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NULL CONTROLLABILLITY OF THE INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACE

JONG YEOUL PARK, MI JIN LEE AND HYO KEUN HAN
Department Mathematics, Pusan National University, Pusan 609-735, Korea

1.Introduction.

Controllability of linear and nonlinear systemes represented by ordinary differential equations in finite-dimension space has been extensively studied. Several authors have extended the concept to infinite-dimension systems represented by evolution equations with bounded operators in Banach spaces(Ref.[4]) for Volterra integro differential systems, Park and Kwun(Ref.[3]) studied the approximate controllability for delay Volterra systems with bounded linear operators in Banach space. Recently, Balachandran, Balasubramaniam and Dauer (Ref.[1]) studied the Local null controllability of nonlinear functional differential systems with unbounded linear operators in Banach space. In this paper, we study the Local null controllability of nonlinear functional differential systems (1) with unbounded linear operators in Banach space. The main tools employed in our analysis are based on the semigroup theory, fractional power operators and Schauder's fixed point theorem. The main result is presented in Section 3 and example is given in Section 4.

2.Preliminaries.

Let X be a Banach space with norm $\|\cdot\|$ and let $C = C([-r,0],X)$ be the Banach space of continuous functions defined on $[-r,0], r > 0$ with supremum norm $\|\cdot\|_C$. If x is continuous function from $[-r,T], T > 0$ to X and $t \in [0,T] = J$, then x_t denotes the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

We consider the functional Integro-differential systems

$$\begin{aligned} \frac{d}{dt}x(t) + A(t)x(t) =&(Bu)(t) + \int_0^t \Big(a(t,s)g(s,x_s) + h(t,s,x_s)\Big)ds \\ &+ f(t,x_t), \quad t \in [0,T] = J \ , \\ x(t) =&\phi(t), \quad t \in [-r,0] \ . \end{aligned} \tag{1}$$

where the state $x(t)$ takes values in the Banach space X and the control function u is given in $L^2(J,U)$, a Banach space of admissible control function with U a

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Banach space. The family $\{A(t) : t \in J\}$ of unbounded linear operators defined on domains $D(A) \subset X$ generates a linear evolution systems, B is a bounded linear operator from U into X, f, g are continuous nonlinear operator on $J \times C$ into X, h is continuous nonlinear operator from $J \times J \times C$ into X, and $\phi \in C = C([-r, 0]; X)$. For the existence of a solution of (1), we need the following assumptions(see Ref.[2]):

(H_1) $A(t)$ is a closed linear operator with a domain $D(A), t \in [0, T]$, that is dense in the Banach space X and independent of t.

(H_2) For each $t \in [0, T]$, the resolvent $R(\lambda, A(t)) = (\lambda - A(t))^{-1}$ of $A(t)$ exists for all λ with $Re\lambda \le 0$ and

$$\|R(\lambda, A(t))\| \le C/(|\lambda| + 1).$$

(H_3) For any $t, s, \tau \in [0, T]$, there exist $0 < \delta < 1$ and $K > 0$ such that

$$\|(A(t) - A(\tau))A^{-1}(s)\| \le K|t - \tau|^{\delta}.$$

(H_4) For any $t \in J$ and some $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, $R(\lambda, A(t))$, is a compact operator.

Conditions (H_1) $-$ (H_3), imply that for each $t \in [0, T]$, the integral

$$A^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} s^{\alpha-1} e^{-sA(t)} ds \tag{2}$$

exists for each $\alpha \in (0, 1]$. The operator (2) is bounded linear operator such that $A^{-\alpha}(t)A^{-\beta}(t) = A^{-(\alpha+\beta)}(t)$. The operator $A^{\alpha}(t) = (A^{-\alpha}(t))^{-1}$ is a closed linear operator with domain $D(A^{\alpha}(t))$ dense in X and such that $D(A^{\alpha}(t)) \subset D(A^{\beta}(t))$, if $\alpha \ge \beta$. $D(A^{\alpha}(t))$ is a Banach space with the norm $\|x\|_{\alpha} = \|A^{\alpha}(t)x\|$, which is denoted by $X_{\alpha}(t)$. Then, the following estimates hold (Ref.[2]):

$$\begin{aligned}\|A^{\nu}(t)A^{-\beta}(\tau)\| &\le K(\beta, \nu)[\|A(t)A^{-1}(\tau)\|]^{\nu} \\ &\le K(\beta, \nu)[K|t - \tau|^{\delta} + 1]^{\nu} \\ &\le K(\beta, \nu)\bar{K},\end{aligned}$$

where $\bar{K} = [1 + 2KT^{\delta}]^{\nu}$ and $0 \le \tau, t \le T, 0 \le \nu < \beta \le 1$. For each $t_0 \in J$, consider the space $C_{\alpha} = C([-r, 0]; X_{\alpha}(t_0))$ with the norm

$$\|\phi\|_{C_\alpha} = \sup_{-r \le \theta \le 0} \|A^{\alpha}(t_0)\phi(\theta)\|.$$

