数理解析研究所講究録
986 巻 1997 年 1-8

Geometry of 7 lines on the real projective plane and the root system of type E_7

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Abstract. The configuration space of 7 points in the real projective plane is studied. In particular, the relationship between the space and the root system of type E_7 is established.

1. Introduction and the main results

The main purpose of this article is to clarify the relationship between the geometry of 7 lines on the real projective plane $\mathbf{P}^2(\mathbf{R})$ and the root system of type E_7.

We first introduce marked 7 lines $(l_1, l_2, \cdots, l_7)$ on $\mathbf{P}^2(\mathbf{R})$. We give conditions on these 7 lines:

1. The 7 lines $l_1, l_2, \cdots, l_7$ are mutually different.
2. No three of $l_1, l_2, \cdots, l_7$ intersect at a point.
3. There is no conic tangent to six of $l_1, l_2, \cdots, l_7$.

The totality of marked 7 lines on $\mathbf{P}^2(\mathbf{R})$ with conditions 1, 2 forms the configuration space $P(3,7)$: the space $P(3,7)$ is defined by

$$P(3,7) = GL(3,\mathbf{R})\backslash M'(3,7)/(\mathbf{R}^{\times})^7,$$

where $M'(3,7)$ is the set of 3×7 real matrices of which no 3-minor vanishes. On the other hand, the totality of marked 7 lines on $\mathbf{P}^2(\mathbf{R})$ with conditions 1, 2, 3 forms a subset of $P(3,7)$ which we denote by $P_0(3,7)$. Both $P(3,7)$, $P_0(3,7)$ are affine open subsets of $\mathbf{R}^6$. Permutations on the 7 lines $l_1, l_2, \cdots, l_7$ induce a biregular S_7-action on $P(3,7)$ (and also that on $P_0(3,7)$). It is stressed here that the S_7-action on $P_0(3,7)$ is naturally extended to a biregular $W(E_7)$-action (cf. [3], [4]). Let $\mathcal{P}_7$ be the set of connected components of $P_0(3,7)$. The $W(E_7)$-action on $P_0(3,7)$ naturally induces that on $\mathcal{P}_7$. Then we are in a position to state one of the main results of this article.

Theorem 1 *The $W(E_7)$-action on $\mathcal{P}_7$ is transitive.*

Before entering into the main text, we are going to explain the motivations of this study briefly. It is an interesting problem to construct a tame compactification of the configurations space $P_{\mathbf{C}}(n,k)$ of marked k hyperplanes on the complex projective space $\mathbf{P}^{n-1}(\mathbf{C})$.

In the case of $n = 2$, there is a nice compactification of $P_{\mathbf{C}}(2,k)$, the so called Terada model. We note here that $P_{\mathbf{C}}(2,k)$ admits S_k-action. What happens if we treat $P(2,k)$ instead of $P_{\mathbf{C}}(2,k)$, where $P(2,k)$ is the configuration space of k-points of $\mathbf{P}^1(\mathbf{R})$. In [8], it is shown that the S_k-action on the totality of connected components of $P(2,k)$ is transitive and that each connected component in question is described in terms of "*juzu*" introduced there. On the other hand, in the case $(n,k) = (3,6)$, there is a tame compactification of the configuration space of marked six lines of $\mathbf{P}^2(\mathbf{C})$ with conditions 1, 2, 3 above constructed in [2] which is called Naruki's cross ratio variety and denoted by $\mathcal{C}$. As an application of the results of [2], [3], already shown is the result which we are going to explain (cf. [7]). Let $P_0(3,6)$ be the space defined similarly to $P_0(3,7)$, changing the 7 lines with 6 lines. We note that $P_0(3,6)$ admits a $W(E_6)$-action (cf. [2], [3]). Then the result for the case $P_0(3,6)$ given in [8] similar to the case of Terada models is that there are 432 connected components in $P_0(3,6)$ and that the $W(E_6)$-action on the set is transitive. Theorem 1 is an analogue of these results to the case of 7 lines on the real projective plane.

The second main result is the classification of the S_7-orbital structure of $\mathcal{P}_7$. This will be done in terms of tetradiagrams for the root system of type E_7. In particular, we conclude that $\mathcal{P}_7$ is decomposed into 14 S_7-orbits (cf. Theorem 4).

