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A DE LA VALLÉE POUSSIN TYPE THEOREM IN BANACH SPACES

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1. Introduction

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic, continuous functions f on the real line $\mathbb{R}$ with the norm

$$\|f\|_\infty = \max\{|f(t)| : |t| \leq \pi\}.$$

Let $\mathbb{N}$ be the set of all positive integers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}_0$, we denote by $\mathfrak{T}_n$ the set of all trigonometric polynomials of degree at most n. For a given $f \in C_{2\pi}$, we define

$$E_n(C_{2\pi}; f) = \inf\{\|f - g\|_\infty : g \in \mathfrak{T}_n\},$$

which is called the best approximation of degree n to f with respect to $\mathfrak{T}_n$.

Let $a \in \mathbb{N}, a \geq 2$ and let $\Omega \neq 0$ be a non-negative, monotone decreasing function on $[a, \infty)$ satisfying the conditions

$$\lim_{x \to \infty} \Omega(x) = 0 \tag{1}$$

and

$$\int_a^\infty \frac{\Omega(x)}{x}\, dx < \infty. \tag{2}$$

Then the classical theorem of de la Vallée Poussin states that: Let $f \in C_{2\pi}$ and $r \in \mathbb{N}_0$. If

$$E_n(C_{2\pi}; f) = O\left(\frac{\Omega(n)}{n^r}\right) \qquad (n \to \infty),$$

then f is r-times continuously differentiable on $\mathbb{R}$ and

$$\omega(C_{2\pi}; f^{(r)}, \delta) = O\left(\delta \int_a^{a/\delta} \Omega(x)\, dx + \int_{1/\delta}^{\infty} \frac{\Omega(x)}{x}\, dx\right) \qquad (\delta \to +0),$$

where

$$\omega(C_{2\pi}; f^{(r)}, \delta) = \sup\{\|f^{(r)}(\cdot - t) - f^{(r)}(\cdot)\|_\infty : |t| \le \delta\}$$

denotes the modulus of continuity of $f^{(r)}$ (cf. [5]).

A statement analogous to this result also holds for the Banach space $L^p_{2\pi}$ consisting of all 2π-periodic, p-power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p\, dt\right)^{1/p} \qquad (1 \le p < \infty)$$

using the integral modulus of continuity (cf. [17]). Furthermore, in [3] these results were generalized by means of the higher order moduli of continuity and yielded the inverse theorems of Bernstein-type on the degree of the best approximation with respect to $\mathfrak{T}_n$ (cf. [2], [9], [10], [20]).

The purpose of this paper is to extend the above-mentioned results to arbitrary Banach spaces, and in particular, homogeneous Banach spaces (cf. [8], [11], [18]) which include $C_{2\pi}$ and $L^p_{2\pi}, 1 \le p < \infty$, as particular cases. For this purpose, we consider the following setting:

Let X be a complex Banach space with norm $\|\cdot\|_X$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let $\mathbb{Z}$ denote the set of all integers, and let $\{P_j : j \in \mathbb{Z}\}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

(P-1) The projections $P_j, j \in \mathbb{Z}$, are mutually orthogonal, i.e., $P_jP_n = \delta_{j,n}P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.

(P-2) $\{P_j : j \in \mathbb{Z}\}$ is fundamental, i.e., the linear span of $\cup_{j\in\mathbb{Z}}P_j(X)$ is dense in X.

(P-3) $\{P_j : j \in \mathbb{Z}\}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

For each $n \in \mathbb{N}_0$, let M_n be the linear span of $\{P_j(X) : |j| \leq n\}$, which is a closed linear subspace of X. For a given $f \in X$, we define

$$E_n(X; f) = \inf\{\|f - g\|_X : g \in M_n\},$$

which is called the best approximation of degree n to f with respect to M_n. Clearly,

$$E_0(X; f) \geq E_1(X; f) \geq \cdots \geq E_n(X; f) \geq E_{n+1}(X; f) \geq \cdots \geq 0,$$

and Condition (P-2) implies that

$$\lim_{n\to\infty} E_n(X; f) = 0 \qquad \text{for every } f \in X.$$

In this paper, we derive certain smoothness properties of an element $f \in X$ from the hypothesis that $\{E_n(X; f) : n \in \mathbb{N}_0\}$ tends to zero with a given rapidity. We refer to [16] for detailed treatments and [15] (cf. [13], [14]) for the study of the direct theorems of Jackson-type (cf. [7]) which estimates the magnitude of $E_n(X; f)$ in terms of the moduli of continuity of higher orders of f with respect to a strongly continuous group of multiplier operators on X associated with Fourier series expansions corresponding to $\{P_j : j \in \mathbb{Z}\}$.