(H_5) Let $b_1, b_3 : J \to R^+, b_2 : J \times J \to R^+$ be continuous functions such that

$$\begin{aligned}&\|g(t, \phi) - g(t, \bar{\phi})\| \le b_1(t)\|\phi - \bar{\phi}\|_{C_\alpha} \ , \\ &\|h(t, s, \phi) - h(t, s, \bar{\phi})\| \le b_2(t, s)\|\phi - \bar{\phi}\|_{C_\alpha} \ , \\ &\|f(t, \phi) - f(t, \bar{\phi})\| \le b_3(t)\|\phi - \bar{\phi}\|_{C_\alpha} \ , \\ &g(t, 0) = 0, \ \ h(t, s, 0) = 0, \ \ f(t, 0) = 0\end{aligned}$$

for $t, s \in J, \phi, \bar{\phi} \in C_{\alpha}$

(H_6) The function $a(t, s)$ is Hölder continuous with exponent α i.e., there exists a positive constant a_0 such that

$$|a(t_1, s_1) - a(t_2, s_2)| \le a_0(|t_1 - t_2|^{\alpha} + |s_1 - s_2|^{\alpha})$$

for $t_1, t_2, s_1, s_2 \in J, \ 0 < \alpha \le 1$.

Let f, g are continuous nonlinear operator on $J \times C_\alpha$ into X, h is continuous nonlinear operator on $J \times J \times C_\alpha$ into X. Then, with the conditions $(H_1)-(H_6)$, there exist a continuous function $x : [-r, T] \to D(A^\alpha(t_0))$ such that

$$\begin{aligned} x(t) =& W(t,0)\phi(0) + \int_0^t W(t,s)\Big[(Bu)(s) + \int_0^s \Big(a(s,\tau)g(\tau,x_\tau) \\ & + h(s,\tau,x_\tau)\Big)d\tau + f(s,x_s)\Big]ds, \quad t \in J \ , \\ x(t) =& \phi(t), \quad t \in [-r, 0] \ , \end{aligned} \tag{3}$$

where $\{W(t,s) : 0 \le s \le t \le T\}$ is the linear evolution system generated by $A(t)$. Note that the solution exists only locally (Ref.[6]). Statements $(H_1) - (H_6)$ imply that there exists a family of bounded linear operators

$$\{Z(t,s) : 0 \le s \le t \le T\}$$

with

$$\|Z(t,s)\| \le C|t-s|^{\delta-1}$$

and such that the operator-valued function $W(t,\tau)$ can be defned for $0 \le \tau \le t \le T$ by

$$W(t,\tau) = e^{-(t-\tau)A(\tau)} + \int_\tau^t e^{-(t-s)A(s)}Z(s,\tau)ds \ .$$

Here, the linear opeators

$$\{e^{-\tau A(t)}; \tau \ge 0\}$$

form an analytic semigroup generated by $-A(t)$.

The family of linear operators

$$\{W(t,\tau); 0 \le \tau \le t \le T\}$$

is strongly jointly continuous in τ, t and maps X into $D(A)$ if $t > \tau$.

Further, it satisfies the following relations;

$$\begin{aligned} &(\partial/\partial t)W(t,\tau) = -A(t)W(t,\tau), \quad t \in (\tau, T], \\ &W(\tau,\tau) = I, \\ &\|e^{-tA(\tau)}\| \le K, \quad t,\tau \in [0,T], \\ &\|A(\tau)e^{-tA(\tau)}\| \le (K/t), \quad t,\tau \in [0,T], \\ &\|A(t)W(t,\tau)\| \le (K/|t-\tau|), \quad 0 \le \tau \le t \le T, \\ &\|A^\beta(t)e^{-\tau A(t)}\| \le (K(\beta)/\tau^\beta)e^{-\omega\tau}, \ t > 0, \ \beta \ge 0, \ \omega > 0, \\ &\|A^\beta(t)W(t,\tau)\| \le K(\beta)|t-\tau|^{-\beta}, 0 < \beta < 1 + t, \ \text{for some } t > 0. \end{aligned} \tag{4}$$

Finally, assumption (H_4) implies that $A^{-\beta}(t)$ is compact for all $\beta > 0$ and that the inclusion $X_\alpha(t) \subset X_\beta(t)$ is compact for $\alpha > \beta \ge 0$. The results given above for semigroups of linear operators, evolution systems and fractional powers of operators can be found in Friedman (Ref.[2]) and Pazy (Ref.[5]).

