数理解析研究所講究録
985 巻 1997 年 42-57

Rough Sets and Convex Subsets in a Linear Space

Nobuaki Kuroki
Department of Mathematics
Joetsu University of Education
Joetsu-shi, Niigata-ken, Japan 943

黒木 伸明
上越教育大学

Abstract

In this paper we give the notion of a congruence on a linear space V and prove that it can be identied with the notion of a subspace of V. And we give some elementary properties of rough sets (the lower and the upper approximations) of a subset, a subspace and a convex subset of V with respect to a subspace.

Keywords: Congruence relation, subspace, rough set, lower approximation, upper approximation, convex subset,

1 Introduction

The notion of rough sets was introduced by Z. Pawlak in his paper [2]. Let μ be a equivalence relation on a given set S. We denote by $[a]_\mu$ the μ-equivalence class containing a of S. Then for a nonempty subset A of S, the sets

$$\mu_-(A) = \{x \in S :\ [x]_\mu \subseteq A\}$$

$$\mu^-(A) = \{x \in S :\ [x]_\mu \cap A \neq \emptyset\}$$

is called the *lower approximation* and the *upper approximation* of A, respectively. And

$$\mu(A) = (\mu_-(A), \mu^-(A))$$

is called the *rough set* of A. So the notion of the rough set $\mu(A)$ is an extended notion of the set A.

We shall apply the notion of rough sets to the elementary theory of a linear space V.

In section 2 we define the notion of a congruence relationon V. Let $C(V)$ the set of all congruence relations on V, and let $S(V)$ the set of all subspaces of V. Then we shall prove that there exists a one-to-one mapping from $S(V)$ onto $C(V)$. This means that we can identify the notion of a congruence on a linear space V with the notion of a subspace of V.

We give some properties of the lower and the upper approximations of subsets of V in section 3, of subspaces of V in section 4, and of convex subsets of V in section 5.

2 Congruences on a linear space

Let R be the set of all real numbers, and V a linear space over R. By a *congruence* on V we mean a equivalence relation μ such that

$$(a,b) \in \mu \text{ implies } (a+x, b+x) \in \mu \text{ and } (ka, kb) \in \mu$$

for all $a,\ b,\ x \in V$ and all $k \in R$.

Let μ and ν be two binary relations on V. Then the product $\mu \circ \nu$ of μ and ν is defined by

$$(\mu \circ \nu) = \{(a,b) \in V \times V : \ (a,x) \in \mu, \ \ (x,b) \in \nu \ \text{ for some } \ x \in V\}.$$

In this section we shall give some properties of congruences on V.

Theorem 1 *Let μ and ν be congruences on a linear space V. Then*

$$\mu \circ \nu = \nu \circ \mu.$$

□

Theorem 2 *Let μ and ν be any congruences on a linear space V over R. Then the product $\mu \circ \nu$ is also a congruence on V.*

Proof. It follows from Theorem 1 that $\mu \circ \nu$ is an equivalence relation on V. In order to see that $\mu \circ \nu$ is congruence, let $(a, b) \in \mu \circ \nu$, and $\forall x \in V$ and $\forall k \in R$. Then there exists an element $y \in V$ such that $(a, y) \in \mu$ and $(y, b) \in \nu$. Since μ and ν are both congruence, we have

$$(a + x, y + x) \in \mu \quad and \quad (y + x, b + x) \in \nu.$$

Then we have

$$(a + x, b + x) \in \mu \circ \nu.$$

And also since

$$(ka, ky) \in \mu \quad and \quad (ky, kb) \in \nu.$$

we have

$$(ka, kb) \in \mu \circ \nu.$$

This means that $\mu \circ \nu$ is congruence. □

Theorem 3 *Let W be a subspace of a linear space V over R. We define a binary relation μ_w on V as follows:*

$$\mu_w = \{(a, b) \in V \times V : \ a - b \in W\}.$$

Then μ_w is a congruence on V.

Proof. As is well-knon, and is easily seen, μ_w is an equivalence relation on V. To see that μ_w is a congruence, let $(a, b) \in \mu_w$, and $x \in V$, $k \in R$. Then we have

$$(a + x) - (b + x) = a - b \in W,$$

and so

$$(a + x, b + x) \in \mu_w.$$

And, since W is a subspace of V, we have

$$ka - kb = k(a - b) \in W,$$

and so

$$(ka, kb) \in \mu_w.$$

This implies that μ_w is a congruence on V. This completes the proof.

