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Implementing Sturm's Algorithm and Its Application

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Abstract. Sturm's theorem is useful to count and to isolate real roots for polynomials. In this paper we show, by introducing a method of integral power of two, how to implement this algorithm more efficiently. We also give some results of isolation and computation time by Maple and Risa/Asir. The experiment shows that the new method is more efficient for polynomials with more real roots. Finally, two application of the algorithm are given. One is to solve ill-condition polynomial system, the other is to plot implicit curves.

1. Introduction

1.1. Isolation of Roots

Let A be a polynomial over Q. The isolation of real roots is to compute a sequence of disjoint intervals with rational endpoints, each containing exactly one real zero of A and together containing all real zeros of A.

1.2. Sturm Sequence

For a univariate polynomial $P(x)$ with integer coefficients, the Sturm sequence is a sequence $p_0(x), p_1(x), \ldots, p_r(x)$.

$$\begin{aligned} p_0(x) &= P(x) \\ p_1(x) &= P'(x) \\ &\cdots \\ p_i(x) &= -remainder(p_{i-2}(x), p_{i-1}(x)) \end{aligned}$$

The coefficients of Sturm sequence can be simplified by the method of sub-resultants.

1.3. Sturm's Theorem

For rational numbers a, b such that $a < b$ and a, b are not roots of $P(x)$, we define $v(P,a)$ and $v(P,b)$ as the number of sign changes in the Sturm sequence respectively:

$$(p_0(a), p_1(a), \ldots, p_r(a))$$
$$(p_0(b), p_1(b), \ldots, p_r(b))$$

Sturm's theorem shows that the number of real roots of $P(x)$ in intervals (a, b) is equal to $v(P, a) - v(P, b)$.
Sturm's theorem gives a basis of root isolation. Based on this theorem, an algorithm of real root isolation for univariate polynomials is given.

2. To Implement Real Roots Isolation Algorithm Efficiently

2.1. Question Proposed

In practical programming, there appears some problems:

- For polynomials with high degree and with much real roots , the representation of isolated intervals becomes very complicated.
- The isolation costs much more in time.

Example 1:Isolating the real roots of polynomial $P1(x)$ and $P2(x)$ respectively.

$$P1(x) = \prod_{i=1}^{20}(x+i)$$

$$P2(x) = \prod_{i=1}^{20}(x+i) + 2^{(-23)}x^{19}$$

Using the Sturm's algorithm, we get the isolated results respectively as follows. Each result are included by a set of intervals. Each interval includes one and only one real root in it.

$$\Big\{\left(\frac{-735}{64},\frac{-1365}{128}\right),\left(\frac{-1365}{128},\frac{-315}{32}\right),\left(\frac{-105}{64},0\right),\left(\frac{-1155}{64},\frac{-2205}{128}\right),$$
$$\left(\frac{-2205}{128},\frac{-525}{32}\right),\left(\frac{-105}{4},\frac{-315}{16}\right),\left(\frac{-525}{32},\frac{-1995}{128}\right),\left(\frac{-1995}{128},\frac{-945}{64}\right),$$
$$\left(\frac{-105}{32},\frac{-315}{128}\right),\left(\frac{-315}{128},\frac{-105}{64}\right),\left(\frac{-105}{8},\frac{-1575}{128}\right),\left(\frac{-735}{128},\frac{-315}{64}\right),$$
$$\left(\frac{-105}{16},\frac{-735}{128}\right),\left(\frac{-1575}{128},\frac{-735}{64}\right),\left(\frac{-315}{64},\frac{-105}{32}\right),\left(\frac{-525}{64},\frac{-945}{128}\right),$$
$$\left(\frac{-945}{128},\frac{-105}{16}\right),\left(\frac{-315}{32},\frac{-525}{64}\right),\left(\frac{-315}{16},\frac{-1155}{64}\right),\left(\frac{-945}{64},\frac{-105}{8}\right)\Big\}$$

$$\Big\{\left(\frac{-12331253767}{2147483648},\frac{-5284823043}{1073741824}\right),\left(\frac{-1761607681}{268435456},\frac{-12331253767}{2147483648}\right),$$
$$\left(\frac{-15854469129}{2147483648},\frac{-1761607681}{268435456}\right),\left(\frac{-8808038405}{1073741824},\frac{-15854469129}{2147483648}\right),$$
$$\left(\frac{-5284823043}{536870912},\frac{-8808038405}{1073741824}\right),\left(\frac{-1761607681}{1073741824},0\right),$$
$$\left(\frac{-5284823043}{2147483648},\frac{-1761607681}{1073741824}\right),\left(\frac{-1761607681}{536870912},\frac{-5284823043}{2147483648}\right),$$

$$\left(\frac{-1761607681}{67108864}, \frac{-1761607681}{134217728}\right), \left(\frac{-5284823043}{1073741824}, \frac{-1761607681}{536870912}\right)\}$$

So we show a new method to implement Sturm's algorithm.

