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SPIRAL CONVERGENCE OF SOR DURAND-KERNER'S METHOD

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ABSTRACT. It is proved that SOR Durand-Kerner's method has spiral trajectories of approximants toward multiple roots.

1. INTRODUCTION

Durand-Kerner's method is an iterative algorithm for finding zeros of a monic complex polynomial $p(x)$ of degree $d > 1$. It was proposed by Weierstrass[8], Durand[2], Dochev[1], Kerner[5] and Prešić[6]. Let $\vec{x} = (x_0, \dots, x_{d-1})$ be a point in the complex Euclidean space $\mathbb{C}^d_x$. Let $I_k = \{0, 1, \dots, k-1\}$, $I'_k = \{1, \dots, k-1\}$ be finite sets of indices. Let $\pi_i : \mathbb{C}^d_x \to \mathbb{C}$, $\pi_i \vec{x} = x_i$, $i \in I_d$, be the projection to the i-th coordinate. If f is a self-map of $\mathbb{C}^d_x$, we denote the iteration of f by f^k: $f^0(\vec{x}) = \vec{x}$, $f^{k+1}(\vec{x}) = f(f^k(\vec{x}))$.

In this paper we consider Durand-Kerner's method with 'successive-over-relaxation'. It can be defined in several ways. SOR Durand-Kerner's method is:

- the iteration of the rational mapping

$$\sigma f : \mathbb{C}^d_x \to \mathbb{C}^d_x$$

where $f : \mathbb{C}^d_x \to \mathbb{C}^d_x$ is the rational function defined by

$$\pi_i f = \begin{cases} x_0 - \lambda \frac{p(x_0)}{(x_0 - x_1) \cdots (x_0 - x_{d-1})}, & i = 0, \\ x_i, & i \in I'_d \end{cases}$$

and $\sigma : \mathbb{C}^d_x \to \mathbb{C}^d_x$ is the linear automorphism

$$\sigma(x_0, \dots, x_{d-1}) = (x_1, \dots, x_{d-1}, x_0).$$

- the iteration of the mapping

$$F = f_{d-1} \cdots f_0 : \mathbb{C}^d_x \to \mathbb{C}^d_x$$

where $f_i = \sigma^{-i} f \sigma^i$, $i \in I_d$.

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· the recursive formula

$$x_{i+d} = x_i - \lambda \frac{p(x_i)}{(x_i - x_{i+1}) \cdots (x_i - x_{i+d-1})}$$

with initial values $x_0, \dots, x_{d-1}$ that generates the sequence of complex numbers $\{x_i\}_{i=0,1,\dots}$.

Remark 1. The constant $\lambda \in \mathbb{R}$ is called the relaxation parameter. The case with $\lambda = 1$ is especially called Gauss-Seidel Durand-Kerner's method.

Remark 2. We have $F = (\sigma f)^d$ because $\sigma^{-d+1} = \sigma$. Each f_i leaves x_j invariant if $j \neq i$: $\pi_j f_i = \pi_j$.

Let r_i, $i \in I_\nu$, be the roots of $p(x)$ with multiplicities m_i, so that $\sum_{i=0}^{\nu-1} m_i = d$. Let R the set of mappings $\rho : I_d \to I_\nu$ such that $\#\rho^{-1}(i) = m_i$ for $i \in I_\nu$. If $\rho \in R$ is given, let $\theta_i : I_{m_i} \to I_d$, $i \in I_\nu$, be the injective mapping such that image$(\theta_i) = \rho^{-1}(i)$, and $\theta_i(j) < \theta_i(k)$ for $j < k$.

Let

$$\ell_d(\gamma) = (1-\gamma)(1-\gamma^2)\cdots(1-\gamma^d), \qquad \gamma \in \mathbb{C}.$$

Then for each primitive d-th root of unity ζ, there exists a function $\lambda \mapsto \gamma_\zeta(\lambda)$ defined for $0 < \lambda < \epsilon$ with ϵ small such that $\ell_d \gamma_\zeta = id$, $\lim_{\lambda \to 0} \gamma_\zeta(\lambda) = \zeta$ and

$$\gamma_\zeta(\lambda) = \zeta - \frac{\zeta}{d^2}\lambda + O(|\lambda|^2) \quad \text{as } \lambda \to 0.$$

We will prove the following theorems.