Definition 2.1. *The system (1) is said to be locally null controllable on the interval [0,T], if for every continuous initial function $\phi \in C$, there exists a control $u \in L^2([0,T],U)$ such that the local solution $x(t)$ of system (1) satisfies $x(T) = 0$.*

3. Main Result.

Theorem 3.1. *If conditions* $(H_1) \sim (H_6)$ *hold and the linear operator* V *from* U *into* X, *given by*

$$Vu = \int_0^T W(T,s)Bu(s)ds$$

defines an invertible operator V^{-1} *on* $L^2([0,T];U)/\ker V$ *such that there exist positive constants* N_1, N_2 *satisfying*

$$\|B\| \le N_1, \quad \|V^{-1}\| \le N_2,$$

then the system (1) is locally null controllable on J.

Proof. Using the hypothesis, define the control

$$u(t) = -V^{-1}\Big[W(T,0)\phi(0) + \int_0^T W(T,s)\Big\{\int_0^s \Big(a(s,\tau)g(\tau,x_\tau) + h(s,\tau,x_\tau)\Big)d\tau + f(s,x_s)\Big\}ds\Big](t) \ .$$

Now, it is shown that, when using this control, the operator defined by

$$\begin{aligned}
(\Phi x)(t) &= \phi(t), \quad t \in [-r,0], \\
(\Phi x)(t) &= W(t,0)\phi(0) \\
&\quad - \int_0^t W(t,\eta)BV^{-1}\Big[W(T,0)\phi(0) + \int_0^T W(T,s)\Big\{\int_0^s \Big(a(s,\tau)g(\tau,x_\tau) \\
&\quad + h(s,\tau,x_\tau)\Big)d\tau + f(s,x_s)\Big\}ds\Big](\eta)d\eta \\
&\quad + \int_0^t W(t,s)\Big\{\int_0^s \Big(a(s,\tau)g(\tau,x_\tau) + h(s,\tau,x_\tau)\Big)d\tau \\
&\quad + f(s,x_s)\Big\}ds, \quad t \in J,
\end{aligned}$$

has a fixed point. This fixed point is a solution of equation (1). Clearly, $(\Phi x)(T) = 0$, which means that control u steers the nonlinear functional differential system from the initial function ϕ to 0 in time T provided we can obtain a fixed point of the nonlinear operator Φ.
Let

$$B_c = \{\psi \in C_\alpha ; \|\psi\|_{C\alpha} \le c\} \ ,$$

where c is a constant. It is easy to observe from hypotheses $(H_5), (H_6)$ there exists a constant N_3 such that $|a(t,s)| \le N_3, \ t,s \in J$.

Since $b_1(t), b_2(t,s)$ and $b_3(t)$ are continuous on their compact domains, there exist constants $P_i \ge 0$, (i=1,2,3) such that $|b_1(t)| \le P_1, |b_2(t,s)| \le P_2$ and $|b_3(t)| \le P_3$. By virtue of the continuity of functions g, h, f, there exist constants K_1, K_2

and K_3 such that $\|g(\tau, x_\tau)\| \le K_1, \|h(s,\tau,x_\tau)\| \le K_2$ and $\|f(s,x_s)\| \le K_3$ for $\tau, s \in J,\ x_\tau \in B_c$. Define the function $\bar{\phi} \in C([-r,T]; X_\alpha(t_0))$ by

$$\bar{\phi}_0 = \phi \ ,$$
$$\bar{\phi}(t) = W(t,0)\phi(0), \quad t \in J \ .$$

Choose $d < c$ so that

$$K(\beta,\alpha)\bar{K}K(\beta)(T/1-\beta)((N_3K_1 + K_2)T + K_3)$$
$$\times \{N_1N_2K(\beta,\alpha)\bar{K}K(\beta)(T/1-\beta)+1\}$$
$$\le d,$$

where

$$\|\bar{\phi}_t\| \le c - d, \quad t \in J \ .$$

Define

$$Y_0 = \{x \in C([-r,T]; X_\alpha(t_0)); x_0 = 0, \|x_t\|_{C\alpha} \le d, t \in J\}.$$