2. Moduli of continuity and Bernstein-type inequality

For any $f \in X$, we associate its (formal) Fourier series expansion with respect to $\{P_j : j \in \mathbb{Z}\}$

$$f \sim \sum_{j=-\infty}^{\infty} P_j(f).$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j : j \in \mathbb{Z}\}$ of complex numbers such that for every $f \in X$,

$$T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j P_(f),$$

and the following notation is used:

$$T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j \tag{3}$$

(cf. [4], [11], [12], [21]).

Let $M[X]$ denote the set of all multiplier operators on X, which is a commutative closed subalgebra of $B[X]$ containing the identity operator I. Let $\{T_t : t \in \mathbb{R}\}$ be a family of operators in $M[X]$ satisfying

$$\|T_t\|_{B[X]} \leq 1 \qquad \text{for all } t \in \mathbb{R} \tag{4}$$

and having the expansions

$$T_t \sim \sum_{j=-\infty}^{\infty} e^{-ijt} P_j \qquad (t \in \mathbb{R}).$$

Then $\{T_t : t \in \mathbb{R}\}$ becomes a strongly continuous group of operators in $B[X]$ and we have

$$G^r(P_j(g)) = (-ij)^r P_j(g) \qquad (j \in \mathbb{Z},\, g \in X,\, r \in \mathbb{N}) \tag{5}$$

and

$$G^r(f) \sim \sum_{j=-\infty}^{\infty} (-ij)^r P_j(f) \qquad (f \in D(G^r), r \in \mathbb{N}),$$

where G is the infinitesimal generator of $\{T_t : t \in \mathbb{R}\}$ with domain $D(G)$ (cf. [11; Proposition 2]). For the basic theory of semigroups of operators on Banach spaces we refer to [1] and [6].

For each $r \in \mathbb{N}_0$ and $t \in \mathbb{R}$, we define

$$\Delta_t^0 = I, \quad \Delta_t^r = (T_t - I)^r = \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} T_{mt} \qquad (r \geq 1),$$

which stands for the r-th iteration of $T_t - I$. Then Δ_t^r belongs to $M[X]$ and

$$\|\Delta_t^r\|_{B[X]} \leq 2^r, \qquad \Delta_t^r \quad \sim \quad \sum_{j=-\infty}^{\infty} (e^{-ijt} - 1)^r P_j.$$

If $r \in \mathbb{N}_0$, $f \in X$ and $\delta \geq 0$, then we define

$$\omega_r(X; f, \delta) = \sup\{\|\Delta_t^r(f)\|_X : |t| \leq \delta\},$$

which is called the r-th modulus of continuity of f with respect to $\{T_t : t \in \mathbb{R}\}$. This quantity has the following properties ([15; Lemma 1]):

Lemma 1. *Let $r \in \mathbb{N}$ and $f \in X$*

(a)

$$\omega_r(X; f, \delta) \leq 2^r \|f\|_X \qquad (\delta \geq 0).$$

(b) $\omega_r(X; f, \cdot)$ is a non-decreasing function defined on $[0, \infty)$ and $\omega_r(X; f, 0) = 0$.

(c)

$$\omega_{r+s}(X; f, \delta) \leq 2^r \omega_s(X; f, \delta) \qquad (s \in \mathbb{N}_0,\ \delta \geq 0).$$

In particular, we have

$$\lim_{\delta \to +0} \omega_r(X; f, \delta) = 0.$$

(d)

$$\omega_r(X; f, \xi\delta) \leq (1 + \xi)^r \omega_r(X; f, \delta) \qquad (\xi, \delta \geq 0).$$

(e) If $0 < \delta \leq \xi$, then

$$\omega_r(X; f, \xi)/\xi^r \leq 2^r \omega_r(X; f, \delta)/\delta^r.$$

(f) If $f \in D(G^r)$, then

$$\omega_{r+s}(X; f, \delta) \leq \delta^r \omega_s(X; G^r(f), \delta) \qquad (s \in \mathbb{N}_0,\ \delta \geq 0).$$

(g) $\omega_r(X; \cdot, \delta)$ is a seminorm on X.