Theorem 1. *Let $d \geq 2$, $0 < \lambda < \epsilon$ with ϵ small, ζ a primitive d-th root of unity, and $\gamma_\zeta(\lambda)$ the function defined as above. There exists a complex manifold $W \subset \mathbb{C}^d_x$ holomorphically isomorphic to the punctured disk $\mathbb{D}^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ such that each $\vec{x}_0 \in W$ has a backward orbit $\vec{x}_{-n} \in W$, $-n \leq 0$, with $\sigma f(\vec{x}_{-(n+1)}) = \vec{x}_{-n}$, $\lim_{-n \to -\infty} \pi_0 \vec{x}_{-n} = \infty$, and*

$$\lim_{-n \to -\infty} \frac{\pi_0 \vec{x}_{-n}}{\pi_0 \vec{x}_{-(n+1)}} = \gamma_\zeta(\lambda).$$

Remark. Existence of the spiral trajectory $\{\pi_0 \vec{x}_{-nd}\}_{-n=0,-1,\dots}$ of 'period' d was observed by Kanno et al. [4].

Theorem 2. *Let $d \geq 2$, $0 < \lambda < \epsilon$ with ϵ small, $\rho \in R$, ζ_i a primitive m_i-th root of unity, θ_i and $\gamma_{\zeta_i}(\lambda)$ the functions defined as above. Denote by $\gamma_i(\lambda) = \gamma_{\zeta_i}(\lambda)$. There is an open set $U \subset \mathbb{C}^d_x$ containing the point $\vec{r}_\rho = (r_{\rho(0)}, \dots, r_{\rho(d-1)})$ on its boundary, such that for each initial value $\vec{x} \in U$ we have*

$$\lim_{n \to \infty} F^n(\vec{x}) = \vec{r}_\rho$$

and, for each $i \in I_\nu$,

$$\lim_{n\to\infty} \frac{\pi_{\theta_i(j)} F^n(\vec{x}) - r_i}{\pi_{\theta_i(j-1)} F^n(\vec{x}) - r_i} = \gamma_i(\lambda), \qquad j \in I'_{m_i},$$
$$\lim_{n\to\infty} \frac{\pi_{\theta_i(0)} F^n(\vec{x}) - r_i}{\pi_{\theta_i(m_i-1)} F^{n-1}(\vec{x}) - r_i} = \gamma_i(\lambda).$$

Our argument is based on the Unstable Manifold Theorem and the deformation of the phase space $\mathbb{C}^d_x$. In section 2 we recall the dynamics of σf in the simplest but important case $p(x) = x^d$ that was studied in [9]. In section 3 we study the dynamics at infinity and prove Theorem 1. In section 4 we study the dynamics close to the root $\vec{r}_\rho$ and prove Theorem 2.

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2. The case $p(x) = x^d$

In [9], we proved the Theorems above in the case $p(x) = x^d$. We take the coordinate change $\vec{y} = \chi(\vec{x})$ defined by

$$y_0 = x_0$$
$$y_i = x_i / x_{i-1}, \qquad i \in I'_d.$$

The rational map $g = \chi\sigma f \chi^{-1} : \mathbb{C}^d_y \to \mathbb{C}^d_y$ is written by

$$\pi_i g(\vec{y}) = \begin{cases} y_0 y_1, & i = 0, \\ y_{i+1}, & 1 \le i \le d-2, \\ \frac{1}{y_1 \cdots y_{d-1}} \left(1 - \lambda \frac{1}{(1-y_1)\cdots(1-y_1\cdots y_{d-1})}\right), & i = d-1. \end{cases} \tag{1}$$

The origin of $\mathbb{C}^d_x$ is blowed-up to the hyperplane

$$\alpha = \left\{\vec{y} \in \mathbb{C}^d_y \mid y_0 = 0\right\} \tag{2}$$

which is forward invariant under g. A point $\vec{y} \in \mathbb{C}^d_y$ is fixed under g if and only if $\vec{y} = \vec{\gamma}$ where $\vec{\gamma} = (0, \gamma, \dots, \gamma) \in \alpha$ and γ is a root of the equation $\lambda = \ell_d(\gamma)$.

Lemma. *Let* $d \ge 2$, $p(x) = x^d$, *and* $0 < \lambda < \epsilon$ *with* ϵ *small. A point* $\vec{y} \in \mathbb{C}^d_y$ *is a stable fixed point under* g *if and only if* $\vec{y} = \vec{\gamma}_\zeta(\lambda) \in \alpha$ *where* ζ *is a primitive* d*-th root of unity and*

$$\vec{\gamma}_\zeta(\lambda) = (0, \gamma_\zeta(\lambda), \dots, \gamma_\zeta(\lambda)).$$

The multipliers of $\vec{\gamma}_\zeta(\lambda)$ *under* $g|\alpha$ *are written by* t_k, $k \in I'_d$, *where*

$$t_k = \zeta^k - \frac{k\zeta^k}{d^2}\lambda + O(|\lambda|^2) \qquad \text{as} \quad \lambda \to 0.$$

Proof. [9].

