数理解析研究所講究録
987 巻 1997 年 164-168

CONSTRUCTION OF MORSE FLOWS

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Let Ω be a bounded domain in $\mathbf{R}^m$, $m \geq 2$, with Lipschitz boundary $\partial\Omega$. In Sobolev space $H^1(\Omega, \mathbf{R}^M)$, $M \geq 1$, we consider the variational functional

$$\text{(P)} \qquad F(u) = \int_\Omega A^{\alpha\beta}(x, u(x)) D_\alpha u^i(x) D_\beta u^i(x) dx,$$

where $u = (u^i)$, $D_\alpha u^i = \partial u^i/\partial x_\alpha$ $(i = 1, 2, \cdots, M; \alpha = 1, 2, \cdots, m)$. The summation convention is used, the Greek indices running from 1 to m and the Latin ones from 1 to M.

The coefficients $A^{\alpha\beta}$, $A^{\alpha\beta} = A^{\beta\alpha}$, are assumed to be measurable in x and continuously differentiable in u and further to satisfy the conditions: there exist positive constant λ, a and L satisfying

$$A^{\alpha\beta}(x, u)\xi^i_\alpha \xi^i_\beta \geq \lambda|\xi|^2,$$
$$|A^{\alpha\beta}(x, u)| \leq L,$$
$$|A^{\alpha\beta}_{u^i} \xi^j_\alpha \xi^j_\beta| \leq 2a|\xi|^2 \quad (i = 1, 2, \cdots, M)$$

for a.e. $x \in \Omega$, $u \in S_B = \{u \in \mathbf{R}^m; |u| < B\}$ and all $\xi = (\xi^i_\alpha) \in \mathbf{R}^{mM}$ with $|\xi| = (\xi^i_\alpha \xi^i_\alpha)^{1/2}$, where B is a positive number introduced relating to the size of the initial data. "Morse flows are prescribed by the equations

$$\frac{\partial u}{\partial t} = D_\alpha(A^{\alpha\beta}(x, u) D_\beta u) - \frac{1}{2} A^{\alpha\beta}_u(x, u) D_\alpha u^i D_\beta u^i,$$

where $A^{\alpha\beta}_u$ denotes the gradient of $A^{\alpha\beta}$ with respect to the variable u.

The aim of this paper is, by discussing the variational problem (P) as an illustration, to investigate the studies of the method proposed in the paper [10] to construct Morse flows.

To accomplish our objective, we perform local estimates for solutions to difference partial differential equations of elliptic-parabolic type, so that we obtainthe higher integrability of their gradients by virtue of 'Gehring theory [7]', the establishment of which requires some refinement of the theory in [1,4] in order to be applicable to treating the solutions to difference partial differential equations. To carry out our scheme, we have to deal with step-functions with respect to the time variable, which makes it a little complicated to gain the local estimates.

We shall set up to construct Morse flows. Let u_0 be a given mapping from the Sobolev space $H^1(\Omega, \mathbf{R}^M)$ and T be a positive number. For a positive integer N, we put

$$h = T/N \quad \text{and} \quad t_n = nh \quad (n = 0, 1, \cdots, N).$$

We set

$$H^1_{u_0}(\Omega, \mathbf{R}^M) = \{u \in H^1(\Omega, \mathbf{R}^M) \ ; \ u - u_0 \in H^1_0(\Omega, \mathbf{R}^M)\},$$

$H^1_0(\Omega, \mathbf{R}^M)$ being the closure of $C^\infty_0(\Omega, \mathbf{R}^M)$ in $H^1(\Omega, \mathbf{R}^M)$. $H^1_{u_0}(\Omega, \mathbf{R}^M)$ is an affine space closed with respect to the weak topology of $H^1(\Omega, \mathbf{R}^M)$.

By a Morse flow to problem (P) with initial and boundary data u_0, we will mean a mapping $u \in L^\infty(0, T, H^1_{u_0}(\Omega)) \cap H^1(0, T, L^2(\Omega))$ which satisfies

$$\text{(0.1)} \qquad \iint_Q \left(\frac{\partial u}{\partial t}\varphi + A(x,u)(Du, D\varphi)\right) dt\, dx = -\frac{1}{2}\iint_Q \varphi A_u(x,u)(Du, Du) dt\, dx$$

for any $\varphi \in L^2(0, T, H^1_0(\Omega, \mathbf{R}^M)) \cap L^\infty(Q, \mathbf{R}^M)$ and $\lim_{t \to +0} u(t) = u_0$ in $L^2(\Omega, \mathbf{R}^M)$.

