mid f(x)+\varepsilon\geq f(\bar{x})$ for any $x\in K\}$ by $A(\varepsilon)$.

Clearly, we have $A(\varepsilon)\neq\emptyset$ under the above assumption.

To estimate the approximate solution set, we define the ρ-distance and the Hausdorff distance:

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Definition 1.2. For $C \subset \boldsymbol{R}^n$,

$$d(x, C) = inf\{\| x - y \| | y \in C\}$$

denotes the distance from x to C. For any $C, D \subset \boldsymbol{R}^n, \rho \geq 0$, we set

$$C_\rho = C \cap \rho B$$

where $B = \{x \in \boldsymbol{R} | \| x \| \leq 1\}$:unit ball.
For, $\rho \geq 0$, the ρ-distance is defined to be

$$d_\rho(C, D) = max\{e(C_\rho, D), e(D, C_\rho)\}$$

where $e(C, D) = sup_{x \in C} d(x, D)$, and the Hausdorff distance between C and D is

$$haus(C, D) = max\{e(C, D), e(D, C)\}.$$

2 The unconstrained case

In this section we introduce the result of [1]. In [1], Attouch and Wets investigated the Lipschitz continuity of the approximate solution set for the unconstrained programming problems. The problem is as follows:

minimize $F(x)$ where $F : \boldsymbol{R}^n \to \boldsymbol{R}$.

For this problem, the approximate solution set is defined to be

$$\varepsilon - argminF = \{\bar{x} \mid infF + \varepsilon \geq F(\bar{x})\}$$

where $infF = inf\{F(x) \mid x \in \boldsymbol{R}\}$. Also, we denote level set of a function f by

$$lev_\alpha F = \{x \in \boldsymbol{R}^n \mid F(x) \leq \alpha\}.$$

To show the Lipschitz continuity, the important lemma was proved in [1].

Lemma 2.1. [1, Lemma4.1.] Suppose that there exists $\rho_o > 0$ such that

$$(\varepsilon - argminF)_{\rho_0} \neq \emptyset \text{ for all } \varepsilon > 0.$$

Then, for all $\alpha > infF$ and $\eta \geq 0$,

$$\text{for all } \hat{x} \in lev_{(\alpha+\eta)}F, \;\; d(\hat{x}, lev_\alpha F) \leq \eta \frac{\| \hat{x} \| + \rho_0}{(\eta + \alpha) - infF}$$

which in turn implies that for all $\rho \geq \rho_0$,

$$d_\rho((\alpha + \eta) - argminF, \alpha - argminF) \leq \eta \frac{\rho_0 + \rho}{(\eta + \alpha) - infF}.$$

3 The constrained case

We apply lemma 2.1. to the constrained programming problems (P).

Lemma 3.1. Suppose that there exists $\rho_o > 0$ such that

$$A(\varepsilon)_{\rho_0} \neq \emptyset \text{ for all } \varepsilon > 0.$$

Then, for all $inf_K f + \varepsilon > inf_K f$,

$$\text{for all } \hat{x} \in A(\varepsilon_2), d(\hat{x}, A(\varepsilon_1)) \leq (\varepsilon_2 - \varepsilon_1)\frac{\| \hat{x} \| + \rho_0}{\varepsilon_1}$$

which in turn implies that for all $\rho \geq \rho_0$,

$$d_\rho(A(\varepsilon_2), A(\varepsilon_1)) \leq (\varepsilon_2 - \varepsilon_1)\frac{\rho_0 + \rho}{\varepsilon_1}.$$

However the above assumption does not hold in the following easy example.

Example 3.1. Let $f(x_1, x_2) = 2^{x_1 + x_2}$:convex and $g(x_1, x_2) = (x_1, x_2)$:convex .
Then, we have

$$A(0) = \emptyset \text{ and } A(\varepsilon) = \{x \mid x \leq 0 \text{ and } 2^{x_1 + x_2} \leq \varepsilon\}.$$

So, it holds

$$\| \bar{x} \| \rightarrow +\infty \text{ where } \bar{x} \in A(\varepsilon) \text{ as } \varepsilon \rightarrow 0.$$

We would like to change the assumption and show the similar result.

Proposition 3.1. We suppose that the strong Slater condition is satisfied.i.e. there are $x_s \in \boldsymbol{R}^n, \delta > 0$ such that

$$\delta \tilde{B} \subset H(x_s) + \boldsymbol{R}_+^{(m+1)}.$$

where $H(x) = (g(x), f(x) - inf_K f - \varepsilon_1), \tilde{B} \subset \boldsymbol{R}_+^{(m+1)}$: unitball.
Also, suppose there exists $C > 0$ such that

$$sup_{x_0 \in A(\varepsilon_2) \setminus A(\varepsilon_1)} \| x_0 - x_s \| \leq C.$$

Then, we have

$$(A(\varepsilon_2), A(\varepsilon_1)) \leq \frac{(\varepsilon_2 - \varepsilon_1)C}{\delta}.$$

Remark. The assumption of proposition 3.1. is satisfied in example 3.1. Let $\varepsilon_2 = 0.5, \varepsilon_1 = 0.25$. So, there exist $x_s = (-2,-2)$ and $\delta = 0.125$ such that the strong Slater condition is satisfied. Since $A(\varepsilon_2) = \{x \mid x \leq 0, x_2 \leq -x_1 - 1\}, A(\varepsilon_1) = \{x \mid x \leq 0, x_2 \leq -x_1 - 2\}$, we have
$sup_{x_0 \in A(\varepsilon_2)\setminus A(\varepsilon_1)} \parallel x_0 - x_s \parallel \leq \parallel (-1,0) - (-2,-2) \parallel = \sqrt{5}$ and $haus(A(0.5), A(0.25)) \leq \frac{(0.5-0.25)\sqrt{5}}{0.125}$.

The above strong Slater condition is equivalent to the ordinary one.

Proposition 3.2. [9]The strong Slater condition is satisfied if and only if the Slater condition be done.

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