Remark: $[x]_{\mu_w} = x + W$,.

The following property can be easily seen.

Theorem 4 *Let μ be a congruence on a linear space V over R. We define a subset W_μ of V as follows:*

$$W_\mu = \{a \in V : (a, 0) \in \mu\}.$$

Then W_μ is a subspace of V.

We denote by $C(V)$ the set of all congruences on a linear space V, and by $S(V)$ the set of all subspaces of V. Then we have the following:

Theorem 5 *Let V be a linear space over R. Then there exists a one-to-one mapping ψ from $S(V)$ onto $C(V)$.*

Proof. We define amapping $\psi : S(V) \longrightarrow C(V)$ as follows:

$$\psi(W) = \mu_w$$

for all $W \in S(V)$. Then it can easily seen that ψ is a one-to-one onto mapping. □

Remark: Theorem 5 shows that we can identify the notion of a subspace with a congruence in a linear space.

Theorem 6 *Let W, U be subspaces of a linear space V over R. Then*

$$\mu_w \cap \mu_u = \mu_{w \cap u}.$$

□

Theorem 7 *Let W and U be subspaces of a linear space V. Then*

$$\mu_w \circ \mu_u = \mu_{w+u}.$$

Proof. It is clear that $\mu_w \circ \mu_u \subseteq \mu_{w+u}$. Conversely, let $(a, b) \in \mu_{w+u}$. Then $a - b \in W + U$, and so there exist elements $x \in W$ and $y \in U$ such that $a - b = x + y$. Then, since $a - (b + y) = x \in W$, we have $(a, b + y) \in \mu_w$. Since $a - (b + x) = y \in U$, we have $(b + y, b) = (a - x, b) \in \mu_u$. Therefore we have $(a, b) \in \mu_w \circ \mu_u$, and so

$$\mu_{w+u} \subseteq \mu_w \circ \mu_u.$$

Therefore we obtain that

$$\mu_w \circ \mu_u = \mu_{w+u}.$$

□

3 The lower and upper approximations with respect to a subspace in a linear spaces

As is proved in Theorem 4 that there exists a one-to-one mapping between C(V) and S(V). Therefore we can identify the notion of congruences with subspaces of a linear space V.
Let W be a subspace of a linear space V. Let A be a nonempty subset of V. Then the sets

$$W_-(A) = \{x \in V : \ x + W \subseteq A\},$$
$$W^-(A) = \{x \in V : \ (x + W) \cap A \neq \emptyset\}$$

is called respectively the *lower and the upper approximations* of the set A with respect to the subspace W.

The following properties can be easily seen:

Theorem 8 *Let W and U be subspaces of a linear space V. Let A and B be any nonempty subsets of V. Then,*
(1) $W_-(A) \subseteq A \subseteq W^-(A)$;
(2) $W^-(A \cup B) = W^-(A) \cup W^-(B)$;
(3) $W_-(A \cap B) = W_-(A) \cap W_-(B)$;
(4) $A \subseteq B$ *implies* $W_-(A) \subseteq W_-(B)$;
(5) $A \subseteq B$ *implies* $W^-(A) \subseteq W^-(B)$;
(6) $W_-(A \cup B) \supseteq W_-(A) \cup W_-(B)$;
(7) $W^-(A \cap B) \subseteq W^-(A) \cap W^-(B)$;
(8) $U \subseteq W$ *implies* $W_-(A) \subseteq U_-(A)$;
(9) $W \subseteq U$ *implies* $W^-(A) \subseteq U^-(A)$.

□

Theorem 9 *Let W be a subspace of a linear space V. Let A and B be nonempty subsets of V. Then*
(1) $W^-(A) + W^-(B) = W^-(A + B)$.
(2) $W_-(A) + W_-(B) \subseteq W_-(A + B)$.