2. Review on the root system Δ of type E_7

We first recall the definition of the root system of type E_7. Let $\tilde{E}$ be an 8-dimensional Euclidean space with a standard basis $\{\varepsilon_j; 1 \le j \le 8\}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\tilde{E}$ defined by

$$\langle \varepsilon_j, \varepsilon_k \rangle = \delta_{jk}$$

and let E be its linear subspace orthogonal to $\varepsilon_7 + \varepsilon_8$. We define the following sixty-three vectors of E:

$$\begin{array}{lcll}
\gamma_1 & = & \varepsilon_8 - \varepsilon_7, & \\
\gamma_j & = & -\varepsilon_{j-1} + \gamma_0, & 1 < j < 8 \\
\gamma_{1j} & = & -\varepsilon_{j-1} + \gamma_0, & 1 < j < 8 \\
\gamma_{jk} & = & \varepsilon_{j-1} - \varepsilon_{k-1}, & 1 < j < k < 8 \\
\gamma_{1jk} & = & -\varepsilon_{j-1} - \varepsilon_{k-1}, & 1 < j < k < 8 \\
\gamma_{ijk} & = & -\varepsilon_{i-1} - \varepsilon_{j-1} - \varepsilon_{k-1} + \gamma_0, & 1 < i < j < k < 8
\end{array}$$

where

$$\gamma_0 = \frac{1}{2}\sum_{j=1}^{8} \varepsilon_j - \varepsilon_7.$$

Note that

$$\gamma_i \perp \gamma_{jk}, \quad \gamma_i \perp \gamma_{ijk}, \quad \gamma_{ij} \perp \gamma_{kl}, \quad \gamma_{ij} \perp \gamma_{ijk}, \quad \gamma_{ij} \perp \gamma_{klm}, \quad \gamma_{ijk} \perp \gamma_{ilm}.$$

The totality Δ of $\pm\gamma_j, \pm\gamma_{jk}, \pm\gamma_{ijk}$ forms a root system of type E_7. It is clear that

$$\gamma_{12}, \quad \gamma_{123}, \quad \gamma_{23}, \quad \gamma_{34}, \quad \gamma_{45}, \quad \gamma_{56}, \quad \gamma_{67} \tag{1}$$

can serve as a system of positive simple roots; its extended Dynkin diagram is given as

$$\begin{array}{ccccccccccccc} \gamma_1 & -- & \gamma_{12} & -- & \gamma_{23} & -- & \gamma_{34} & -- & \gamma_{45} & -- & \gamma_{56} & -- & \gamma_{67} \\ & & & & & & | & & & & & & \\ & & & & & & \gamma_{123} & & & & & & \end{array} \tag{2}$$

The set $\{\gamma_i, \gamma_{jk}, \gamma_{ijk}\}$ is the totality of positive roots of Δ.

Let s_i, s_{ij}, s_{ijk} be the reflections on E with respect to $\gamma_i, \gamma_{ij}, \gamma_{ijk}$. These reflections act on Δ as

$$s_i: \quad \gamma_j \leftrightarrow \gamma_{ij}, \quad \gamma_{jk} \leftrightarrow \gamma_{jk}, \quad \gamma_{jkl} \leftrightarrow \gamma_{mnp}, \quad \{i,j,k,l,m,n,p\} = \{1,2,\ldots,7\},$$

$$s_{ij}: \quad \text{permutation of the indices } i \text{ and } j,$$

$$s_{123}: \ \gamma_1 \leftrightarrow \gamma_1, \quad \gamma_4 \leftrightarrow \gamma_{567}, \quad \gamma_{12} \leftrightarrow \gamma_{12}, \quad \gamma_{14} \leftrightarrow \gamma_{234}, \quad \gamma_{45} \leftrightarrow \gamma_{45}, \quad \gamma_{145} \leftrightarrow \gamma_{145}$$

modulo signs. We define two reflection groups

$$G_1 = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{67} \rangle \cong S_7,$$

$$G = \langle G_1, s_{123} \rangle \cong W(E_7),$$

where S_7 is the symmetric group on seven numerals $\{1,2,\ldots,7\}$, $W(E_7)$ the Weyl group of type E_7 and $\langle a, b, \ldots \rangle$ denotes the group generated by $a, b, \ldots$ Note that G acts on Δ transitively on Δ.

