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On Dynamics of Hyperbolic Rational Semigroups and Hausdorff Dimension of Julia sets

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Abstract

We consider about subsemigroups of End$\overline{\mathbb{C}}$. We can define Julia sets, Fatou sets, etc. We show the backward selfsimilarity of Julia sets of finitely generated rational semigroups. If the post critical set of the semigroup is contained in a domain of Fatou set, the Julia set is a self similar set. Next, hyperbolic rational semigroups have no wandering domains under a general assumption. If the hyperbolic rational semigroup is finitely generated and satisfies some conditions, the limit functions of the semigroup on the Fatou set are only constant functions that take their values on post critical set. When the generators of a finitely generated hyperbolic rational semigroup are perturbed, the hyperbolicity is kept and the Jilia sets depend cotiniously on the generators. Further more, if the finitely generated rational semigroup is hyperbolic and if the inverse images by the generators of the Julia set are mutually disjoint, then the Julia set moves by holomorphic motion.

Next we consider about finitely generated rational semigroups satisfying a strong open set condition. We show that if a semigroup satisfies the strong open set condition, the Julia set has no interior points, and furthur more, if the semigroup is hyperbolic, the Hausdorff dimension of Julia set is strictly lower than 2. The value δ of the dimension coincides with the unique value that allows us to construct δ-conformal measure. The δ-Hausdorff measure of the Julia set is a finite value strictly bigger than zero.

Next we construct the generalized Lyubich measure on Julia set when the semigroups are hyperbolic or satisfy strong open set condition. The measure also can be considered as the generalized Bernoulli measure on self similar set. And using that measure we try to estimate the Hausdorff dimension of Julia set from below.

1 Introduction

For a Riemann surface S, let End(S) denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being functional composition. A *rational semigroup* is a subsemigroup of End$(\overline{\mathbb{C}})$ without any constant elements. Similarly, an *entire semigroup* is a subsemigroup of End$(\mathbb{C})$ without any constant elements. A rational semigroup G is called a *polynomial semigroup* if each $g \in G$ is a polynomial. When a ratio nal or entire semigroup G is generated by $\{f_1, f_2, \dots f_n, \dots\}$, we denote this situation by

$$G = \langle f_1, f_2, \dots f_n, \dots \rangle.$$

The rational or entire semigroup generated by a single function g is denoted by $\langle g \rangle$. We denote the n th iterate of f by f^n.

Definition 1.1 *Let G be a rational semigroup.*

$$F(G) \stackrel{\text{def}}{=} \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}$$

$$J(G) \stackrel{\text{def}}{=} \overline{\mathbb{C}} \setminus F(G)$$

$F(G)$ is called Fatou set *for G and $J(G)$ is called* Julia set *for G. Similarly, Fatou set and Julia set for entire semigroup are defined.*

Definition 1.2 *Let G be a rational semigroup and z be a point of $\overline{\mathbb{C}}$. The backward orbit $O^-(z)$ of z and the set of exceptional points $E(G)$ are defined by:*

$$O^-(z) \stackrel{\text{def}}{=} \{w \in \overline{\mathbb{C}} \mid \text{there is some } g \in G \text{ such that } g(w) = z\},$$

$$E(G) \stackrel{\text{def}}{=} \{z \in \overline{\mathbb{C}} \mid \sharp O^-(z) \leq 2\}.$$

Definition 1.3 *A subsemigroup H of a semigroup G is said to be of finite index if there is a finite collection of elements $\{g_1, g_2, \ldots, g_n\}$ of G such that $G = \cup_{i=1}^n g_i H$. Similarly we say that a subsemigroup H of G has cofinite index if there is a finite collection of elements $\{g_1, g_2, \ldots, g_n\}$ of G such that for every $g \in G$ there is $j \in \{1, 2, \ldots, n\}$ such that $g_j g \in H$.*

Lemma 1.1 *Let G be a rational semigroup.*

1. *For any $f \in G$,*

$$f(F(G)) \subset F(G), \; f^{-1}(J(G)) \subset J(G)$$

$$F(G) \subset F(\langle f \rangle), \; J(\langle f \rangle) \subset J(G)$$

2. *If $G = \langle f_1, \ldots, f_n \rangle$, then*

$$F(G) = \cap_{i=1}^n f_i^{-1}(F(G)), \; J(G) = \cup_{i=1}^n f_i^{-1}(J(G))$$

If a set K satisfies that $K = \cup_{i=1}^n f_i^{-1}(K)$, we say that K has backward self similarity. Next lemma was shown in [HM1].

Lemma 1.2 *Let G be a rational semigroup.*

1. *If a subsemigroup H of G is of finite or cofinite index, then*

$$J(H) = J(G).$$

In particular, when G is a rational semigroup generated by finite elements $\{f_1, f_2, \ldots f_n\}$ and m is an integer, if we set

$$H_m = \{g = f_{j_1} \cdots f_{j_k} \in G \mid m \text{ devides } k\},$$

$$I_m = \{g \in G \mid g \text{ is a product of some elements of word length } m\}$$

then

$$J(G) = J(H_m) = J(I_m).$$

Here we say an element $f \in G$ is word length m if m is the minimum integer such that

$$f = f_{j_1} \cdots f_{j_m}.$$

2. *If $J(G)$ contains at least three points, then $J(G)$ is a perfect set.*

3. *If there is some $g \in G$ such that* $\deg(g) \geq 2$ *or there is some $g \in G$ such that* $\deg(g) = 1$ *and order of g is infinite, then*

$$E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp O^-(z) < \infty\}, \ \sharp E(G) \leq 2.$$