Proof. (1) Let c be any element of $W^-(A+B)$. Then $(c+W)\cap(A+B) \neq \emptyset$. Thus there exists an element $x \in (c + W) \cap (A + B)$, and so $x \in c + W$ and $x \in A + B$. Then $x = a + b$ with $a \in A$ and $b \in B$, and

$$c \in x + W = (a + b) + W = (a + W) + (b + W).$$

Thus $c = y + z$ with $y \in a + W$ and $z \in b + W$. Then $a \in (y + W) \cap A$ and $b \in (z + W) \cap B$. Therefore $y \in W^-(A)$ and $z \in W^-(B)$. Thus we have

$$c = y + z \in W^-(A) + W^-(B),$$

and so

$$W^-(A + B) \subseteq W^-(A) + W^-(B).$$

Conversely, let c be any element of $W^-(A) + W^-(B)$. Then $c = a + b$ with $a \in W^-(A)$ and $b \in W^-(B)$. Thus there exist elements x and y in V such that

$$x \in (a + W) \cap A \quad \text{and} \quad y \in (b + W) \cap B,$$

and so

$$x \in a + W, \quad x \in A, \quad y \in b + W \quad \text{and} \quad y \in B.$$

Then

$$x + y \in (a + W) + (b + W) = (a + b) + W = c + W,$$

and

$$x + y \in A + B.$$

Thus we have

$$x + y \in (c + W) \cap (A + B).$$

Thus

$$W^-(A) + W^-(B) \subseteq W^-(A + B).$$

Therefore we obtain that

$$W^-(A) + W^-(B) = W^-(A + B).$$

(2) Let c be any element of $W_-(A) + W_-(B)$. Then $c = a + b$ with $a \in W_-(A)$ and $b \in W_-(B)$. Thus

$$a + W \subseteq A \quad \text{and} \quad b + W \subseteq B,$$

and so

$$c + W = (a + b) + W = (a + W) + (b + W) \subseteq A + B.$$

Thus $c \in W_-(A + B)$, and so

$$W_-(A) + W_-(B) \subseteq W_-(A + B),$$

which completes the proof. □

Theorem 10 *Let W and U be subspaces of a linear space V. Let A be a nonempty subset of V. Then*
(1) $(W \cap U)^-(A) \subseteq W^-(A) \cap U^-(A)$.
(2) $(W \cap U)_-(A) = W_-(A) \cap U_-(A)$.

Proof. (1) Let $c \in (W \cap U)^-(A)$. Then $(c + (W \cap U)) \cap A \neq \emptyset$. Thus there exists an element $a \in (c + (W \cap U)) \cap A$, and so

$$a \in c + (W \cap U) \ \text{ and } \ a \in A.$$

This implies that

$$a \in c + W, \ \ a \in A \ \text{ and } \ a \in c + U, \ \ a \in A.$$

This means that

$$c \in W^-(A) \ \text{ and } \ c \in U^-(A),$$

and so we have

$$c \in W^-(A) \cap U^-(A).$$

Thus we obtain that

$$(W \cap U)^-(A) \subseteq W^-(A) \cap U^-(A).$$

(2)

$$\begin{aligned} c \in (W \cap U)_-(A) &\Leftrightarrow c + (W \cap U) \subseteq A \\ &\Leftrightarrow c + W \subseteq A \ \text{ and } \ c + U \subseteq A \\ &\Leftrightarrow c \in W_-(A) \ \text{ and } \ c \in U_-(A) \\ &\Leftrightarrow c \in W_-(A) \cap U_-(A). \end{aligned}$$

Therefore we obtain that

$$(W \cap U)_-(A) = W_-(A) \cap U_-(A).$$

□

Theorem 11 *Let W be a subspace of a linear space V over R, and $k \in R(k \neq 0)$. If A is a nonempty subset of V, then*

$$W^-(kA) = kW^-(A).$$

Proof. Let c be any element of $W^-(kA)$. Then $(c+W)\cap kA \neq \emptyset$, and so there exists an element $a \in (c+W)\cap kA$. Then $a \in c+W$ and $a \in A$. Thus $a = kb$ for some $b \in A$. Then we have

$$\begin{aligned} c &\in a+W = kb+W = kb+k(1/k)W \\ &\subseteq kb+kW = k(b+W). \end{aligned}$$