3. Connected components of $\mathcal{C}'_{\mathbf{R}}(\Delta, D_4)$

We start with introducing a 3×7 matrix

$$X = \begin{pmatrix} t & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -2 & 5 & 6 \\ 1 & 0 & 1 & 1 & 4 & -1 & -6 \end{pmatrix}.$$

Then by taking the entries of the vector $(1 \quad x \quad y) \cdot X$, we obtain seven lines of the real projective plane regarding x, y as inhomogeneous coordinates:

$$\begin{array}{lcl} L_1 & : & x + y + t = 0 \\ L_2 & : & x = 0 \\ L_3 & : & y = 0 \\ L_4 & : & x + y + 1 = 0 \\ L_5 & : & -2x + 4y + 1 = 0 \\ L_6 & : & 5x - y + 1 = 0 \\ L_7 & : & 6x - 6y + 1 = 0 \end{array} \tag{3}$$

We assume that t is real and sufficiently large.

By the seven lines above, we obtain ten triangles (T_j) $(j = 1, 2, \cdots, 10)$ surrounded by the three lines

(T_1)	$L_1L_2L_6$	γ_{126}
(T_2)	$L_3L_6L_7$	γ_{367}
(T_3)	$L_4L_5L_6$	γ_{456}
(T_4)	$L_2L_3L_7$	γ_{237}
(T_5)	$L_2L_3L_5$	γ_{235}
(T_6)	$L_1L_2L_4$	γ_{124}
(T_7)	$L_1L_3L_4$	γ_{134}
(T_8)	$L_1L_3L_5$	γ_{135}
(T_9)	$L_4L_5L_7$	γ_{457}
(T_{10})	$L_1L_5L_7$	γ_{157}

(4)

Corresponding to these ten triangles, there is a 6-polytope P of $\mathcal{C}'_{\mathbf{R}}(\Delta, D_4)$ which we are going to define.

Theorem 2 *There is a simply connected and connected component P of $\mathcal{C}'_{\mathbf{R}}(\Delta, D_4)$ surrounded by the following hypersurfaces:*

A_1	$Y_{126}, Y_{367}, Y_{456}, Y_{235}, Y_{134}, Y_{157},$
A_2	$Y_{237}, Y_{124}, Y_{135}, Y_{457},$
A_3	$Z_{26,145}, Z_{237,456}, Z_{45,267}, Z_{37,246}, Z_{126,457}, Z_{14,236},$ $Z_{57,146}, Z_{12,467}, Z_{67,123}, Z_{46,357}, Z_{23,567}, Z_{124,367},$
A_4	$W_{1,37,45}, W_{3,26,57}, W_{3,12,45}, W_{1,23,46}, W_{5,14,67}, W_{5,12,37},$
A_5	$W_{1,23,45}, W_{5,14,37}, W_{3,12,57},$
A_6	$X_{123}, X_{145}, X_{357}$

(5)

Let $\mathcal{H}(P)$ be the totality of these hypersurfaces. Then, for each $w \in W(E_7)$, $w \cdot P$ is surrounded by the hypersurfaces contained in $w \cdot \mathcal{H}(P)$.

We consider automorphisms of P in $W(E_7)$. Let s_i, s_{ij}, s_{ijk} be the reflections with respect to the roots $\gamma_i, \gamma_{ij}, \gamma_{ijk}$, respectively. Using this notation, we put

$$\sigma_1 = s_{12}s_{34}s_{57}s_6,$$
$$\sigma_2 = s_{35}s_{24}s_{56}s_2.$$

Then it is easy to show that

$$\sigma_1 \cdot P = \sigma_2 \cdot P = P.$$

Let G_P be the group generated by σ_1, σ_2. Then G_P is isomorphic to S_4, the symmetric group on four letters. This is proved as follows. First we note that

$$\sigma_1^2 = 1, \quad \sigma_2^3 = 1, \quad (\sigma_1\sigma_2)^4 = 1.$$

Putting

$$\tau_1 = \sigma_2\sigma_1\sigma_2^{-1}, \quad \tau_2 = \sigma_1, \quad \tau_3 = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1,$$

we find that the correspondence

$$\tau_1 \to (12), \quad \tau_2 \to (23), \quad \tau_3 \to (34)$$

induces the isomorphism between $< \tau_1, \tau_2, \tau_3 >$ and S_4. On the other hand, since $\tau_2\tau_1\tau_2\tau_3 = \sigma_2^{-1}$, it follows that $G_P =< \tau_1, \tau_2, \tau_3 >$.