4. *If a point z is not in $E(G)$, then for every $x \in J(G)$, x belongs to $\overline{O^-(z)}$. In particular if a point z belongs to $J(G) \setminus E(G)$, then*

$$\overline{O^-(z)} = J(G).$$

5. *If there is some $g \in G$ such that* $\deg(g) \geq 2$ *or there is some $g \in G$ such that* $\deg(g) = 1$ *and order of g is infinite and $J(G)$ contains at least three points, then $J(G)$ is the smallest closed backward invariant set containing at least three points. Here we say that a set A is backward invariant under G if for each $g \in G$, $g^{-1}(A) \subset A$.*

6. *If $J(G)$ contains at least three points, then*

$$J(G) = \overline{\{z \in \overline{\mathbb{C}} \mid z \text{ is a repelling fixed point of some } g \in G\}}$$

Proof [HM1]. □

Remark A similar result of *6.* for entire semigroup can also be stated.

Example 1.1 *For a regular triangle $p_1p_2p_3$, we set $g_i(z) = 2(z - p_i) + p_i$, $i = 1, 2, 3$. And let G be a rational semigroup generated by $\{g_i\}$, not as a group. Then $J(G)$ is the Sierpiński Gasket.*

2 Dynamics of Hyperbolic Rational Semigroups

2.1 Limit Functions

Definition 2.1 *Let G be a rational semigroup. We set*

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g\}}$$

and we say that G is hyperbolic if and only if $P(G) \subset F(G)$.

Definition 2.2 *Let G be a rational semigroup and U be a component of $F(G)$. For every element g of G, we denote by U_g the connected component of $F(G)$ containing $g(U)$. We say that U is a wandering domain if and only if $\{U_g\}$ is infinite.*

Theorem 2.1 *Let G be a rational semigroup and U be a wandering domain. Then there is a constant limit function φ of G on U taking its value ζ in $J(G)$.*

Now we show a sufficient condition such that there is no wandering domain.

Theorem 2.2 *Let G be a rational semigroup and U be a wandering domain. Also let φ be a constant limit function of G on U taking its value ζ in $J(G)$. If there is an element of G such that the degree is at least two, then the value ζ is in $P(G)$.*

Corollary 2.1 *If G is a hyperbolic rational semigroup containing an element of degree at least two, then there is no wandering domain of $F(G)$.*

Similarly we can show the following result.

Theorem 2.3 *In the same situation as Theorem 2.1, assume that every element of G is of degree one. For every point $x \in \overline{\mathbb{C}}$, we denote the closure of G orbit of x by $A(x)$. Then for all $x \in \overline{\mathbb{C}}$ but at most two points of G-fixed points, ζ belongs to $A(x)$.*

Corollary 2.2 *If every element of G is degree one and there is a point $x \in \overline{\mathbb{C}}$ such that $A(x)$ contains at least two points and is included in $F(G)$, then there is no wandering domain of $F(G)$.*

Next we consider about limit functions of a hyperbolic rational semigroup on the Fatou set. By Theorem 2.2, we get one of main results.

Theorem 2.4 *Let G be a finitely generated hyperbolic rational semigroup which contains an element of degree at least two and assume that every element of degree one is not elliptic. Then for every compact subset K of $F(G)$, the G orbit of K can accumulate only to $P(G)$ and every limit function of G on $F(G)$ is a constant function that takes its value in $P(G)$.*

2.2 Continuity of Julia sets

Definition 2.3 *Let E be a metric space. We denote by $Comp^*(E)$ the set of non-empty compact subsets of E. For every $A, B \in \mathrm{Comp}^*(E)$ we set*

$$\partial(A,B) = \sup\{d(x,B) \mid x \in A\}$$

and

$$d_H(A,B) = \max\{\partial(A,B),\ \partial(B,A)\}.$$

It is well known that d_H is a metric on $Comp^*(E)$. We call it the Hausdorff metric.

Next we consider if a Julia set depends continuously on the generators.

Definition 2.4 *Let M be a complex manifold. Suppose the map*

$$(z,a) \in \overline{\mathbb{C}} \times M \mapsto f_{j,a}(z) \in \overline{\mathbb{C}}$$

is holomorphic for each $j = 1, \dots, n$. We set $G_a = \langle f_{1,a}, \cdots, f_{n,a}\rangle$. Then we say that $\{G_a\}_{a \in M}$ is a holomorphic family of rational semigroups.

Definition 2.5 *Let G be a rational semigroup. We say that a compact subset K of $F(G)$ is a confinement set of G if and only if for every $z \in F(G)$, for all but finite elements g of G the point $g(z)$ is included in K.*

Theorem 2.5 *Let $\{G_a\}_{a \in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a}\rangle$. We assume that for a point $b \in M$ there is a confinement set K of G_b. Then the map*

$$a \mapsto J(G_a) \in \mathrm{Comp}^*(\overline{\mathbb{C}})$$

is continuous at the point $a = b$ with respect to Hausdorff metric.

By Theorems 2.4, 2.5, we get the following result.