3. Dynamics at infinity

Here we prove Theorem 1. Let $p(x, x') = x'^d p(x/x')$ be the homogeneous polynomial of degree d of two variables x, x'. We take the coordinate change $\vec{z} = \chi(\vec{x})$ defined by

$$\begin{aligned} z_0 &= 1/x_0 \\ z_i &= x_i/x_{i-1}, \qquad i \in I'_d. \end{aligned}$$

The rational map $h = \chi \sigma f \chi^{-1} : \mathbb{C}^d_z \to \mathbb{C}^d_z$ is written by

$$\pi_i h(\vec{z}) = \begin{cases} z_0/z_1, & i = 0, \\ z_{i+1}, & 1 \leq i \leq d-2, \\ \frac{1}{z_1 \cdots z_{d-1}} \left(1 - \lambda \frac{p(1,z_0)}{(1-z_1)\cdots(1-z_1\cdots z_{d-1})}\right), & i = d-1. \end{cases}$$

The hyperplane $\beta \subset \mathbb{C}^d_z$ defined by $z_0 = 0$ corresponds to the set of 'points at infinity' of $\mathbb{C}^d_x$, and is forward invariant under h.

Since $p(1, 0) = 1$, we have $h|\beta = g|\alpha$ if we identify $\beta \subset \mathbb{C}^d_z$ with $\alpha \subset \mathbb{C}^d_x$. For each primitive d-th root of unity ζ, the point $\vec{\gamma}_\zeta(\lambda) \in \beta$ is a stable fixed point of $h|\beta$, but is a saddle of h with a multiplier $1/\gamma_\zeta(\lambda)$ and the eigenvector tangent to the complex line

$$L_\zeta = \{(y, \gamma_\zeta(\lambda), \ldots, \gamma_\zeta(\lambda)) \mid y \in \mathbb{C}\}.$$

Thus it has a holomorphic unstable manifold V of complex dimension 1 tangent to L_ζ (by an argument of Hirsch-Pugh-Shub [3] adapted to the holomorphic category). We take $W = \chi^{-1}(V - \{\vec{\gamma}_\zeta(\lambda)\})$ and all the assertions in Theorem 1 follows.

4. Dynamics around the root

Here we prove Theorem 2. We denote the rational map g defined in (1) by g_d, and the hyperplane α defined in (2) by α_d.

Let $\sigma_i : \mathbb{C}^d \to \mathbb{C}^d$, $i \in I_\nu$, be the linear automorphism defined by

$$\pi_k \sigma_i(\vec{x}) = x_k, \qquad k \in I_d \text{ with } \rho(k) \neq i,$$

and

$$\pi_{\theta_i(j)} \sigma_i(\vec{x}) = \begin{cases} x_{\theta_i(j+1)}, & 0 \leq j \leq m_i - 2 \\ x_{\theta_i(0)}, & j = m_i - 1. \end{cases}$$

Let $\hat{f}_i = f_{\theta_i(0)}$ for $i \in I_\nu$. It is easily seen that

- $\sigma_i^{m_i} = id$,
- $f_{\theta_i(j)} = \sigma_i^{-j} \hat{f}_i \sigma_i^j$ for $i \in I_\nu$, $j \in I_{m_i}$,
- $\sigma_i f_k = f_k \sigma_i$ if $i \neq \rho(k)$,
- $\sigma_i \hat{f}_j = \hat{f}_j \sigma_i$ if $i \neq j$.

Thus we can re-factor $F = f_{d-1} \cdots f_0$ by the composite of $\sigma_i \hat{f}_i$, $i \in I_\nu$, as

(3) $$F = \sigma_{\rho(d-1)} \hat{f}_{\rho(d-1)} \cdots \sigma_{\rho(0)} \hat{f}_{\rho(0)}.$$

Denote by $(z_{i,0}, \dots, z_{i,m_i-1})$ a point in $\mathbb{C}^{m_i}_{z_i}$, $i \in I_\nu$, and let $M = \mathbb{C}^{m_0}_{z_0} \times \cdots \times \mathbb{C}^{m_{\nu-1}}_{z_{\nu-1}}$. Let $\pi_{i,j} : M \to \mathbb{C}$, $\pi_{i,j}(\vec{z}) = z_{i,j}$, $i \in I_\nu$, $j \in I_{m_i}$, be the projection to the (i,j)-th component. Let $\chi_i : \mathbb{C}^d_x \to \mathbb{C}^{m_i}_{z_i}$, $i \in I_\nu$, be the rational map

$$z_{i,j} = \begin{cases} x_{\theta_i(0)} - r_i, & j = 0, \\ (x_{\theta_i(j)} - r_i)/(x_{\theta_i(j-1)} - r_i), & j \in I'_{m_i}. \end{cases}$$