Then for any $x \in Y_0$, we get $\|g(\tau, \bar{\phi}_\tau + x_\tau)\| \le K_1, \|h(s,\tau,\bar{\phi}_\tau + x_\tau)\| \le K_2$ and $\|f(s, \bar{\phi}_s + x_s)\| \le K_3$, for $\tau, s \in J$ and $x_\tau, x_s \in B_c$, because

$$\|x_\tau + \bar{\phi}_\tau\| \le \|x_\tau\| + \|\bar{\phi}_\tau\| \le d + c - d = c \ .$$

Consider the transformation

$$S : Y_0 \to C([-r,T]; X_\alpha(t_0))$$

defined by

$$(Sx)_0 = 0 \ ,$$
$$(Sx)(t) = -\int_0^t W(t,\eta)BV^{-1}\Big[\int_0^T W(T,s)\Big\{\int_0^s \Big(a(s,\tau)g(\tau,\bar{\phi}_\tau + x_\tau)$$
$$+ h(s,\tau,\bar{\phi}_\tau + x_\tau)\Big)d\tau + f(s,\bar{\phi}_s + x_s)\Big\}ds\Big](\eta)d\eta$$
$$+ \int_0^t W(t,s)\Big\{\int_0^s \Big(a(s,\tau)g(\tau,\bar{\phi}_\tau + x_\tau) + h(s,\tau,\bar{\phi}_\tau + x_\tau)\Big)d\tau$$
$$+ f(s,\bar{\phi}_s + x_s)\Big\}ds, \quad t \in J.$$

Finding a fixed point of S, and thus proving the theorem, is equivalent to finding a fixed point of Φ, and hence the solution (3) for the system (1). It is claimed that $S : Y_0 \to Y_0$. Since $(Sx)_0 = 0$ and

$$\|(Sx)(t)\|_\alpha \le \int_0^t \|A^\alpha(t_0)W(t,\eta)BV^{-1}\Big[\int_0^T W(T,s)\Big\{\int_0^s \Big(a(s,\tau)$$
$$\times g(\tau,\bar{\phi}_\tau + x_\tau) + h(s,\tau,\bar{\phi}_\tau + x_\tau)\Big)d\tau$$
$$+ f(s,\bar{\phi}_s + x_s)\Big\}ds\Big](\eta)\|d\eta$$

$$+\int_0^t \|A^\alpha(t_0)W(t,s)\Big\{\int_0^s\Big(a(s,\tau)g(\tau,\bar{\phi}_\tau+x_\tau)$$
$$+h(s,\tau,\bar{\phi}_\tau+x_\tau)\Big)d\tau+f(s,\bar{\phi}_s+x_s)\Big\}\|ds$$
$$\le N_1N_2\int_0^t \|A^\alpha(t_0)A^{-\beta}(t)A^\beta(t)W(t,\eta)\|$$
$$\times\Big[\int_0^T \|A^\alpha(t_0)A^{-\beta}(T)\|\|A^\beta(T)W(T,s)\|$$
$$\times((N_3K_1+K_2)T+K_3)ds\Big](\eta)d\eta$$
$$+\int_0^t \|A^\alpha(t_0)A^{-\beta}(t)\|\|A^\beta(t)W(t,s)\|((N_3K_1+K_2)T+K_3)ds$$
$$\le N_1N_2K(\beta,\alpha)\bar{K}K(\beta)\int_0^t |t-\eta|^{-\beta}\Big[K(\beta,\alpha)\bar{K}K(\beta)$$
$$\times((N_3K_1+K_2)T+K_3)\int_0^T |T-s|^{-\beta}ds\Big](\eta)d\eta$$
$$+K(\beta,\alpha)\bar{K}K(\beta)((N_3K_1+K_2)T+K_3)\int_0^t |t-s|^{-\beta}ds$$
$$\le K(\beta,\alpha)\bar{K}K(\beta)(T/1-\beta)((N_3K_1+K_2)T+K_3)$$
$$\times\{N_1N_2K(\beta,\alpha)\bar{K}K(\beta)(T/1-\beta)+1\}$$
$$\le d,$$