Then $c = kb$ for some $y \in b+W$, and so $b \in (y+W)\cap A$. Thus $y \in W^-(A)$, and so $c = ky \in kW^-(A)$. Therefore we have

$$W^-(kA) \subseteq kW^-(A).$$

Conversely, let c be any element of $kW^-(A)$. Then $c = ka$ for some $a \in W^-(A)$. Thus there exists an element $x \in (a+W)\cap A$, and so $x \in a+W$ and $x \in A$. Then

$$kx \in k(a+W) = ka+kW \subseteq ka+W,$$

and $kx \in kA$. Thus $kx \in (ka+W)\cap kA$, and so $c = ka \in W^-(kA)$. Therefore we have

$$kW^-(A) \subseteq W^-(kA).$$

Therefore we obtain that

$$W^-(kA) = kW^-(A),$$

which completes the proof. □

4 Rough subspaces in a linear space

Let W be a subspace of a linear space V over R. Let A be a nonempty subset of V. Then

$$W(A) = (W_-(A), W^-(A))$$

is called a *rough set* of A with respect to the subspace W. A nonempty subset A of V is called a *W^--rough subspace* of V if the upper approximation $W^-(A)$ of A is a subspace of V. Similarly, A is called a *W_--rough subspace* of V if $W_-(A)$ is a subspace of V.

Theorem 12 *Let W be a subspace of a linear space V over R. If A is a subspace of V, then it is a W^--rough subspace of V*

Proof. Since W and A are subspaces of V, $0 \in W$ and $0 \in A$, and so

$$0 \in (0 + W) \cap A.$$

Thus $0 \in W^-(A)$. Let $a,\ b \in W^-(A)$. Then

$$(a + W) \cap A \neq \emptyset \text{ and } (b + W) \cap A \neq \emptyset.$$

Then there exist elements $x,\ y \in V$ such that

$$x \in (a + W) \cap A \text{ and } y \in (b + W) \cap A.$$

Thus we have

$$x \in a + W, \quad x \in A, \quad y \in b + W \quad \textit{and} \quad y \in A.$$

Since A is a subspace of V, we have $x + y \in A$. And since W is a subspace of V,

$$x + y \in (a + W) + (b + W) = (a + b) + W.$$

Therefore we have

$$x + y \in ((a + b) + W) \cap A,$$

and so

$$a + b \in W^-(A).$$

Let $a \in W^-(A)$ and $k \in R$. Then there exists an element $x \in V$ such that

$$x \in (a + W) \cap A,$$

and so

$$x \in a + W \quad \text{and} \quad x \in A.$$

Since A is a subspace of V, $kx \in A$. And also W is a spbspace of of V,

$$kx \in k(a + W) = ka + kW \subseteq ka + W,$$

and so

$$kx \in (ka + W) \cap A.$$

Thus we have

$$ka \in W^-(A).$$

Therefore we have $W^-(A)$ is a subspace of V, and A is a W^--rough subspace of V. □

Theorem 13 *Let W be a subspace of a linear space V over R. If A is a sibspace of V such that $W \subseteq A$, then A is a W_--rough subspace of V.*

Proof. Since $0 + W = W \subseteq A$, we have $0 \in W_-(A)$. Let $a,\ b \in W_-(A)$. Then

$$a + W \subseteq A \quad \text{and} \quad b + W \subseteq A.$$

Then, since A is a subspace of V, we have

$$(a + b) + W = (a + W) + (b + W) \subseteq A + A \subseteq A,$$

and so

$$a + b \in W_-(A).$$

Let $a \in W_-(A)$ and $k \in R$. If $k = 0$, then, as is stated above,

$$ka = 0a = 0 \in W_-(A).$$

If $k \neq \emptyset$, then $k(1/k) = 1$. Since $a + W \subseteq A$ and since A is a subspace of V, we have

$$ka + W = ka + k(1/k)W \subseteq ka + kW = k(a + W) \subseteq kA \subseteq A,$$

and so

$$ka \in W_-(A).$$

Therefore $W_-(A)$ is a subspace of V, and A is a W_--rough subspace of V. □

Theorem 14 *Let W and U be subspaces of a linear space V over R. If A is a subspace of V, then*
(1) $W^-(A) + U^-(A) \subseteq (W + U)^-(A)$.
(2) $W_-(A) + U_-(A) \subseteq (W + U)_-(A)$.