If k is a function in $L^1_{2\pi}$ having the Fourier series expansion

$$k(t) \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{k}(j)e^{ijt}$$

with its Fourier coefficients

$$\hat{k}(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)e^{-ijt}\,dt \qquad (j \in \mathbb{Z})$$

and if $T \in B[X]$, then we define the convolution operator $k * T$ by

$$(k * T)(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)T_t(T(f))\,dt \qquad (f \in X),$$

which exists as a Bochner integral (cf. [11]). Obviously, $k * T$ belongs to $B[X]$ and

$$\|k * T\|_{B[X]} \leq \|k\|_1 \|T\|_{B[X]} \tag{6}$$

because of (4). In particular, if T is an operator in $M[X]$ having the expansion (3), then $k * T \in M[X]$ and there holds

$$k * T \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{k}(j)\tau_j P_j, \tag{7}$$

which is an immediate consequence of [15; Lemma 2].

Now we need the following Bernstein-type inequality in order to prove the main theorem.

Lemma 2. *Let* $n \in \mathbb{N}_0$ *and* $r \in \mathbb{N}$. *Then*

$$\|G^r(f)\|_X \leq (2n)^r \|f\|_X \tag{8}$$

for all $f \in M_n$.

Proof. By (5), (8) is trivial for $n = 0$. Let $n \in \mathbb{N}$, and let $k_n(t) = 2nF_n(t)\sin nt$, where

$$F_n(t) = 1 + 2\sum_{j=1}^{n}\left(1 - \frac{1}{n}\right)\cos jt = \frac{1}{n}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2$$

is the Fejér kernel. Since

$$f = \sum_{j=-n}^{n} P_j(f), \tag{9}$$

(5) implies $f \in D(G^r)$ and

$$G^r(f) = \sum_{j=-n}^{n} G^r(P_j(f)) = \sum_{j=-n}^{n} (-ij)^r P_j(f). \tag{10}$$

Since

$$\frac{i}{n} k_n(t) = F_n(t)e^{int} - F_n(t)e^{-int},$$

we obtain

$$\frac{i}{n} \hat{k_n}(j) = \frac{j}{n} \qquad (|j| \leq n).$$

Therefore, in view of (7), (9) and (10), we have

$$(k_n * I)(f) = \sum_{j=-n}^{n} P_j((k_n * I)(f)) = \sum_{j=-n}^{n} \hat{k_n}(j) P_j(f)$$

$$= \sum_{j=-n}^{n} (-ij) P_j(f) = G(f),$$

and so (6) yields

$$\|G(f)\|_X = \|(k_n * I)(f)\|_X \leq \|k_n\|_1 \|f\|_X \leq 2n\|f\|_X.$$

Thus (8) follows from induction on r. $\square$

3. The main theorem

Recall that M_n is the closed linear subspace of X spanned by $\{P_j(X) : |j| \leq n\}$. Here we suppose that for each given $f \in X$ and each $n \in \mathbb{N}_0$, there exists an element $f_n \in M_n$ of the best approximation of f with respect to M_n, i.e.,

$$E_n(X; f) = \|f - f_n\|_X. \tag{11}$$

Remark 1. *If the dimension of M_n is finite, then for any $f \in X$ there exists an element of the best approximation of f with respect to M_n. In particular, if $\{\varphi_j, \varphi_j^*\}_{j \in \mathbb{Z}}$ is a fundamental, total, biorthogonal system (cf. [11; Remark 8]) and if M_n is the linear span of $\{\varphi_j : |j| \leq n\}$, then for every $f \in X$ there exists an element of the best approximation of f with* *respect to M_n. Also, if X is a Hilbert space, then for each $f \in X$ there exists a unique element of the best approximation of f with respect to M_n. For the general theory of the best approximation in normed linear spaces, we refer to [19].*

Theorem 1. *Let Ω be as in Section 1. Let $f \in X$ and $r \in \mathbb{N}_0$. If*

$$E_n(X; f) = O\left(\frac{\Omega(n)}{n}\right) \qquad (n \to \infty), \tag{12}$$

then $f \in D(G^r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(X; G^r(f), \delta) = O\left(\delta^k \int_a^{a/\delta} x^{k-1}\Omega(x)\, dx + \int_{1/\delta}^{\infty} \frac{\Omega(x)}{x}\, dx\right) \quad (\delta \to +0). \tag{13}$$