Theorem 2.6 *Let $\{G_a\}_{a\in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a}\rangle$. And we assume that for every j, $d_j = \deg(f_{j,a})$ is constant independent of a. Then*

1. *Let b be a point of M. Assume that G_b is hyperbolic. And also assume that d_1 is at least two and for every $g \in G_b$ such that $\deg(g)$ is equal to one g is not elliptic. Then there is an open neighborhood W of b such that for every $a \in W$ the rational semigroup G_a is hyperbolic and the map $a \mapsto J(G_a)$ is continuous with respect to the Hausdorff metric.*

2. *Under the same assumption as 1, if the sets $(f_{j,b}^{-1}(J(G)))_j$ are mutually disjoint, then there is an open neighborhood V of b and a continuous map $i : \overline{\mathbb{C}} \times V \to \overline{\mathbb{C}}$ such that for every $z \in \overline{\mathbb{C}}$ the map $a \mapsto i(z,a)$ is holomorphic, and for every $a \in V$ the map $z \mapsto i(z,a)$ is a quasi conformal homeomorphism of $\overline{\mathbb{C}}$ and maps $J(G_b)$ onto $J(G_a)$.*

2.3 Self Similarity of Julia Sets

When G is generated by a rational function f, we know that if all the critical points are in the immediate attractive basin of a fixed point, then the Julia set is a Cantor set. Now we consider about the folloing situation similar to that.

Theorem 2.7 *Let $G = \langle f_1, \ldots, f_n\rangle$ be a finitely generated rational semigroup. Assume that $\deg(f_1)$ is at least two and for each j such that $\deg(f_j)$ is one, the map f_j is not elliptic. If $P(G)$ is included in a connected component U of $F(G)$, then there are simply connected domains $V_1, \ldots, V_k$ and mappings $h_1, \ldots, h_s$ from $W = \cup_j V_j$ to W such that for each j, i the map h_j is a hyperbolic metric contraction from V_i to some $V_{i'}$ with the contraction rate bounded by a constant strictly less than one throughout V_i and*

$$J(G) \subset W, \quad \bigcup_j h_j(J(G)) = J(G).$$

Example 2.1 *Let $G_c = \langle z^2 + c,\ z^2 + ci\rangle$. Then $J(G_c)$ is a Cantor set for sufficiently large positive number c.*

3 Hyperbolicity and Strong Open Set Condition

Definition 3.1 *Let $G = \langle f_1, f_2, \ldots f_n\rangle$ be a finitely generated rational semigroup. We say that G satisfies strong open set condition if and only if there is an open neighborhood O of $J(G)$ such that each set $f_j^{-1}(O)$ is included in O and is mutually disjoint.*

The Julia set of rational semigroup may have interior points in general. For example, the Julia set of $\langle z^2, 2z\rangle$ is the closure of the unit disc. In [HM2], it was shown that if G is a finitely generated rational semigroup, then each super attracting point of any element of $g \in G$ does not belong to the boundary of the Julia set. So we can construct many examples such that the Julia set has interior points. Here we show a sufficient condition such that the Julia set has no interior points.

Theorem 3.1 *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated rational semigroup. We assume that for each (i,j) such that $i \neq j$ the set $f_i^{-1}(J(G) \cap f_j^{-1}(J(G))$ has at most countable points. Then the Julia set $J(G)$ has no interior points.*

Proof Assume $J(G)$ has its interior points and let U be a component of $\mathrm{int}(J(G))$. From Lemma 1.1.2, $U \bigcap \cup_{j=1}^{n} f_j^{-1}(\mathrm{int}J(G))$ is dense in U. So if $U \bigcap f_{i_1}^{-1}(\mathrm{int}J(G))$ is not empty, then $U \bigcap f_{i_1}^{-1}(\mathrm{int}J(G))$ is dense in U and $f_{i_1}(U)$ is included in a component U_1 of $\mathrm{int}(J(G))$. In this way, we can take a sequence $(i_k)_k$ such that for each k the number i_k is in $(1,\dots,n)$ and

$$f_{i_k} \circ \cdots \circ f_{i_1}(U) \subset U_k,$$

where U_k is a component of $\mathrm{int}(J(G))$. Now let (g_j) be a sequence of elements of G. If the sequence contains infinite elements of $(f_{i_k} \circ \cdots \circ f_{i_1})$, then (g_j) is a normal family on U. If (g_j) contains a subsequence (h_l) such that for each l the map h_l does not belong to the sequence $(f_{i_k} \circ \cdots \circ f_{i_1})$, then for each l the set $h_l(U)$ is included in $F(G)$ and so (g_j) is a normal family on U. So U is included in $F(G)$ and this is a contradiction. □

Definition 3.2 *Let G be a polynomial semigroup. We denote by $K(G)$ the clusure of a set $K_1(G)$ such that for each $z \in K_1(G)$ there is a sequence $(g_m)_m$ consisting of mutually distinct elements of G and the sequence $(g_m(z))_m$ is bounded. $K(G)$ is called the filled-in Julia set of G.*

Remark For each $g \in G$ the inverse image $g^{-1}(K(G))$ is included in $K(G)$ and $J(G) \subset K(G)$. If $G = \langle f_1, f_2, \dots f_n\rangle$ is a finitely generated polynomial semigroup, then

$$K(G) = \bigcup_{j=1}^{n} f_j^{-1}(K(G)).$$

Theorem 3.2 *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated polynomial semigroup. Assume that for each (i,j) such that $i \neq j$ the set $f_i^{-1}(J(G) \cap f_j^{-1}(J(G))$ has at most countable points. Then*

$$\partial(K(G)) = J(G).$$

Proof By remark and a similar argument in the proof of Theorem 3.1. □

Now we consider about the expanding property of hyperbolic rational semigroups.