we obtain

$$\|(Sx)_t\|_{C_\alpha}\le d\ .$$

The family $\{(Sx)(t) : x\in Y_0\}$ is an equicontinuous. To show this, let $0\le t_1<t_2\le T$. Then,

$$\|(Sx)(t_1)-(Sx)(t_2)\|_\alpha$$
$$\le\int_0^{t_1}\|A^\alpha(t_0)[W(t_2,\eta)-W(t_1,\eta)]BV^{-1}\Big[\int_0^T W(T,s)$$
$$\times\Big\{\int_0^s\Big(a(s,\tau)g(\tau,\bar{\phi}_\tau+x_\tau)+h(s,\tau,\bar{\phi}_\tau+x_\tau)\Big)d\tau+f(s,\bar{\phi}_s+x_s)\Big\}ds\Big](\eta)\|d\eta$$
$$+\int_{t_1}^{t_2}\|A^\alpha(t_0)W(t_2,\eta)BV^{-1}\Big[\int_0^T W(T,s)$$
$$\times\Big\{\int_0^s\Big(a(s,\tau)g(\tau,\bar{\phi}_\tau+x_\tau)+h(s,\tau,\bar{\phi}_\tau+x_\tau)\Big)d\tau+f(s,\bar{\phi}_s+x_s)\Big\}ds\Big](\eta)\|d\eta$$
$$+\int_0^{t_1}\|A^\alpha(t_0)[W(t_2,s)-W(t_1,s)]$$

$$\times\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}\|ds$$

$$+\int_{t_1}^{t_2}\|A^\alpha(t_0)W(t_2,s)\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)\right.$$

$$\left.+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}\|ds$$

$$\leq\int_0^{t_1-\varepsilon}\|A^\alpha(t_0)[e^{-(t_2-\eta)A(\eta)}-e^{-(t_1-\eta)A(\eta)}]BV^{-1}$$

$$\times\left[\int_0^T A^\alpha(t_0)\Big(e^{-(T-s)A(s)}+\int_s^T e^{-(T-\mu)A(\mu)}Z(\mu,s)d\mu\Big)\right.$$

$$\left.\times\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}ds\right](\eta)\|d\eta$$

$$+\int_0^{t_1-\varepsilon}\|A^\alpha(t_0)\left[\int_\eta^{t_2}e^{-(t_2-\nu)A(\nu)}Z(\nu,\eta)d\nu-\int_\eta^{t_1}e^{-(t_1-\nu)A(\nu)}Z(\nu,\eta)d\nu\right]$$

$$\times BV^{-1}\left[\int_0^T A^\alpha(t_0)\Big(e^{-(T-s)A(s)}+\int_s^T e^{-(T-\mu)A(\mu)}Z(\mu,s)d\mu\Big)\right.$$

$$\left.\times\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}ds\right](\eta)\|d\eta$$

$$+\int_0^{t_1-\varepsilon}\|A^\alpha(t_0)[e^{-(t_2-s)A(s)}-e^{-(t_1-s)A(s)}]$$

$$\times\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}\|ds$$

$$+\int_0^{t_1-\varepsilon}\|A^\alpha(t_0)\left[\int_s^{t_2}e^{-(t_2-\nu)A(\nu)}Z(\nu,s)d\nu-\int_s^{t_1}e^{-(t_1-\nu)A(\nu)}Z(\nu,s)d\nu\right]$$

$$\times\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}\|ds$$

$$+\int_{t_1-\varepsilon}^{t_2}\|A^\alpha(t_0)W(t_2,\eta)BV^{-1}\left[\int_0^T A^\alpha(t_0)W(T,s)\right.$$

$$\left.\times\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}ds\right](\eta)\|d\eta$$

$$+\int_{t_1-\varepsilon}^{t_2}\|A^\alpha(t_0)W(t_2,s)$$

$$\times\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}\|ds$$

$$+\int_{t_1-\varepsilon}^{t_1}\|A^\alpha(t_0)W(t_1,\eta)BV^{-1}\left[\int_0^T A^\alpha W(T,s)\right.$$

$$\left.\times\left\{\int_0^s\Big(a(s,\tau)g(\tau,\bar\phi_\tau+x_\tau)+h(s,\tau,\bar\phi_\tau+x_\tau)\Big)d\tau+f(s,\bar\phi_s+x_s)\right\}ds\right](\eta)\|d\eta$$