Proof. Let f_n be an element of the best approximation of f with respect to M_n. Then by (11) and (12), we have

$$\|f - f_{a^n}\|_X \le C_1 a^{-nr}\Omega(a^n) \qquad (n \ge 1), \tag{14}$$

where C_1 is a positive constant independent of n. Put

$$g_2 = f_{a^2}, \quad g_n = f_{a^n} - f_{a^{n-1}} \qquad (n \ge 3). \tag{15}$$

Then it follows from (14) that

$$\|g_n\|_X \le \|f_{a^n} - f\|_X + \|f - f_{a^{n-1}}\|_X \le C_1 \frac{(1+a^r)\Omega(a^{n-1})}{a^{nr}} \qquad (n \ge 3),$$

and so Lemma 2 yields

$$\|G^r(g_n)\|_X \le 2^r C_1 (1+a^r)\Omega(a^{n-1}) \qquad (n \ge 3). \tag{16}$$

By (2), we have

$$\sum_{n=3}^{\infty} \Omega(a^{n-1}) \le \frac{a}{a-1}\int_a^{\infty} \frac{\Omega(x)}{x}\, dx < \infty,$$

which together with (16) implies that there exists an element $g \in X$ such that

$$g = \sum_{n=2}^{\infty} G^r(g_n). \tag{17}$$

Also, (1), (14) and (15) imply that

$$f = \sum_{n=2}^{\infty} g_n. \tag{18}$$

Since G^r is a closed linear operator, it follows from (17) and (18) that $f \in D(G^r)$ and

$$G^r(f) = \sum_{n=2}^{\infty} G^r(g_n). \tag{19}$$

Therefore, by Lemma 1 (a) and (g), we have that for each $m \geq 2$,

$$\omega_k(X; G^r(f), \delta) \leq \omega_k\left(X; \sum_{n=2}^{m} G^r(g_n), \delta\right) + \omega_k\left(X; \sum_{n=m+1}^{\infty} G^r(g_n), \delta\right)$$

$$\leq \sum_{n=2}^{m} \omega_k(X; G^r(g_n), \delta) + 2^k \sum_{n=m+1}^{\infty} \|G^r(g_n)\|_X = I_1 + I_2,$$

say. By (10), Lemma 1 (f), Lemma 2 and (16), we have

$$\omega_k(X; G^r(g_n), \delta) \leq \delta^k \|G^k(G^r(g_n))\|_X$$

$$\leq \delta^k (2a^n)^k \|G^r(g_n)\|_X \qquad (n \geq 2)$$

$$\leq \delta^k (2a^n)^k 2^r C_1 (1 + a^r) \Omega(a^{n-1}) \qquad (n \geq 3).$$

Thus we obtain that for each $m \geq 2$,

$$I_1 \leq \delta^k (2a^2)^k \|G^r(g_2)\|_X + 2^{k+r} C_1 (1 + a^r) \delta^k \sum_{n=3}^{m} a^{kn} \Omega(a^{n-1})$$

$$\leq C_2 \delta^k \sum_{n=2}^{m} \left(a^{k(n-1)} - a^{k(n-1)-1}\right) \Omega(a^{n-1}),$$

where C_2 is a positive constant independent of δ and m.

Now let $0 < \delta \leq 1/a$, and we choose $m \in \mathbb{N}, m \geq 3$ such that

$$a^{m-2} \leq \delta^{-1} < a^{m-1}.$$

First, we consider the case of $\Omega(a^2) = 0$. Then (14) implies $f = g_2 \in M_{a^2}$, and so (10), Lemma 1 (f) and Lemma 2 yield that

$$\omega_k(X; G^r(f), \delta) \leq (2a^2)^k \|G^r(f)\|_X \delta^k. \tag{20}$$

Also, we have

$$\int_a^{a/\delta} x^{k-1}\Omega(x)\,dx \ge \int_a^{a^{m-1}} x^{k-1}\Omega(x)\,dx \ge \int_a^{a^2} x^{k-1}\Omega(x)\,dx > 0,$$

which together with (20) clearly implies (13).