Theorem 3.3 *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated hyperbolic rational semigroup . Assume that $\deg(f_1)$ is at least two and for each $g \in G$ such that $\deg(g)$ is one the map g is not elliptic. Let K be a compact subset of $\overline{\mathbb{C}} \setminus P(G)$. Then there is a positive number c and is a number $\lambda > 1$ such that for each k*

$$\inf\{\|(f_{i_k} \circ \cdots \circ f_{i_1})'(z)\| \mid z \in (f_{i_k} \circ \cdots \circ f_{i_1})^{-1}(K), (i_k,\dots,i_1) \in (1,\dots,n)^k\}$$

$\geq c\lambda^k$, here we denote by $\|\ \|$ the hyperbolic metric on an open subset of $\overline{\mathbb{C}} \setminus P(G)$. In particular, if W is a simply connected domain which is a relative compact subdomain of $\overline{\mathbb{C}} \setminus P(G)$ and $\mathcal{A}$ is a family of maps on W such that each element h of $\mathcal{A}$ is a well defined branch of g^{-1} where g is an element of G, then each limit function of $\mathcal{A}$ on W is a constant function such that the constant value is in $J(G)$.

Proof We take a relatively compact open subset V of $\overline{\mathbb{C}} \setminus P(G)$ such that V contains $K \cup J(G)$ and for each element g of G the inverse image $g^{-1}(V)$ is included in V. We take the hyperbolic metric in each component of V. From [S3], every limit function of G on $F(G)$ is a constant function such that the constant value is in $P(G)$. So for each large number k the closure of $g_k^{-1}(V)$ is included in V where g_k is any element of G in the form $f_{i_k} \circ \cdots \circ f_{i_1}$. So the inclusion map i from $g_k^{-1}(V)$ to V satisfies that $\|i'(z)\| < 1$ for each $z \in g_k^{-1}(V)$ from the Schwartz lemma, where we denote by $\| \ \|$ the hyperbolic metric on $g_k^{-1}(V)$. The map g_k is a covering map from $g_k^{-1}V$ to V and is an isometry between the hyperbolic metric on $g_k^{-1}V$ and V. So $\|g_k'(z)\| > 1$ for each $z \in g_k^{-1}(V)$, where we denote by $\| \ \|$ the hyperbolic metric on V. □

By Theorems 3.1, 3.3, we get the following result.

Theorem 3.4 *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated hyperbolic rational semigroup satisfying strong open set condition. Then $m(J(G)) = 0$, where we denote by m the Lebesgue measure on $\overline{\mathbb{C}}$.*

4 δ-Conformal Measure

We construct δ-conformal mbeasure on Julia sets of rational semigroups. δ conformal measure on Julia sets of rational function is considered in [Sul] .

Theorem 4.1 *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated hyperbolic rational semigroup satisfying strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. And assume that $\infty \in F(G)$. Let O be the open set in Definition 3.1. Then there is a number $0 < \delta \leq 2$ and there is a probability measure μ such that the support of the measure is included in $J(G)$ and if A is a measurable set included in $f_j^{-1}(O)$ such that f_j is injective on A,*

$$\mu(f_j(A)) = \int_A |f_j'(z)|^\delta d\mu.$$

Also we say that a probability measure is δ-conformal if and only if the measure satisfies the above. And we set

$$\delta(G) = \inf\{\delta \geq 0 \mid \textit{there is a } \delta\textit{-conformal measure on } J(G)\}.$$

Then $\delta(G) > 0$.

Theorem 4.2 *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated hyperbolic rational semigroup satisfying strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. And assume that $\infty \in F(G)$. Let δ be a number satisfying that $0 < \delta \leq 2$ and assume that there is a δ-conformal measure μ on $J(G)$. Then $\delta = \delta(G)$ and*

$$\dim_H(J(G)) = \delta(G), \ 0 < H_{\delta(G)}(J(G)) < \infty,$$

where $\dim_H$ is the Hausdorff dimension and H_α is the $\alpha-$Hausdorff measure.

By Theorem 3.4 , Theorem 4.1 and Theorem 4.2, we get one of main results.

Corollary 4.1 *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated hyperbolic rational semigroup satisfying strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. Then*

$$0 < \dim_H(J(G)) < 2.$$

And if we set $\alpha = \dim(J(G))$, *then*

$$0 < H_\alpha(J(G)) < \infty.$$

Theorem 4.3 *Let* $G = \langle f_1, f_2, \dots f_n\rangle$ *be a finitely generated hyperbolic rational semigroup satisfying strong open set condition. We assume that when* n *is equal to one the degree of* f_1 *is at least two. Let* λ *be the number in Theorem 3.3 when* $K = J(G)$. *Then*

$$\dim_H(J(G)) \leq \frac{\log(\sum_j \deg(f_j))}{\log \lambda}.$$

Example 4.1 *Let* n *be a positive integer such that* $n \geq 4$. *We set* $G = \langle z^n,\ n(z-4)+4\rangle$. *Then* G *is a finitely generated hyperbolic rational semigroup satisfying strong open set condition and*

$$1 \leq \dim_H J(G) \leq \frac{\log(n+1)}{\log(n)}.$$

5 Invariant Measure

We introduce some notations and results from [L]. Let A be a bounded operator in the complex Banach space $\mathcal{B}$. The operator A is called *almost periodic* if the orbit $\{A^m\varphi\}_{m=1}^\infty$ of any vector $\varphi \in \mathcal{B}$ is strongly conditionally compact. The eigenvalue λ and related eigenvector are called unitary if $|\lambda| = 1$. The set of unitary eigenvectors of the operator A will be denoted by $\mathrm{spec}_u A$. We denote by $\mathcal{B}_u$ the closure of the linear span of the unitary eigenvectors of the operator A. And we set

$$\mathcal{B}_0 = \{\varphi \mid A^m\varphi \to 0\ (m \to \infty)\},$$

here the convergence is assumed to be strong.