$$+\int_{t_1-\varepsilon}^{t_1} \|A^{\alpha}(t_0)W(t_1,\eta)\Big\{\int_0^s \Big(a(s,\tau)g(\tau,\bar{\phi}_\tau + x_\tau)$$

$$+h(s,\tau,\bar{\phi}_\tau + x_\tau)\Big)d\tau + f(s,\bar{\phi}_s + x_s)\Big\}\|ds$$

$$\leq K(\alpha,1)\bar{K}((N_3K_1 + K_2)T + K_3)\int_0^{t_1-\varepsilon} \|A(t_0)[e^{-(t_2-\eta)A(\eta)}$$

$$-e^{-(t_1-\eta)A(\eta)}]N_1N_2K(\alpha,1)\bar{K}\Big[\int_0^T A(t_0)\Big(e^{-(T-s)A(s)}$$

$$+C\int_s^T e^{-(T-\mu)A(\mu)}|\mu - s|^{\delta-1}d\mu\Big)ds\Big](\eta)\|d\eta$$

$$+K(\alpha,1)\bar{K}((N_3K_1 + K_2)T + K_3)C\int_0^{t_1-\varepsilon}\Big\|\Big\{\Big[\|A(t_0)\int_\eta^{t_1}[e^{-(t_2-\nu)A(\nu)}$$

$$-e^{-(t_1-\nu)A(\nu)}]|\nu-\eta|^{\delta-1}d\nu\|N_1N_2 + \|\int_{t_1}^{t_2} e^{-(t_2-\nu)A(\nu)}|\nu-\eta|^{\delta-1}d\nu\|\Big]$$

$$\times K(\alpha,1)\bar{K}\|\int_0^T A(t_0)\Big(e^{-(T-s)A(s)}$$

$$+C\int_s^T e^{-(T-\mu)A(\mu)}|\mu - s|^{\delta-1}d\mu\Big)ds\|\Big\}(\eta)\Big\|d\eta$$

$$+K(\alpha,1)\bar{K}((N_3K_1 + K_2)T + K_3)$$

$$\times\int_0^{t_1-\varepsilon} \|A(t_0)[e^{-(t_2-s)A(s)} - e^{-(t_1-s)A(s)}]\|ds$$

$$+K(\alpha,1)\bar{K}((N_3K_1 + K_2)T + K_3)\int_0^{t_1-\varepsilon} C\|\int_s^{t_1} A(t_0)$$

$$\times[e^{-(t_2-s)A(s)} - e^{-(t_2-s)A(s)}]|\nu - s|^{\delta-1}d\nu\|ds$$

$$+K(\alpha,1)\bar{K}K(\alpha)N_1N_2((N_3K_1 + K_2)T + K_3)(|t_2 - t_1 + \varepsilon|^{1-\alpha}/1-\alpha)$$

$$\times(K(\alpha,1)\bar{K}K(\alpha)(T/1-\alpha))$$

$$+K(\alpha,1)\bar{K}K(\alpha)((N_3K_1 + K_2)T + K_3)(|t_2 - t_1 + \varepsilon|^{1-\alpha}/1-\alpha)$$

$$+K(\alpha,1)\bar{K}K(\alpha)N_1N_2((N_3K_1 + K_2)T + K_3)(\varepsilon^{1-\alpha}/1-\alpha)$$

$$\times(K(\alpha,1)\bar{K}K(\alpha)(T/1-\alpha))$$

$$+K(\alpha,1)\bar{K}K(\alpha)((N_3K_1 + K_2)T + K_3)(\varepsilon^{1-\alpha}/1-\alpha)$$

The operator-valued function $A(t)e^{-\eta A(s)}$ is uniformly continuous in (t,η,s) for $0 \leq t \leq T, 0 \leq s \leq T$ and $m \leq \eta \leq T$, where m is any positive number(see Ref.[2]). Hence, the set

$$Y_1 = \{(Sx)(t) : x \in Y_0\}$$

is equicontinuous. Further, since $A^{-\beta}(t_0)$ is compact for all $\beta \in (0,1]$ and