Input:A, a square-free integral polynomial of positive degree.

Bound.$L \leftarrow (-b, b)$ where $b = 2^n \geq max|\alpha_i|$ for all zeros α_i of A.

Sturm sequence: Compute $(A, A', A_3, \cdots, A_r)$ as a Sturm sequence of A by sub-resultant method.

Test.for every interval $(a, b]$ in L do bisect it into $(a, m]$ and $(m, b]$,
if m is a real root of A,

- output $[m, m]$. For each subinterval $(a, m), (m, b]$,compute the number r of real zeros using Sturm's theorem. (For the case of (a, m), the $r = R - 1$, R is the number of real zero in $(a, m]$.)
  - if $r = 0$, drop the interval.
  - if $r = 1$, output the interval.
  - if $r > 1$, keep the interval in L.

else

- for each subinterval $(a, m]$,$(m, b]$ compute the number r of real zeros.
  - if $r = 0$, drop the interval.
  - if $r = 1$, output the interval.
  - if $r > 1$ keep the interval in L.

Output:A list of isolating intervals with rational endpoints for all real zeros of A.

Compared with the way mentioned in [1], here we make the following modification:

- The univariate polynomial is made square-free at first.
- When calculating the boundary of maximal root, we select it as an integral power of 2.
- Using the shape-changed Sturm's theorem(Input is square-free polynomial, output is half-open, half close intervals), we omit to calculate the minimal distance between roots any more.

3. Experimental Results

$$P1(x) = \prod_{i=1}^{20}(x + i)$$

$$P2(x) = \prod_{i=1}^{20}(x + i) + 2^{(-23)}x^{19}$$

$$P3(x) = \prod_{i=1}^{25}(x - i)$$

$$P4(x) = 3x^6 - 5x^4 + 3x^3 - 7x^2 + 2$$
$$P5(x) = \prod_{i=1}^{10}(10x + i)$$

We implement our algorithm in the Maple and Risa/Asir, and get the following results. The roots of $P1(x)$ is :

$$\{[-20,-20],[-18,-18],(-18,-16),[-4,-4],(-20,-18),[-6,-6],$$
$$(-8,-6),(-6,-4),[-2,-2],(-4,-2),(-2,0),[-12,-12],$$
$$[-10,-10],[-16,-16],(-10,-8),[-14,-14],(-12,-10),[-8,-8],$$
$$(-14,-12),(-16,-14)\}$$

The roots of $P2(x)$ is :

$$\left\{\left(-7,\frac{-13}{2}\right),\left(\frac{-13}{2},-6\right),\left(-3,\frac{-5}{2}\right),\left(\frac{-5}{2},-2\right),\left(-9,\frac{-17}{2}\right),\left(-5,\frac{-9}{2}\right),\right.$$
$$\left.\left(\frac{-17}{2},-8\right),(-2,0),\left(\frac{-9}{2},-4\right),(-32,-16)\right\}$$

Time consumed in isolating polynomials above in Maple V.

Time(sec.)	P1	P2	P3	P4	P5
New Method	3.2	32.58	8	0.183	1.88
Primitive Method	13.2	124.83	91.46	0.217	2.12

Time consumed in isolating polynomials above in Risa/Asir.

Sec.	P1	P2	P3	P4	P5
New	0.45/0.07	1.56/0.18	1.02/0.09	0.05/0.02	0.19/0.01
Prim	0.68/0.25	3.37/0.22	1.79/0.31	0.06/0.03	0.19/0.05

4. Some Remarks

- Using new method to isolate roots, we can simplify the results of isolation and avoid the computation of minimal distance between roots. At the same time, the time can be saved.
- Our method is much efficient when isolating polynomials with more real roots.
- In some special cases, for example, root is integer, the root can be obtained exactly in finite steps.

5. Applications

5.1. Solving Polynomials

Sturm's Algorithm can be used to solve ill-conditioned multivariate polynomial systems. Using the numerical methods, usually we couldn't get all of the roots of system correctly. Some roots are lost or not correct. But we can always get all of its real roots correctly using Sturm's algorithm because there are no any approximate computation in it.
For multivariate polynomial system $F_1 = 0, F_2 = 0, \cdots, F_n = 0$, we can derive a univariate polynomial $G(x)$ by calculating the *Gröbner* basis of the ideal $(F_1, \cdots, F_n)$. Then the roots of $G(x) = 0$ can be solved as accurate as we hope using Sturm's algorithm. Then roots of $G(x) = 0$ can substituted back into the other elements of the *Gröbner* basis and the real roots can be determined.