Lemma 1 *By the G_P-action, the hypersurfaces in A_j are transitive ($j = 1, 2, 3, 4, 5, 6$).*

On the other hand, the center of $W(E_7)$ which is isomorphic to $\mathbf{Z}_2$ acts on the 6-polytope P as the identity transformation. Noting that A_2 consists of four hypersurfaces, we obtain the following.

Proposition 1 *The isotropy subgroup of P in $W(E_7)$ is isomorphic to the group $S_4 \times \mathbf{Z}_2$.*

4. 6-polytopes adjacent to P

To study the 6-polytopes adjacent to P, it is better to treat $P' = s_{34}s_{23}s_{236}s_{16}s_{57} \cdot P$ instead of P. Then P' is surrounded by the hypersurfaces in TABLE I below:

TABLE I

A_1	$Y_{12}, Y_{127}, Y_7, Y_{56}, Y_{34}, Y_{567}$,
A_2	$Y_{67}, Y_{23}, Y_{45}, Y_1$,
A_3	$Z_{67,345}, Z_{67}, Z_{345}, Z_{45,123}, Z_{12}, Z_{234,567}, Z_{567}, Z_{456}, Z_{17}, Z_{123}, Z_{234}, Z_{23,456}$,
A_4	$W_{12,345}, W_{34,567}, W_{127}, W_{567}, W_{56,234}, W_{123,456}$,
A_5	$W_{167}, W_{1,23,67}, W_{123}$,
A_6	X_0, X_{167}, X_{123}

(6)

Lemma 2 *The following hold.*

$$\begin{aligned}
\mathcal{H}(P') \cap \mathcal{H}(s_{12} \cdot P') &= \{H \in \mathcal{H}(P'); s_{12} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{Y_{12}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(s_{67} \cdot P') &= \{H \in \mathcal{H}(P'); s_{67} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{Y_{67}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(s_{13} \cdot P') &= \{H \in \mathcal{H}(P'); s_{13} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{Z_{123}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(s_{14}s_{23} \cdot P') &= \{H \in \mathcal{H}(P'); s_{14}s_{23} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{W_{567}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(s_{47}s_{56} \cdot P') &= \{H \in \mathcal{H}(P'); s_{47}s_{56} \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{W_{123}} \neq \emptyset\} \\
\mathcal{H}(P') \cap \mathcal{H}(us_1 \cdot P') &= \{H \in \mathcal{H}(P'); us_1 \cdot H = H\} \\
&= \{H \in \mathcal{H}(P'); \overline{H} \cap \overline{X_0} \neq \emptyset\}
\end{aligned}$$

where $u = (1765432) \in W(E_7)$.

As a consequence of Lemma 2, we find the following.

Theorem 3 *The following statements hold:*

(i) The 6-polytopes P' *and* $s_{12} \cdot P'$ *are adjacent to each other and* $\overline{P'} \cap s_{12} \cdot \overline{P'} \subset Y_{12}$.

(ii) The 6-polytopes P' *and* $s_{67} \cdot P'$ *are adjacent to each other and* $\overline{P'} \cap s_{67} \cdot \overline{P'} \subset Y_{67}$.

(iii) The 6-polytopes P' *and* $s_{13} \cdot P'$ *are adjacent to each other and* $\overline{P'} \cap s_{13} \cdot \overline{P'} \subset Z_{123}$.

(iv) The 6-polytopes P' *and* $s_{14}s_{23} \cdot P'$ *are adjacent to each other and* $\overline{P'} \cap s_{14}s_{23} \cdot \overline{P'} \subset W_{567}$.

(v) The 6-polytopes P' *and* $s_{47}s_{56} \cdot P'$ *are adjacent to each other and* $\overline{P'} \cap s_{47}s_{56} \cdot \overline{P'} \subset W_{123}$.

(vi) The 6-polytopes P' *and* $us_1 \cdot P'$ *are adjacent to each other and* $\overline{P'} \cap us_1 \cdot \overline{P'} \subset X_0$.

Theorem 1 is a consequence of the theorem above.