Theorem 5.1 *If* $A : \mathcal{B} \to \mathcal{B}$ *is an almost periodic operator in the complex Banach space* $\mathcal{B}$, *then*

$$\mathcal{B} = \mathcal{B}_u \oplus \mathcal{B}_0.$$

Corollary 5.1 *Let* $A : \mathcal{B} \to \mathcal{B}$ *be an almost periodic operator in the complex Banach space* $\mathcal{B}$. *Assume that* $\mathrm{spec}_u A = \{1\}$ *and the point* $\lambda = 1$ *is a simple eigenvalue. Let* $h \neq 0$ *be an invariant vector of the operator* A. *Then there exists an* A^* *invariant functional* $\mu \in \mathcal{B}^*$, $\mu(h) = 1$, *such that*

$$A^m\varphi \to \mu(\varphi)h \quad m \to \infty.$$

Proof [L] . □

We now construct invariant measures on Julia sets of hyperbolic rational semigroups. Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated rational semigroup. For each compact set K of $\overline{\mathbb{C}}$ we denote by $C(K)$ all continuous complex valued functions on K. It is a Banach space with supremum norm. Assume that K is backward invariant under G. For each j and for each element φ we set

$$(A_j\varphi)(z) = \frac{1}{\deg(f_j)} \sum_{\zeta \in f_j^{-1}(z)} \varphi(\zeta),$$

where z is any point of K. Then $A_j\varphi$ is an element of $C(K)$ and A_j is a bounded operator on $C(K)$. We set

$$\mathcal{W} = \{(a_1, \dots, a_n) \in \mathbf{R}^n \mid \sum_j a_j = 1,\ a_j \geq 0\}.$$

And for each $a \in \mathcal{W}$ we set

$$(B_a\varphi)(z) = \sum_{j=1}^{n} a_j(A_j\varphi)(z).$$

Then B_a is a bounded operator on $C(K)$.

Theorem 5.2 *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated hyperbolic rational semigroup. Assume G has an element of degree at least two. Let $a \in \mathcal{W}$ be a point satisfying that there is a number i such that $a_i \neq 0$ and f_i is not an elliptic element of $Aut(\overline{\mathbb{C}})$. Then there is a probability measure μ_a on $\overline{\mathbb{C}}$ such that for each compact set K included in $\overline{\mathbb{C}} \setminus P(G)$*

$$\|B_a^m\varphi - \mu_a(\varphi)\mathbf{1}\|_K \to 0 \quad m \to \infty, \tag{1}$$

where we denote by $\mathbf{1}$ the constant function taking its value 1 and $\|\ \|_K$ is supremum norm on K. Also

$$supp(\mu_a) = J(\langle f_{i_1}, \dots, f_{i_k}\rangle),$$

where $\{i_1, \dots, i_k\} = \{j \mid a_j \neq 0\}$.

To prove Theorem 5.2, we need the following two lemmas.

Lemma 5.1 *If K is a backward invariant compact subset of $\overline{\mathbb{C}} \setminus P(G)$, then B_a is an almost periodic operator on $C(K)$.*

Lemma 5.2 *Let K be a backward invariant compact subset of $\overline{\mathbb{C}} \setminus P(G)$. If $B_a\varphi = \lambda\varphi$, $|\lambda| = 1$, then $\lambda = 1$ and φ is constant. That is, $(C(K))_u = \overline{\mathbb{C}} \cdot \mathbf{1}$.*

Proof of Lemma 5.1 By Ascoli Arzela theorem it is sufficient to show that for each element $\varphi \in C(K)$ the family $\{B_a^m\varphi\}_m$ is equicontinuous on K because $\|B_a^m\varphi\|_K \leq \|\varphi\|_K$ for each m and so the family $\{B_a^m\varphi\}_m$ is uniformly bounded. Let z be a point of K and let U be a simply connected open neighborhood of z included in $\overline{\mathbb{C}} \setminus P(G)$. Then for each $g \in G$ we can take well defined branches of g^{-1} on U. The family $\{S \mid$ a branch of g^{-1} on U, $g \in G\}$ is normal on U and equicontinuous on U. So $\{B_a^m\varphi\}_m$ is equicontinuous. □

Proof of Lemma 5.2 Let z be a point of K such that

$$|\varphi(z)| = \sup_{w \in K} |\varphi(w)|.$$

Then

$$\begin{aligned} |\varphi(z)| &= |(B_a\varphi)(z)| \\ &= |\sum_j a_j(A_j\varphi)(z)| \\ &\leq \sum_j a_j \frac{1}{\deg(f_j)} \sum_{\zeta \in f_j^{-1}(z)} |\varphi(\zeta)| \\ &\leq \sum_j a_j|\varphi(z)| = \varphi(z). \end{aligned}$$

So if ζ is a point of $f_i^{-1}(z)$, then $|\varphi(\zeta)| = |\varphi(z)|$ and so $\varphi(\zeta) = \lambda\varphi(z)$. Since if ζ is a point of $f_i^{-r}(z)$, then $\varphi(\zeta) = \lambda^r\varphi(z)$. Now for each point ζ of $J(\langle f_i\rangle)$ there is a sequence $(\zeta_m)_m$ such that for each m the point ζ_m belongs to $f_i^{-m}(z)$ and

$$\zeta_m \to \zeta.$$

Then

$$\lambda^m \varphi(z) = \varphi(\zeta_m) \to \varphi(\zeta) \quad m \to \infty.$$