Theorem. *Let $u_0 \in L^\infty \cap H^1(\Omega, \mathbf{R}^M)$ have the trace $u_0|_{\partial\Omega} = \hat{u}_0|_{\partial\Omega}$ for some mapping $\hat{u}_0 \in W^{1,p_0}(\Omega, \mathbf{R}^M)$ with some $p_0 > 2$ and $||\hat{u}_0||_\infty < B$. Suppose $\lambda > 2aB$. Then there exists a Morse flow to problem* (P) *with initial and boundary data u_0.*

Beginning with $u_0 \in H^1(\Omega, \mathbf{R}^M)$, we introduce a fuctional, for each n $(n = 1, 2, \cdots, N)$,

$$\text{(0.2)} \qquad F_n(u) = F(u) + \frac{1}{h}\int_\Omega |u - u_{n-1}|^2 dx$$

and determine u_n as a minimizer of F_n in $H^1_{u_0}(\Omega, \mathbf{R}^M)$. The existence of the minimizer u_n follows from the lower semi-continuity and the coercivity of F_n in $H^1_{u_0}(\Omega, \mathbf{R}^M)$.

Through the minimality u_n of the functional F_n defined in (0.2), we construct a Cauchy polygon $u_h(t), 0 \le t \le T$, called an approximate solution to (0.1), in $H^1_{u_0}(\Omega, \mathbf{R}^M)$ by

(0.3)

$$u_{(h)}(t) = \frac{t_n - t}{h}u_{n-1} + \frac{t - t_{n-1}}{h}u_n \quad \text{for } t \in (t_{n-1}, t_n] \ (n = 1, 2, \cdots, N),$$

$$u_{(h)}(t) = u_0 \quad \text{for } t \in [-h, 0].$$

We set

$$u^*_{(h)}(t) = u_{(h)}(t_n) \quad \text{for } t \in (t_{n-1}, t_n] \; (n = 1, 2, \cdots, N), \tag{0.4}$$
$$u^*_{(h)}(t) = u_0 \quad \text{for } t \in [-h, 0].$$

We take Euler-Langrange operator at u_n of F_n $(n = 1, 2, \cdots, N)$ in $H^1_{u_0}(\Omega, \mathbf{R}^M)$ and get the identity

$$\int_\Omega \left(\frac{\partial u_{(h)}(t)}{\partial t}\varphi + A(x, u^*_{(h)}(t)(Du^*_{(h)}(t), D\varphi) \right) dx = \tag{0.5}$$
$$= -\frac{1}{2}\int_\Omega \varphi A_u(x, u^*_{(h)}(t))(Du^*_{(h)}(t), D\varphi_{(h)}(t))dx$$

for any $t \in (0, T]$ and for any $\varphi \in C^\infty_0(\Omega, \mathbf{R}^M)$, where $\varphi_{(h)}(t)$, $0 \leq t \leq T$, is such a mapping determined from φ as in (0.4).

Upon comparing u_{n-1} with the minimizer u_n in the functional F_n, we infer

$$F(u_n) + \frac{1}{h}\int_\Omega |u_n - u_{n-1}|^2 dx \leq F(u_{n-1}),$$

from which the following Lemma 1 will be obtained.

We notice that such estimates are available for the variational problem (P) with the coefficients $A(x, u)$ on which there are imposed the assumptions prescribed in this paper and, in principle, do work for any variational problems.