5. Diagrams corresponding to 6-polytopes

Let P' be the 6-polytope introduced in the previous section. From TABLE I, we find that there are 10 hypersurfaces in A_1, A_2 corresponding to positive roots of type E_7 below:

TABLE II

A_1	$\gamma_{12}, \gamma_{127}, \gamma_7, \gamma_{56}, \gamma_{34}, \gamma_{567}$
A_2	$\gamma_{67}, \gamma_{23}, \gamma_{45}, \gamma_1$

(7)

From these 10 roots, it is possible to construct a diagram similar to Dynkin diagrams. Namely, the diagram in question consists of 10 circles attached with the 10 roots in TABLE II. Two circles are connected by a segment if and only if the roots attached to them are not orthogonal each other. Then we find that there are 12 segments in this diagram.

In the same way, to each 6-polytope in $\mathcal{P}_7$, there associates a diagram which is called a *tetradiagram* for the reason that in the case of P', the 4 circles corresponding to the roots in A_2 of TABLE II are regarded as 4 vertices of a tetrahedron and the 6 circles corresponding to the roots in A_1 of TABLE II are regarded as 6 middle points of the sides of the referred tetrahedron.

To avoid the complexity, we neglect the $\pm$ signs of tetradiagrams. In this manner, we finally obtain a tetra-diagram corresponding to each $P'' \in \mathcal{P}_7$. We denote by $\mathcal{TP}_7$ the set of tetra-diagrams.

We here note that the classification of the arrangements of 7 lines $(l_1, l_2, \cdots, l_7)$ on the real projective plane with conditions 1, 2, 3 is accomplished if we determine the decomposition of $\mathcal{TP}_7$ into S_7-orbits. This will be done in the next section.

6. Tetradiagrams

We have already introduced tetradiagrams in the previous section. In this section, we treat them in the language of root systems systematic manner.

Definition 1 *Let $a_i\,(i=1,2,3,4)$, $b_{ij}\,(1 \le i < j \le 4)$ be roots of Δ. (Assume that $b_{ij} = b_{ji}$ for all i, j.) Then the set*

$$\mathsf{A} = \{a_i; i = 1,2,3,4\} \cup \{b_{ij}; 1 \le i < j << 4\}$$

is called a tetrahedral set if the following conditions hold:

(i) $\langle a_i, a_j \rangle = 0 \quad (i \neq j)$.

(ii) $\langle b_{ij}, b_{i'j'} \rangle = 0 \quad (\{i,j\} \neq \{i',j'\})$.

(iii) $|\langle a_i, b_{jk} \rangle| = 0$ *if and only if* $i \notin \{j,k\}$.

Example 1 *The set*

$$\mathsf{U} = \{\gamma_{345}, \gamma_{123}, \gamma_{136}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}$$

is a tetrahedral set. In particular, the correspondence

$$\begin{array}{llllll}
\gamma_{345} \longrightarrow \mathrm{A}_1 & & \gamma_{135} \longrightarrow \mathrm{B}_{12} & & \gamma_{124} \longrightarrow \mathrm{B}_{23} \\
\gamma_{123} \longrightarrow \mathrm{A}_2 & & \gamma_{167} \longrightarrow \mathrm{B}_{13} & & \gamma_{236} \longrightarrow \mathrm{B}_{24} \\
\gamma_{146} \longrightarrow \mathrm{A}_3 & & \gamma_{347} \longrightarrow \mathrm{B}_{14} & & \gamma_{257} \longrightarrow \mathrm{B}_{34} \\
\gamma_{256} \longrightarrow \mathrm{A}_4 & & & &
\end{array}$$

induces a tetradiagram for U.

For a tetrahedral set $\mathsf{A} = \{a_i\} \cup \{b_{ij}\}$, we put

$$\tilde{\mathsf{A}} = \{\pm a_i\} \cup \{\pm b_{ij}\}$$

and call it an extended tetrahedral set. Let A' be also a tetrahedral set. Then A and A' are *equivalent* if and only if $\tilde{\mathsf{A}} = \tilde{\mathsf{A}}'$. In this case, we confuse a tetradiagram for A and that for A', for simplicity.

The G_1-orbit structure will be also important. For this purpose, we will introduce fourteen extended tetrahedral sets in TABLE III. If A is a tetrahedral set whose extended tetrahedral set has the name X in TABLE III, we denote by O_{X} the S_7-orbit of $\tilde{\mathsf{A}}$. If B is a tetrahedral set such that $\tilde{\mathsf{B}} \in O_{\mathrm{X}}$, we call B a tetrahedral set of type X. Similarly, we call a tetradiagram for B that of type X.