So $\lambda = 1$. Now we will show that φ is constant. We put $\varphi = \Re\varphi + i\Im\varphi$. Then

$$B_a(\Re\varphi) = \Re\varphi, \; B_a(\Im\varphi) = \Im\varphi.$$

Let z be a point of K such that

$$|\Re\varphi(z)| = \sup_{w \in K} |\Re\varphi(w)|.$$

By a similar argument we can show that $\varphi(\zeta) = \varphi(z)$ for each $\zeta \in f_i^{-1}(z)$. Let ζ be any point of $J(\langle f_i \rangle)$. Let $(\zeta_m)_m$ be a sequence such that for each m the point ζ_m belongs to $f_i^{-m}(z)$ and $\zeta_m \to \zeta$. Then $\varphi(\zeta_m) \to \varphi(\zeta)$ so $\varphi(z) = \varphi(\zeta)$. In the same way we can show that if x is the minimum point of the function $\Re\varphi$, then $\varphi(x) = \varphi(\zeta)$, where ζ is any point of $J(\langle f_i \rangle)$. So $\Re\varphi$ is constant and by the same argument $\Im\varphi$ is also constant. Whence φ is constant. □

Proof of Theorem 5.2 we can assume that for each j, $a_j \neq 0$. By Lemma 5.1, Lemma 5.2, and Corollary 5.1, if K is a backward invariant compact subset of $\overline{\mathbb{C}} \setminus P(G)$, then there is a probability measure $\mu_{a,K}$ on K such that for each $\varphi \in C(K)$

$$\|B_a^m \varphi - \mu_{a,K}(\varphi)\,\mathbf{1}\|_K \to \infty \quad m \to \infty. \tag{2}$$

We consider $\mu_{a,K}$ as a probability measure on $\overline{\mathbb{C}}$. Then $\mu_{a,K}$ is independent of K which is backward invariant under G and is included in $\overline{\mathbb{C}} \setminus P(G)$ because $J(G)$ is included in K by Lemma 1.2.5 and (2) holds. For each $\varphi \in C(\overline{\mathbb{C}})$ we put $\mu_a(\varphi) = \mu_{a,K}(\varphi)$. Then μ_a is a probability measure on $\overline{\mathbb{C}}$. Now let L be any compact subset of $\overline{\mathbb{C}} \setminus P(G)$. There is a compact subset K of $\overline{\mathbb{C}} \setminus P(G)$ which contains L and is backward invariant under G. Then by (2),

$$\|B_a^m \varphi - \mu_a(\varphi)\,\mathbf{1}\|_L \to \infty \quad m \to \infty.$$

Now we will show that $\operatorname{supp} \mu_a = J(G)$. If we set $K = J(G)$, then $\mu_a = \mu_{a,K}$. So $\operatorname{supp} \mu_a \subset J(G)$. To prove $\operatorname{supp} \mu_a \supset J(G)$ it is sufficient to show that for each $z \in J(G)$ and for each $\varphi \in C(\overline{\mathbb{C}})$ such that $\varphi \geq 0$, $\varphi(z) > 0$

$$\int_{\overline{\mathbb{C}}} \varphi \, d\mu_a > 0.$$

We set

$$U = \{\zeta \in J(G) \mid \varphi(\zeta) > 0\}.$$

By Lemma 1.2.6, there is a point $z_0 \in U$ such that z_0 is a repelling fixed point of an element $g \in G$. Then there is an open neighborhood U_0 of z_0 included in U such that $g(U_0) \supset U_0$. When $\deg(g)$ is at least two, then $E(\langle g \rangle)$ is included in $P(G) \subset F(G)$ and

$$J(G) \subset \bigcup_{m=1}^{\infty} g^m(U_0).$$

Whence there is a positive integer N such that for each $\zeta \in J(G)$

$$g^{-N}(\zeta) \cap U \neq \emptyset. \tag{3}$$

When $\deg(g)$ is equal to one if we change g to some another element of G, we can show that (3) holds for an element $g \in G$ and for an integer N. Now for each $z \in J(G)$

$$B_a^N \varphi(z) > 0,$$

so there is a positive number c such that for each $z \in J(G)$

$$B_a^N \varphi(z) \geq c.$$

Whence for each integer $m \geq N$ and for each $z \in J(G)$

$$B_a^m \varphi(z) \geq c,$$

and so

$$\int \varphi \, d\mu_a \geq c > 0.$$

□

Theorem 5.3 *Let μ_a be the probability measure constructed in Theorem 5.2. Then*

1. *$\mu_a = \sum a_j A_j^* \mu_a$. And if μ is a probability measure on $\overline{\mathbb{C}}$ such that supp $\mu \subset \overline{\mathbb{C}} \setminus P(G)$ and $\mu = \sum a_j A_j^* \mu$, Then $\mu = \mu_a$.*
2. *Let b be a point of $\mathcal{W}$ and assume that there is an integer i such that $a_i \neq 0$ and f_i is not an elliptic element of Aut $\overline{\mathbb{C}}$. Then the map $a \mapsto \mu_a$ is continuous at b with respect to the weak topology.*
3. *Let $a \in \mathcal{W}$ be a point. If there is an integer j such that $a_j \neq 0$ and $\deg(f_j)$ is at least two, then μ_a is non atomic.*

Theorem 5.4 *Let a be a point of $\mathcal{W}$. Assume that there is a number j_0 such that $a_{j_0} \neq 0$ and f_{j_0} is not elliptic element of Aut$\overline{\mathbb{C}}$. Then*