Proof. (1) Let c be any element of $W^-(A) + U^-(A)$. Then $c = a + b$ with $a \in W^-(A)$ and $b \in U^-(A)$. Then

$$(a + W) \cap A \neq \emptyset \quad \text{and} \quad (b + U) \cap A,$$

and so there exist elements $x,\ y \in V$ such that

$$x \in (a + W) \cap A \quad \text{and} \quad y \in (b + U) \cap A.$$

Thus we have

$$x \in a+W, \quad x \in A, \quad y \in b+W \quad \text{and} \quad y \in A.$$

Since W is a subspace of V,

$$\begin{aligned} x+y &\in (a+W)+(b+U) \\ &= (a+(W+b))+U \\ &= (a+(b+W))+U \\ &= ((a+b)+W)+U \\ &= (a+b)+(W+U) \\ &= c+(W+U). \end{aligned}$$

Since A is a subspace of V, $x+y \in A$. Thus we have

$$x+y \in (c+(W+U)) \cap A,$$

and so

$$c \in (W+U)^{-}(A).$$

Therefore we obtain that

$$W^{-}(A)+U^{-}(A) \subseteq (W+U)^{-}(A).$$

(2) Let c be any element of $W_{-}(A)+U_{-}(A)$. Then $c=a+b$ with $a \in W_{-}(A)$ and $b \in U_{-}(A)$. Thus

$$a+W \subseteq A \quad \text{and} \quad b+U \subseteq B.$$

Then, since W and A are subspaces of V, we have

$$\begin{aligned} (a+b)+(W+U) &= (a+(b+W))+U \\ &= (a+(W+b))+U \\ &= ((a+W)+B)+U \\ &= (a+W)+(b+U) \\ &\subseteq A+A \\ &\subseteq A, \end{aligned}$$

and so

$$c = a + b \in (W+U)_-(A).$$

Thus we obtain that

$$W_-(A) + U_-(A) \subseteq (W+U)_-(A).$$

□

Theorem 15 *Let W and U be subspaces of a linear space V over R. If A is a subspace of V, then*

$$(W+U)^-(A) \subseteq (W^-(A)+U) \cap (U^-(A)+W).$$

Proof. Let c be any element of $(W+U)^-(A)$. Then $(c+(W+U)) \cap A \neq \emptyset$. Then there exists an element $x \in V$ such that

$$x \in (c+(W+U)) \cap A.$$

Thus we have

$$x \in c+(W+U) \quad \text{and} \quad x \in A.$$

Then $x = c+a+b$ for some $a \in W$ and $b \in U$. Note that, since W and U are subspaces of V, $-a \in W$ and $-b \in U$. Then we have

$$x = c+a+b \in c+W+b = c+b+W,$$

and so

$$x \in (c+b+W) \cap A.$$

Thus we have

$$c+b \in W^-(A),$$

and so

$$c \in W^-(A) + (-b) \subseteq W^-(A) + U.$$

Similarly, it can be seen that

$$c \in U^-(A) + W,$$

and so

$$c \in (W^-(A)+U) \cap (U^-(A)+W).$$

Therefore we obtain that

$$(W+U)^-(A) \subseteq (W^-(A)+U) \cap (U^-(A)+W).$$

□

5 Convex subsets

Let S be a nonempty subset of a linear space V over R. Then S is called to be *convex* if for any $a,\ b \in S$ and $0 \leq \lambda \leq 1$,

$$\lambda a + (1-\lambda)b \in S.$$

In this section we give some properties of the upper approximation of convex subsets of a linear space V.

Theorem 16 *Let W be a subspace of a linear space V over R. If S is a convex subset of V, Then $W^-(S)$ is convex.*

Proof. Let $a,\ b \in S$, and $0 \leq \lambda \leq 1$. Then

$$(a+W)\cap S \neq \emptyset \quad \text{and} \quad (b+W)\cap S \neq \emptyset,$$

Theu there exist elements $x,\ y \in S$ such that

$$x \in a+W \qquad y \in b+W.$$

Then, since W is a subspace of V,

$$\lambda x \in \lambda(a+W) = \lambda a + \lambda W \subseteq \lambda a + W$$

and

$$(1-\lambda)y \in (1-\lambda)(b+W) = (1-\lambda)y + (1-\lambda)W \subseteq (1-\lambda)y + W.$$

Thus we have

$$\lambda x + (1-\lambda)y \in (\lambda a + W) + ((1-\lambda)b + W) = (\lambda a + (1-\lambda)b) + W.$$

Since S is convex, we hace

$$\lambda x + (1-\lambda)y \in S.$$

Thus we have

$$\lambda x + (1-\lambda)y \in ((\lambda a + (1-\lambda)b) + W) \cap S.$$

and so

$$\lambda a + (1 - \lambda)b \in W^-(S).$$

This means that $W^-(S)$ is convex, which completes the proof.