$(H_4), A^{-1}(t_0)$ is compact. Also,

$$\begin{aligned}
&\|A^{\beta}(t_0)(Sx)(t)\|_{\alpha} \\
&\leq \| \int_0^t W(t,\eta)BV^{-1}\Big[\int_0^T A^{\beta}(t_0)W(T,s)\Big\{\int_0^s \Big(a(s,\tau)g(\tau,\bar{\phi}_\tau + x_\tau) \\
&\qquad +h(s,\tau,\bar{\phi}_\tau + x_\tau)\Big)d\tau + f(s,\bar{\phi}_s + x_s)\Big\}ds\Big](\eta)d\eta\| \\
&\quad + \| \int_0^t A^{\beta}(t_0)W(t,s)\Big\{\int_0^s \Big(a(s,\tau)g(\tau,\bar{\phi}_\tau) + h(s,\tau,\bar{\phi}_\tau + x_\tau)\Big)d\tau \\
&\qquad +f(s,\bar{\phi}_s + x_s)\Big\}ds\| \\
&\leq K(\beta,1)\bar{K}K(\beta)N_1N_2\int_0^t (t-\eta)^{-\beta}\Big[K(\beta,1)\bar{K}K(\beta) \\
&\quad \times((N_3K_1+K_2)T+K_3)\int_0^T |T-s|^{-\beta}ds\Big](\eta)d\eta \\
&\quad + K(\beta,1)\bar{K}K(\beta)((N_3K_1+K_2)T+K_3)\int_0^t (t-s)^{-\beta}ds \\
&\leq K(\beta,1)\bar{K}K(\beta)((N_3K_1+K_2)T+K_3)(T/1-\beta) \\
&\quad \times (N_1N_2\bar{K}\bar{K}(\alpha)(T/1-\beta)+1)
\end{aligned}$$

for any β with $0 \leq \alpha < \beta < 1, t \in [-r,T]$. Thus, the set $\{A^{\beta}(t_0)(Sx)(t)\}$ is bounded in X. Now, since the mapping $A^{-\beta} : X \to X_{\alpha}(t_0)$ is compact for each $\beta > \alpha$, it follows that the set Y_1 is precompact. Therefore, by the Arzela-Ascoli theorem, Y_1 is a precompact set of C_{α}.

Now, we will show the continuity of the mapping S from Y_0 into $C([-r,T];X_{\alpha})$, we suppose that

$$\sup_{0\leq s\leq T} \|x(s) - \bar{x}(s)\|_{\alpha} < \delta$$

then, for any $0 \leq t \leq T$

$$\begin{aligned}
&\|(Sx)(t) - (S\bar{x})(t)\|_{\alpha} \\
&\leq \int_0^t \|A^{\alpha}(t_0)W(t,\eta)BV^{-1}\Big[\int_0^T A^{\alpha}(t_0)W(T,s) \\
&\quad \times \Big[\Big\{\int_0^s \Big(a(s,\tau)g(\tau,\bar{\phi}_\tau + x_\tau) + h(s,\tau,\bar{\phi}_\tau + x_\tau)\Big)d\tau \\
&\quad +f(s,\bar{\phi}_s + x_s)\Big\} - \Big\{\int_0^s \Big(a(s,\tau)g(\tau,\bar{\phi}_\tau + x_\tau) \\
&\quad +h(s,\tau,\bar{\phi}_\tau + x_\tau)\Big)d\tau + f(s,\bar{\phi}_s + \bar{x}_s)\Big\}\Big]ds\Big](\eta)\|d\eta
\end{aligned}$$

$$+\int_0^t \|A^\alpha(t_0)W(t,s)\Big[\Big\{\int_0^s \Big(a(s,\tau)g(\tau,\bar{\phi}_\tau+x_\tau)$$
$$+h(s,\tau,\bar{\phi}_\tau+x_\tau)\Big)d\tau+f(s,\bar{\phi}_s+x_s)\Big\}-\Big\{\int_0^s\Big(a(s,\tau)g(\tau,\bar{\phi}_\tau+\bar{x}_\tau)$$
$$+h(s,\tau,\bar{\phi}_\tau+\bar{x}_\tau)\Big)d\tau+f(s,\bar{\phi}_s+\bar{x}_s)\Big\}\Big]\|ds$$
$$\le K(\alpha,1)\bar{K}K(\alpha)N_1N_2\int_0^t|t-\eta|^{-\alpha}\Big[K(\alpha,1)\bar{K}K(\alpha)((N_3\varepsilon+\varepsilon)T+\varepsilon)$$
$$\times\int_0^T(T-s)^{-\alpha}ds\Big](\eta)d\eta+K(\alpha,1)\bar{K}K(\alpha)((N_3\varepsilon+\varepsilon)T+\varepsilon)$$
$$\times\int_0^t|t-s|^{-\alpha}ds$$
$$\le K(\alpha,1)\bar{K}K(\alpha)(T^{1-\alpha}/1-\alpha)(N_3\varepsilon+\varepsilon)T+\varepsilon)(N_1N_2K(\alpha,1)\bar{K}K(\alpha)$$
$$\times(T/1-\alpha)+1).$$