1. *$\mu_a(f_j^{-1}(J(G))) \geq a_j$, for each number j.*
2. *Assume that for each (i,j) such that $i \neq j$, $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G)) = \emptyset$. Then $\mu_a(f_j^{-1}(J(G))) = a_j$, for each number j.*
3. *Assume that there is a number k such that $a_k \neq 0$ and $\deg(f_k) \geq 2$. Also assume that for each (i,j) such that $i \neq j$, the set $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ has at most countable points. Then $\mu_a(f_j^{-1}(J(G))) = a_j$, for each number j.*

Theorem 5.5 *In the same assumption as Theorem 5.2,*

1. *if for each number i the map f_i is not elliptic element of Aut $\overline{\mathbb{C}}$ and for each (i,j) such that $i \neq j$ the set $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ is empty, then the map*

$$a \mapsto \mu_a$$

   is topological embedding from $\mathcal{W}$ into the space of all probability measures on $\overline{\mathbb{C}}$ with respect to the weak topology.

2. *if $G \cap$ Aut $\overline{\mathbb{C}} = \emptyset$ and for each (i,j) such that $i \neq j$ the set $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ has at most countable points, then the map*

$$a \mapsto \mu_a$$

   is topological embedding from $\mathcal{W}$ into the space of all probability measures on $\overline{\mathbb{C}}$ with respect to the weak topology.

Proof By 2., 3. of Theorem 5.4. □

Theorem 5.6 *Let M be a complex manifold. Suppose for each $j = 1, \ldots, n$ the map*

$$(z, u) \in \overline{\mathbb{C}} \times M \mapsto f_{j,u}(z) \in \overline{\mathbb{C}}$$

is holomorphic. We set $G_u = \langle f_{1,u}, \cdots, f_{n,u} \rangle$. And we assume that for every j, $d_j = \deg(f_{j,u})$ is constant independent of u. Let v be a point of M. Assume that G_v is a hyperbolic rational semigroup not included in Aut $\overline{\mathbb{C}}$ and each $g \in G_v \cap$ Aut $\overline{\mathbb{C}}$ is a hyperbolic element. Let a be a point of $\mathcal{W}$. Then there is an open neighborhood V of v in M such that for each $u \in V$ we can construct the probabilty measure $\mu_{a,u}$ in Theorem 5.2 with respect to the hyperbolic rational semigroup G_u and the map

$$u \mapsto \mu_{a,u}$$

is continuous from V to the space of all probability measures on $\overline{\mathbb{C}}$ with respect to the weak topology.

Now we construct invariant measures on Julia sets of rational semigroups satisfying strong open set condition.

Theorem 5.7 *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated rational semigroup satisfying strong open set condition. When $n = 1$, we assume that f_1 is not elliptic element of Aut $\overline{\mathbb{C}}$. Let O be an open set in Definition 3.1. Then for each $a \in \mathcal{W}$ there is a probability measure μ_a on $\overline{\mathbb{C}}$ such that for each compact subset K of O which is backward invariant under G*

$$\|B_a^m \varphi - \mu_a(\varphi) 1\|_K \to 0, \; m \to \infty,$$

where φ is any element of $C(\overline{\mathbb{C}})$. Also

$$supp(\mu_a) = J(\langle f_{i_1}, \ldots, f_{i_k} \rangle),$$

where $\{i_1, \ldots, i_k\} = \{j \mid a_j \neq 0\}$. Also

$$\mu_a = \sum_{j=1}^{n} a_j A_j^* \mu_a$$

and the map

$$a \mapsto \mu_a$$

is a topologacal embedding from $\mathcal{W}$ into the space of all probability measures on $\overline{\mathbb{C}}$ with respect to the weak topology.

Proof The proof is similar to that of Theorem 5.2 so we only have to show that the family $\{B_a^m \varphi\}_m$ is equicontinuous on K for each $\varphi \in C(K)$. □

Theorem 5.8 *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated rational semigroup satisfying strong open set condition. When $n = 1$, we assume that f_1 is not elliptic element of Aut $\overline{\mathbb{C}}$. Let a be a point of $\mathcal{W}$. Assume that there is a number j such that $a_j \neq 0$ and $\deg(f_j) \geq 2$. Then μ_a is non atomic.*

6 Estimate of Hausdorff Dimension of Julia sets

Using invariant measures in Theorem 5.7, we get one of main results which gives a lower estimate of Hausdorff dimension of the Julia set of a rational semigroup which satisfies strong open set condition.

Theorem 6.1 *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup satisfying strong open set condition. When $n = 1$, we assume that f_1 is not elliptic element of Aut $\overline{\mathbb{C}}$. Assume that $\infty \in F(G)$ and we set*

$$M = \max_{j=1,\dots,n} \max_{z \in f_j^{-1}(J(G))} |f_j'(z)|.$$

Then

$$\dim_H J(G) \geq \frac{\log(\sum_{j=1}^n \deg(f_j))}{\log M},$$

where we denote by $\dim_H$ the Hausdorff dimension.