Lemma 1. *Let $u_{(h)}$ be an approximate solution to* (0.1). *Suppose $u_0 \in H^1(\Omega, \mathbf{R}^M)$. Then there hold the estimates*

$$\sup_{0 \leq t \leq T} \int_\Omega A(x, u^*_{(h)}(t))(Du^*_{(h)}(t), Du^*_{(h)}(t))dx \leq \int_\Omega A(x, u_0)(Du_0, Du_0)dx$$

and

$$\int_0^T \int_\Omega \left| \frac{\partial u_{(h)}}{\partial t} \right|^2 dx\, dt \leq \int_\Omega A(x, u_0)(Du_0, Du_0)dx.$$

Lemma 2. *Suppose $u_0 \in L^\infty \cap H^1(\Omega, \mathbf{R}^M)$ and $\lambda > 2aB$ with $B > \sup\{|u_0(x)|; x \in \Omega\}$. Then it holds that*

$$\sup\{|u_n(x)|\, ;\, x \in \Omega, n = 1, 2, \cdots, N\} < B.$$

For the proof of Lemma 2, we have only to note that there holds the inequality

$$\int_\Omega \left(\frac{v_n - v_{n-1}}{h} v_n^{(k)} + A(x, u_n)(Dv_n, Dv_n^{(k)}) \right) dx \leq 0$$

with $v_n = |u_n|^2$ and $v_n^{(k)} = \max(v_n - k, 0)$, $|u_0|^2_\infty \le k < B^2$. Taking a summation of both sides of the inequality with respect to n from 1 to l, we find

$$\int_\Omega |v_l^{(k)}|^2 dx \le \int_\Omega |v_0^{(k)}|^2 dx = 0 \quad (l = 1, 2, \cdots, N),$$

which implies the conclusion of Lemma 2.

Lemma 3. *Let $u_{(h)}$ be an approximate solution to* (0.1) *with initial and boudary data $u_0 \in L^\infty \cap W^{1,p_0}(\Omega, \mathbf{R}^M)$ for some $p_0 > 2$. Suppose $\lambda > 2aB$ with $B = \sup\{|u_0(x)|; x \in \Omega\}$. Then the estimate*

$$\int_0^T \int_\Omega |Du_{(h)}|^p dx\, dt \le C \int_Q |Du_0|^p dz$$

holds for any p, $2 \le p < \min\{2 + \varepsilon_0, p_0\}$, where ε_0 and C are positive numbers independent of h and $u_{(h)}$.

For the proof of Lemma 3, we carry out local estimates for the solutions to difference partial differential equations, which follow the ones due to Giaquinta-Struwe [6], and lead to the higher integrability of the gradients of $u_{(h)}$. In performing the estimation, we take an approach which distinguishes two cases: $\rho^2 \ge h$ and $\rho^2 < h$ according to the size of the treated local domain with the diameter ρ. The equations (0.5) in the domain with the former restiriction have the characteristic of parabolic differential equations, while those in the domain with the latter restriction have the characteristic of elliptic ones. The first case can be treated analogously for parabolic differential equations. The second case requires that we make some device to acquire regularity estimates, in space and time variables, not depending on approximate solutions to (0.1). For details, see [8].

For a topological space X, $A \subset\subset X$ means that the subset A has the closure compact in X.

Lemma 1 gives us the following:

$$\{u_{(h)}\}_{(h>0)} \subset\subset L^\infty(0, T, H^1(\Omega, \mathbf{R}^M)) \text{ with the weak-star topology}$$

and

$$\{u_{(h)}\}_{(h>0)} \subset\subset H^1(0, T, L^2(\Omega, \mathbf{R}^M)) \text{ with the weak topology.}$$

Moreover, since $u_{(h)}$ satisfies the identity (0.5), we have, in light of Lemma 1,2 and 3,

$$\{u_{(h)}\}_{(h>0)} \subset\subset L^2(0, T, H^1(\Omega, \mathbf{R}^M)).$$

These properties enable us to select a sequence $\{h_j, j = 1, 2, \cdots\}$, $h_j \to 0$ as $j \to \infty$, and a mapping u such that

$$\lim_{j\to\infty} u_{(h_j)} = \lim_{j\to\infty} u^*_{(h_j)} = u$$

in such spaces as stated just before. This mapping u turns out to belong to $L^\infty(Q, \mathbf{R}^M) \cap H^1(0, T, L^2(\Omega, \mathbf{R}^M)) \cap L^2(0, T, H^1_{u_0}(\Omega, \mathbf{R}^M))$ and to satisfy the identity (0.1) and the initial-boudary condition

$$u(t) \in H^1_{u_0}(\Omega, \mathbf{R}^M) \quad \text{for almost every } t \in (0, T)$$

and

$$\lim_{t\to 0} u(t) = u_0 \quad \text{in } L^2(\Omega).$$

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