Example2

$$\begin{cases} F1 = (x^2 + y^2 - 1)(xy - 0.25) + 0.0001xy \\ F2 = (x^2 + y^2 - 1)(x - y) - 0.00001(x + 1) \end{cases}$$

We derive a univariate polynomial $G(y)$ from the ideal (F1,F2):

$$\begin{aligned} G(y) &= 221000000000y^8 - 200000000000y^7 - 391237900000y^6 + 30002210000y^5 \\ &+199050502500y^4 - 11251424879y^3 - 28765850000y^2 + 1249993750y \end{aligned}$$

The following is the real roots (x, y) we got.it is a approximate representation of set (x, y).

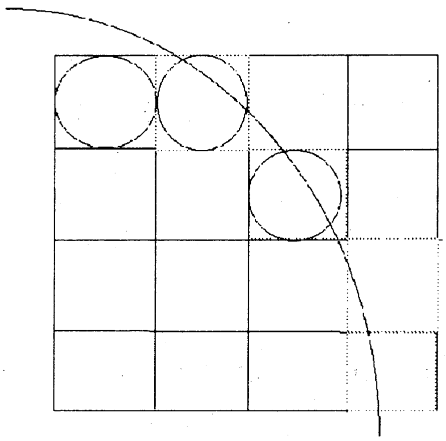
$$\begin{gathered} (0.9990732, 0.0432840),\ (-0.0262280, 0.9996512), \\ (-1.000000, 0.0), (-0.0227383, -0.9997464), \\ (0.6645091, 0.7471453), (-0.7141434, -0.6998564), \\ (0.5000350, 0.5000650),\ (-0.5000550, -0.5000450) \end{gathered}$$

5.2. Plotting Implicit Curves

The Sturm's algorithm can also be used to plot implicit curves. The basic idea is:Suppose the area to plot is covered by a circle.

$$\begin{cases} f(x, y) = 0 \\ (x - a)^{(2n)} + (y - b)^{(2n)} = r^{(2n)} \end{cases}$$

If the curve $f(x, y)$ have common roots with this circle, it means the curves pass through this area. Then the area can be divided into four small areas. In each small areas, we can decide if the curves pass through it or not. If the curve does not pass through, the small areas can be discard. This procedure can be done many times until the small area is as small as we hope. See Fig.1. And we can see an example in Fig.2.



Figl.

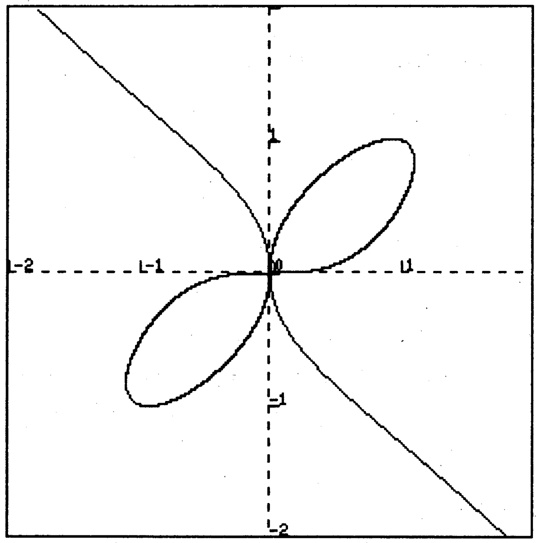


Fig2. $x^5 - 2x^2y + y^5$

Future Development:

- How to use this algorithm in robotics problems. Because many problems in robotics is related to whether there are some real roots or in a system.
- How to use this algorithm to CAD. Basically, the CAD is isolate real roots in multivariate case.

参 考 文 献

[1] Davenport, J.H., Siret,Y., Tournier,E.: Computer Algebra. (Systems and algebraic Computation) 1th. ed. Academic Press 1988, pp. 105-111.

[2] Collins,G.E., Loos,R.: Real Zeros of Polynomials. Computer Algebra Symbolic and Algebraic Computation 2th ed. Springer-Verlag, Wien,New York, 1983, pp. 83-94.

[3] Loos,R. Generalized Polynomial Remainder Sequences. Computer Algebra Symbolic and Algebraic Computation 2th ed. Springer-Verlag, Wien, New York, 1983, pp. 115-137.

[4] Akritas, A.G.:Elements of Computer Algebra with Applications. John Wiley & Sons. 1989, pp. 341-349.

[5] Knuth, D.E.: The Art of Computer Programming, Chapter 4, Vol.2, 2th ed. (Semi-numerical Algorithms). Reading, Mass.: Addison-Wesley 1981.

[6] Maple V manuals. Springer-Verlag. 1992.