We take the coordinate change

$$\chi = \chi_0 \times \cdots \times \chi_{\nu-1} : \mathbb{C}^d_x \to M.$$

The rational mapping $h_i = \chi \sigma_i \hat{f}_i \chi^{-1} : M \to M$ is written by

$$\pi_{k,j} h_i(\vec{z}) = z_{k,j}, \qquad k \in I_\nu,\ j \in I_{m_k}, \text{ with } k \neq i$$

and

$$\pi_{i,j} h_i(\vec{z}) = \begin{cases} z_{i,0} z_{i,1}, & j = 0, \\ z_{i,j+1}, & 1 \leq j \leq m_i - 2, \\ \frac{1}{z_{i,1} \cdots z_{i,m_i-1}} \left(1 - \lambda H_i(\vec{z}) / \prod_{k=1}^{m_i-1} (1 - z_{i,1} \cdots z_{i,k})\right), & j = m_i - 1 \end{cases}$$

where

$$H_i(\vec{z}) = \frac{\prod_{k \in I_\nu, k \neq i} (r_i - r_k + z_{i,0})^{m_k}}{\prod_{k \in I_\nu, k \neq i} \prod_{l=0}^{m_k - 1} (r_i - r_k + z_{i,0} - z_{k,0} \cdots z_{k,l})}.$$

By (3) we have

$$\chi F \chi^{-1} = h_{\rho(d-1)} \cdots h_{\rho(0)}.$$

Let $\beta_i \subset \mathbb{C}^{m_i}_{z_i}$ be the hyperplane defined by $z_{i,0} = 0$. The product $B = \beta_0 \times \cdots \times \beta_{\nu-1} \subset M$ corresponds under χ to the point $\vec{r}_\rho \in \mathbb{C}^d_x$, and is forward invariant under every h_i, $i \in I_\nu$. Since $H_i(\vec{z}) = 1$ on B, $i \in I_\nu$, we have

$$h_i | B = id \times \cdots \times (g_{m_i} | \alpha_{m_i}) \times \cdots \times id, \qquad i \in I_\nu,$$

if we idently $\beta_i \subset \mathbb{C}^{m_i}_z$ with $\alpha_{m_i} \subset \mathbb{C}^{m_i}_y$. Note that h_i's are commutative on B: $h_i h_j | B = h_j h_i | B$, $i, j \in I_\nu$.

A point $\vec{z} \in B$ is fixed under every $h_i | B$ if and only if $\vec{z} = \vec{\gamma}_0 \times \cdots \times \vec{\gamma}_{\nu-1}$ where $\vec{\gamma}_i = (0, \gamma_i, \dots, \gamma_i) \in \beta_i$ and γ_i is a root of the equation $\lambda = \ell_{m_i}(\gamma)$. A point $\vec{z} \in B$ is a stable fixed point of every $h_i | B$, $i \in I_\nu$, if and only if $\vec{z} = \vec{\gamma}_{\zeta_0} \times \cdots \times \vec{\gamma}_{\zeta_{\nu-1}}$ where ζ_i, $i \in I_\nu$, is a primitive m_i-th root of unity. Such fixed point $\vec{\gamma}_{\zeta_0} \times \cdots \times \vec{\gamma}_{\zeta_{\nu-1}}$ is also a stable fixed point of every h_i, $i \in I_\nu$, with a multiplier $\gamma_{\zeta_i}(\lambda)$ and the eigenvector tangent to the complex line $\vec{\gamma}_{\zeta_0} \times \cdots \times L_{\zeta_i} \times \cdots \times \vec{\gamma}_{\zeta_{\nu-1}}$. Thus it has an attracting region V. We take $U = \chi^{-1}(V - \{\vec{\gamma}_{\zeta_0} \times \cdots \times \vec{\gamma}_{\zeta_{\nu-1}}\})$ and all the assertions in Theorem 2 follows.

5. Discussion

In section 4, we only studied the points $\vec{z} \in B$ that is fixed under 'every' $h_i|B$, $i \in I_\nu$. It is desirable that our argument be extended to the stable fixed points of the mapping $h_{\nu-1}^{m_{\nu-1}} \cdots h_0^{m_0}|B$ which will also have the spiral trajectories in the space $\mathbb{C}_x^d$.

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