Next, let $\Omega(a^2) > 0$. Then we have

$$\sum_{n=2}^{m}\left(a^{k(n-1)} - a^{k(n-1)-1}\right)\Omega(a^{n-1}) \le (a^{2k} - a^{2k-1})\Omega(a)$$

$$+\sum_{n=3}^{m}\int_{a^{k(n-1)-1}}^{a^{k(n-1)}} \Omega(x^{1/k})\,dx \le \frac{\Omega(a)}{\Omega(a^2)}\int_{a^{2k-1}}^{a^{2k}} \Omega(x^{1/k})\,dx$$

$$+\int_{a^{2k-1}}^{a^{k(m-1)}} \Omega(x^{1/k})\,dx \le \left(\frac{\Omega(a)}{\Omega(a^2)}+1\right)\int_{a^k}^{a^{k(m-1)}} \Omega(x^{1/k})\,dx$$

$$\le \left(\frac{\Omega(a)}{\Omega(a^2)}+1\right)\int_{a^k}^{(a/\delta)^k} \Omega(x^{1/k})\,dx.$$

Therefore putting $u = x^{1/k}$, we get

$$I_1 \le C_2\left(\frac{\Omega(a)}{\Omega(a^2)}+1\right)k\delta^k\int_a^{a/\delta} u^{k-1}\Omega(u)\,du.$$

Also, by (16) we have

$$I_2 \le 2^{k+r}C_1(1+a^r)\sum_{n=m+1}^{\infty}\Omega(a^{n-1})$$

$$\le 2^{k+r}C_1(1+a^r)\frac{a}{a-1}\int_{a^{m-1}}^{\infty}\frac{\Omega(x)}{x}\,dx$$

$$\le 2^{k+r}C_1(1+a^r)\frac{a}{a-1}\int_{1/\delta}^{\infty}\frac{\Omega(x)}{x}\,dx.$$

Hence, we arrive at

$$\omega_k(X;G^r(f),\delta) \le C_3\left(\delta^k\int_a^{a/\delta} x^{k-1}\Omega(x)\,dx + \int_{1/\delta}^{\infty}\frac{\Omega(x)}{x}\,dx\right),$$

where C_3 is a positive constant independent of δ. This implies (13) and the proof of the theorem is now complete. □

Applying Theorem 1 to the case where

$$\Omega(x) = \frac{1}{x^\alpha}, \quad \alpha > 0,$$

we have the following corollary.

Corollary 1. *Let $\alpha > 0, f \in X$ and $r \in \mathbb{N}_0$. If*

$$E_n(X; f) = O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (n \to \infty),$$

then $f \in D(G^r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(X; G^r(f), \delta) = \begin{cases} O(\delta^\alpha) & (\alpha < k,\ \delta \to +0) \\ O(\delta^k |\log \delta|) & (\alpha = k,\ \delta \to +0) \\ O(\delta^k) & (\alpha > k,\ \delta \to +0). \end{cases}$$

In the remaining part of this section, we restrict ourselves to the case where X is a homogeneous Banach space, i.e., X satisfies the following properties:

(H-1) X is a linear subspace of $L^1_{2\pi}$ with a norm $\|\cdot\|_X$ under which it is a Banach space.

(H-2) X is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $C > 0$ such that $\|f\|_1 \le C\|f\|_X$ for all $f \in X$.

(H-3) The translation operator T_t defined by

$$T_t(f)(\cdot) = f(\cdot - t) \qquad (f \in X),$$

is isometric on X for each $t \in \mathbb{R}$.

(H-4) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on $\mathbb{R}$.

Typical examples of homogeneous Banach spaces are $C_{2\pi}$ and $L^p_{2\pi}, 1 \le p < \infty$. For other examples see [11] (cf. [8], [18]).

Now we define the sequence $\{P_j : j \in \mathbb{Z}\}$ of projection operators in $B[X]$ by

$$P_j(f)(\cdot) = \hat{f}(j)e^{ij\cdot} \qquad (f \in X),$$

which satisfies Conditions (P-1), (P-2) and (P-3) just as Section 1 (cf. [8], [11]). Notice that $M_n = \mathfrak{T}_n$ and for all $f \in X$ we have

$$\Delta^0_t(f) = f, \quad \Delta^r_t(f)(\cdot) = \sum_{m=0}^{r} (-1)^{r-m}\binom{r}{m} f(\cdot - mt) \qquad (t \in \mathbb{R}, r \in \mathbb{N}).$$

Consequently, in the above setting all the results obtained in this paper hold, and so by Theorem 1, we have the inverse theorem of the generalized de la Vallée Poussin type (cf. [3], [5], [17]) in arbitrary homogeneous Banach spaces. Furthermore, for $r=0$ and $0<\alpha<1$, Corollary 1 gives a generalization of [18; Theorem 9.4.5.1] in the context of the use of the k-th modulus of continuity $\omega_k(X;f,\delta)$. Also, for $k=2$, Corollary 1 establishes the theorem of Zygmund type (cf. [22]) in arbitrary homogeneous Banach spaces.

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