A nonemply subset C of V is called a *cone* if for all $a, \in C$ and for all $\lambda \geq 0, \quad \lambda a \in C$.

Theorem 17 *Let W be a subspace of V over R. If C is a cone, then $W^-(C)$ is a cone.*

Proof. Let a be any element of $W^-(C)$ and $\lambda \geq 0$. Then

$$(a + W) \cap C \neq \emptyset.$$

Thus there exists an element $x \in C$ such that $x \in a + W$. Then we have

$$\lambda x \in \lambda(x + W) = \lambda x + \lambda W \subseteq \lambda x + W.$$

Since C is a cone, $\lambda x \in C$. Thus we have

$$\lambda x \in (\lambda a + W) \cap C.$$

This implies that $\lambda a \in W^-(C)$, which means that $W^-(C)$ is a cone. □

Theorem 18 *Let W be a subspace of a linear space V over R and C a convex cone of V. Then $W^-(C)$ is a convex cone.*

□

6 The kernel of a linear mapping

Let V and V' be two linear spaces over R, and $f : V \to V'$ a linear mapping. We denote by $0'$ the zero of V'. Then the set

$$Ker(f) = \{x \in V : f(x) = 0'\}$$

is called the *kernel* of f.

As is easily seen, $Ker(f)$ is a subspace of V. The following can be easily seen.

Lemma 1 *Let $f : V \to V'$ be a linear mapping. Then*

$$\mu_{Ker(f)} = \{(a, b) \in V \times V : f(a) = f(b)\}.$$

□

Lemma 2 *Let $f : V \to V'$ be a linear mapping. Then for a nonempty subset A of V,*

$$f(A) = f(A + Ker(f)).$$

Proof. Let y be any element of $f(A)$. Then $f(a) = y$ for some $a \in A$. We note that $0 \in Ker(f)$. Thus we have

$$y = f(a) = f(a + 0) \in f(A + Ker(f)),$$

and so

$$f(A) \subseteq f(A + Ker(f)).$$

Conversely, let y be any element of $f(A + Ker(f))$. Then $f(a) = y$ for some $a \in A + Ker(f)$. Thus $a = b + c$ with $b \in A$ and $c \in Ker(f)$. Then

$$y = f(a) = f(b + c) = f(b) + f(c) = f(b) + 0' = f(b) \in f(A),$$

and so

$$f(A + Ker(f)) \subseteq f(A).$$

Therefore we obtain that

$$f(A) = f(A + Ker(f)),$$

which completes the proof. □

Theorem 19 *Let $f : V \to V'$ be a linear mapping, and W a subspace of V. Then for a nonempty subset A of V,*

$$f(A) \subseteq f(W^-(A)) \subseteq f(A + N).$$

Proof. By Theorem 8(1), $A \subseteq W^-(A)$, and so $f(A) \subseteq f(W^-(A))$. To see $f(W^-(A) \subseteq f(A + W)$, let y be any element of $f(W^-(A))$. Then $f(a) = y$ for some $a \in W^-(A)$. Then there exists an element $x \in V$ such that $x \in$

$(a+W)\cap A$. Thus $x \in a+W$ and $x \in A$. Then $x = a+b$ for some $b \in W$, that is, $a = x-b$. Since W is a spbspace of V, $-b \in W$. Then we have

$$y = f(a) = f(x-b) \in f(A+W),$$

and so

$$f(W^{-}(A)) \subseteq f(A+W),$$

which completes the proof. □

Theorem 20 *Let $f : V \to V'$ be a linear mapping. Then for a nonempty subset A of V,*

$$f(A) = f(Ker(f)^{-}(A)).$$

Proof. By Lemma 1 and Theorem 19 we have

$$f(A) \subseteq f(Ker(f)^{-}(A)) \subseteq f(A + Ker(f)) = f(A).$$

Therefore we obtain that

$$f(A) = f(Ker(f)^{-}(A)),$$

which completes the proof.

References

[1] N. Kuroki; Rough ideals in semigroups. (to appear in Information Sciences an international journal)

[2] Z. Pawlak, Rough sets, *Int. J. Inf. Comp. Sci.* 11:341-356(1982)

[3] Z. Pawlak, *Rough Sets Theoretical Aspects of Reasoning about Data*, Kluwer Academic, Norwell, MA, 1991.

[4] S. Roman, *Advanced Linear Algebra*, Springer-Verlag, 1992