We are going to give a classification of S_7-orbital structure of extended tetrahedral sets. For this purpose, we define

$$L = \{\mathrm{A}, \mathrm{B1}, ..., \mathrm{B5}, \mathrm{C1}, ..., \mathrm{C4}, \mathrm{D1}, ...\mathrm{D4}\}.$$

Theorem 4 *The set $\mathcal{T}$ of extended tetrahedral sets is decomposed into fourteen S_7-orbits O_{X} $(X \in L)$.*

It is clear from the definition that $\mathcal{T}$ and $\mathcal{TP}_7$ are isomorphic. Therefore the classification of S_7-orbits of $\mathcal{T}$ is accomplished by Theorem 4.

TABLE III

Name	Extended tetrahedral set	Isotropy
A	$\pm\{\gamma_{345},\gamma_{123},\gamma_{146},\gamma_{256},\gamma_{135},\gamma_{167},\gamma_{347},\gamma_{124},\gamma_{236},\gamma_{257}\}$	$\mathbf{Z}_3$
B1	$\pm\{\gamma_{345},\gamma_{123},\gamma_{146},\gamma_{256},\gamma_{25},\gamma_{167},\gamma_{347},\gamma_{34},\gamma_{16},\gamma_{257}\}$	$\mathbf{Z}_3$
B2	$\pm\{\gamma_{345},\gamma_{123},\gamma_{146},\gamma_{256},\gamma_{14},\gamma_{2},\gamma_{57},\gamma_{124},\gamma_{236},\gamma_{257}\}$	1
B3	$\pm\{\gamma_{14},\gamma_{25},\gamma_{146},\gamma_{256},\gamma_{135},\gamma_{167},\gamma_{347},\gamma_{124},\gamma_{236},\gamma_{257}\}$	1
B4	$\pm\{\gamma_{345},\gamma_{123},\gamma_{146},\gamma_{256},\gamma_{6},\gamma_{167},\gamma_{23},\gamma_{17},\gamma_{236},\gamma_{45}\}$	1
B5	$\pm\{\gamma_{2},\gamma_{123},\gamma_{47},\gamma_{256},\gamma_{135},\gamma_{167},\gamma_{347},\gamma_{124},\gamma_{236},\gamma_{257}\}$	1
C1	$\pm\{\gamma_{15},\gamma_{24},\gamma_{36},\gamma_{7},\gamma_{25},\gamma_{167},\gamma_{17},\gamma_{34},\gamma_{346},\gamma_{6}\}$	1
C2	$\pm\{\gamma_{25},\gamma_{14},\gamma_{146},\gamma_{256},\gamma_{345},\gamma_{5},\gamma_{27},\gamma_{34},\gamma_{16},\gamma_{257}\}$	1
C3	$\pm\{\gamma_{36},\gamma_{7},\gamma_{15},\gamma_{24},\gamma_{246},\gamma_{167},\gamma_{347},\gamma_{356},\gamma_{145},\gamma_{257}\}$	$\mathbf{Z}_3$
C4	$\pm\{\gamma_{6},\gamma_{37},\gamma_{146},\gamma_{256},\gamma_{157},\gamma_{26},\gamma_{347},\gamma_{34},\gamma_{267},\gamma_{15}\}$	1
D1	$\pm\{\gamma_{56},\gamma_{123},\gamma_{13},\gamma_{256},\gamma_{136},\gamma_{157},\gamma_{57},\gamma_{124},\gamma_{24},\gamma_{1}\}$	1
D2	$\pm\{\gamma_{47},\gamma_{123},\gamma_{2},\gamma_{256},\gamma_{14},\gamma_{146},\gamma_{57},\gamma_{6},\gamma_{236},\gamma_{23}\}$	1
D3	$\pm\{\gamma_{46},\gamma_{123},\gamma_{146},\gamma_{23},\gamma_{14},\gamma_{147},\gamma_{367},\gamma_{7},\gamma_{25},\gamma_{257}\}$	1
D4	$\pm\{\gamma_{37},\gamma_{6},\gamma_{146},\gamma_{256},\gamma_{157},\gamma_{136},\gamma_{57},\gamma_{124},\gamma_{1},\gamma_{24}\}$	1

(Here $\pm\{\gamma_{345},\gamma_{123},\gamma_{146},\ldots\}$ is an abbreviation of $\{\pm\gamma_{345},\pm\gamma_{123},\pm\gamma_{146},\ldots\}$.)

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