Proof Let $\mu = \mu_a$ be the probability measure constructed in Theorem 5.7 where

$$a_j = \frac{\deg(f_j)}{\sum_{i=1}^n \deg(f_i)}.$$

We fix a number t satisfying

$$0 < t < \frac{\log(\sum_{j=1}^n \deg(f_j))}{\log M}$$

and we take a number a such that

$$t < a < \frac{\log(\sum_{j=1}^n \deg(f_j))}{\log M}.$$

Let $\epsilon > 0$ be a small number and for each j we denote by $J_{j,\epsilon}$ the ϵ neighborhood of $f_j^{-1}(J(G))$. We take small ϵ such that for each j and for each $z \in f_j^{-1}(J(G))$

$$|f_j'(z)| < (\sum_{j=1}^n \deg(f_j))^{\frac{1}{a}}.$$

We set

$$C = \bigcup_{j=1}^n \{\text{ critical points of } f_j\}$$

and

$$C' = C \cup J(G).$$

we can assume that

$$C' = \cup_{j=1}^n J_{j,\epsilon} \bigcap C.$$

We fix any positive integer p. Because G satisfies strong open set condition, there is no super attracting fixed point of any element of G in $J(G)$. So there is a positive number η such that $\eta < \epsilon$ and if $w \in C'$ is a critical point of an element $g \in G$ of word length $l < p$, then

$$|w - g(w)| > \eta.$$

Also there is a positive number ρ such that $\rho < \frac{\eta}{2}$ and for each j if w is a critical point of f_j, then for each point $z \in D(w, \rho)$

$$|f_j'(z)| < 1.$$

Let C'_ρ be the ρ neighborhood of C'. There is a positive number δ such that $\delta < \frac{\rho}{2}$ and for each $z \in J(G) \setminus C'_\rho$ and for each j the map f_j is injective on

$D(z,\ \delta)$. Now let ζ be any point of $J(G)$ and r be a small positive number. We take a positive integer s such that

$$\delta((\sum_{j=1}^{n}\deg(f_j))^{\frac{1}{a}})^{-s-1} < r < \delta((\sum_{j=1}^{n}\deg(f_j))^{\frac{1}{a}})^{-s}.$$

There is the unique element $g \in G$ of word length s such that $g(\zeta) \in J(G)$. We can assume that $\deg(f_1) = \max_j \deg(f_j)$. Then the equation $g(z) = g(\zeta)$ has at most

$$(\deg(f_1))^{(\sum_{j=1}^{n} 2\deg(f_j)-2)\frac{s}{p}}$$

roots in $D(\zeta,\ r)$ counting multiplicities. Then for each $m \in \mathbb{N}$

$$\begin{aligned}\mu_{s+m}^{g(\zeta)}(D(\zeta,\ r)) &\leq \frac{(\sum_{j=1}^{n}\deg(f_j))^m \cdot (\deg(f_1))^{(\sum_{j=1}^{n} 2\deg(f_j)-2)\frac{s}{p}}}{(\sum_{j=1}^{n}\deg(f_j))^{s+m}} \\ &= \frac{(\deg(f_1))^{(\sum_{j=1}^{n} 2\deg(f_j)-2)\frac{s}{p}}}{(\sum_{j=1}^{n}\deg(f_j))^{s}}.\end{aligned}$$

Let $m \to \infty$ and we get

$$\begin{aligned}\mu(D(\zeta,\ r)) &\leq \frac{(\deg(f_1))^{(\sum_{j=1}^{n} 2\deg(f_j)-2)\frac{s}{p}}}{(\sum_{j=1}^{n}\deg(f_j))^{s}} \\ &\leq (\sum_{j=1}^{n}\deg(f_j))^{-s(1-\frac{\sum_{j=1}^{n} 2\deg(f_j)-2}{p})}.\end{aligned}$$

If we take p such that

$$a(1-\frac{\sum_{j=1}^{n} 2\deg(f_j)-2}{p}) > t,$$

then we get

$$\mu(D(\zeta,\ r)) \leq (\sum_{j=1}^{n}\deg(f_j))^{-\frac{st}{a}}.$$

Hence

$$\mu(D(\zeta,\ r)) \leq (\frac{r}{\delta})^t(\sum_{j=1}^{n}\deg(f_j))^{\frac{t}{a}} \tag{4}$$

and the statement of our theorem follows. □

Next we consider the case a hyperbolic rational semigroup G does not satisfy strong open set condition but satisfy open set condition and $J(G)$ is finitely ramified. In this case we also consider the lower estimate of Hausdorff dimension of the Julia set. Here we get the next interesting example in which the Julia set is similar to the Sierpiński Gasket.

Example 6.1 *Let* $G = \langle f_1,\ f_2,\ f_3\rangle$ *where*

$$f_1(z) = z^2,\ f_2(z) = 3(z-4)+4,\ f_3(z) = 3(z-4i)+4i.$$

G is hyperbolic. For small positive number c we set $G_c = \langle f_{1,c},\ f_{2,c},\ f_{3,c}\rangle$ *where*

$$f_{1,c}(z) = z^2,\ f_{2,c}(z) = (3+c)(z-4)+4,\ f_{3,c}(z) = (3+c)(z-4i)+4i.$$

Then G_c is also hyperbolic. G does not satisfy strong open set condition but G_c satisfies. $J(G_c) = J(\langle\{f_{i,c} \circ f_{j,c}\}_{i,j}\rangle)$ and by Theorem 6.1,

$$\begin{aligned}\dim_H J(G_c) &\geq \frac{\log 16}{\log \max_{i,j} \max_{z \in (f_{i,c} \circ f_{j,c})^{-1}(J(G))} |(f_{i,c} \circ f_{j,c})'(z)|} \\ &\to \frac{\log 16}{\log 12}, \ c \to 0.\end{aligned}$$

By Theorem 2.6.1, the map $c \mapsto J(G_c)$ is continuous with respect to the Hausdorff metric. And by Theorem 5.6 and (4) in Theorem 6.1, we get

$$\dim_H J(G) \geq \frac{\log 16}{\log 12}.$$

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