Since $g: J\times C_\alpha\to X, h: J\times J\times C_\alpha\to X$ and $f: J\times C_\alpha\to X$ are continuous and there exists a constant N_3 such that $|a(t,s)|\le N_3$ for $t,s\in J$. Hence, by the Schauder's fixed point theorem, the mapping S has a fixed point.

4.Example .

Consider the parabolic integro-differential equation of the form

$$\begin{aligned} y_t(x,t)&=a(x,t)y_{xx}+b(x,t)u(t)\\ &\quad+\int_0^t\Big[c(t,s)g(s,y(s-r,x))+h(t,s,y(s-r,x))\Big]ds\\ &\quad+f(t,y(t-r,x)),\quad x\in[0,1]=I, t\in[0,T]=J,\\ y(t,0)&=y(t,1)=0,\quad t\in J,\\ y(t,x)&=\phi(t,x),\quad -r\le t\le 0,\quad x\in I,\end{aligned}\tag{5}$$

where $y_t-a(x,t)y_{xx}$, is a uniformly parabolic differential operator. Here, $a(x,t)$ and $b(x,t)$ are continuous on I and uniformly Hölder continuous in t. The functions c,g,h and f in (5) satisfy the following conditions;

(i) $c: J\times J\to R$ is Hölder continuous with exponent α,

(ii) The functions $g,f: J\times R\to R, h: J\times J\times R\to R$ are continuous such that

$$\begin{aligned}&|g(t,x)-g(t,\bar{x})|\le L_1|x-\bar{x}|,\\ &|h(t,s,x)-h(t,s,\bar{x})|\le L_2|x-\bar{x}|,\\ &|f(t,x)-f(t,\bar{x})|\le L_3|x-\bar{x}|,\\ &|g(t,0)|=|h(t,s,0)|=|f(t,0)|=0\end{aligned}$$

for $t,s\in J$ and $x,\bar{x}\in R$, where L_1,L_2,L_3 are nonnegative constants. Let $X=L^2[0,1]$ and U be subset of X. Under the assumtions, $A: X\to X$ defined by

$$A(t)y=-a(x,t)y''$$

with domain $D(A) = \{y \in L^2[0,1]; y, y'$ are absolutely continuous, $y'' \in X, y(0) = y(1) = 0\}$ generates an evolution system $W(t,s)$ satisfying conditions $(H_1) - (H_4)$ (see Ref. [2]). Assume that there exists a linear operator V from U into X defined by

$$Vu = \int_0^T W(T,s)b(x,s)u(s)ds,$$

such that invertible operator V^{-1} exists in $L^2(J,U)/\ker V$ and is uniformly bounded and the $W(t,s)$ is compact operator for $t, s \in J$. Then, for some $t_0 \in J$, the operator $A^{1/2}(t_0)$ can be defined by

$$A^{1/2}(t_0)y = a(t_0,x)^{1/2}y', \quad y \in D(A^{1/2}(t_0)) \ ,$$

on $D(A^{1/2}(t_0)) = \{y \in X; y$ is absolutely continuous, $y' \in X, y(0) = y(1) = 0\}$. Define the mapping $G, F : J \times C_{1/2} \to X$ and $H : J \times J \times C_{1/2} \to X$ by $G(t,\phi)(x) = g(t, \phi(-r)x), H(t,s,\phi)(x) = h(t,s,\phi(-r)x)$ and $F(t,\phi)(x) = f(t,\phi(-r)x)$. Then the equation (5) can be formulated abstractly as

$$\begin{aligned} y'(t) + A(t)y(t) = (Bu)(t) + \int_0^t \Big[c(t,s)G(s,y_s) \\ + H(t,s,y_s)\Big] ds + F(t,y_t), \quad t \in J \ , \\ y(t) = \phi(t), \quad -r \le t \le 0 \ . \end{aligned} \tag{6}$$

Thus by hypotheses and using of Theorem 3.1, the system (6) is null controllable with respect to the operator $A^{1/2}(t_0)$ for some $t_0 \in J$.

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