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On the Local Connectivity of the Boundary of Unbounded Periodic Fatou Components of Transcendental Functions

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1 Definitions, Notations and Results

Let f be a transcendental entire function, $F_f \subset \mathbb{C}$ the Fatou set of f and $J_f := \mathbb{C} \setminus F_f$ the Julia set of f. We call a connected component of F_f *a Fatou component*. It is well known that a Fatou component U is either *eventually periodic* (i.e. there exists a k_0 such that $f^{k_0}(U)$ is periodic) or *a wandering domain* (i.e. $f^m(U) \cap f^n(U) = \emptyset$ for every $m, n \in \mathbb{N}$ $(m \neq n)$) and if it is *periodic* (i.e. there exists an $n_0 \in \mathbb{N}$ with $f^{n_0}(U) \subseteq U$), there are four possibilities;

1. There exists a point $z_0 \in U$ with $f^{n_0}(z_0) = z_0$ and $|(f^{n_0})'(z_0)| < 1$ and every point $z \in U$ satisfies $f^{n_0 k}(z) \to z_0$ as $k \to \infty$. The point z_0 is called *an attracting periodic point* and the domain U is called *an attractive basin.*

2. There exists a point $z_0 \in \partial U$ with $f^{n_0}(z_0) = z_0$ (it is possible that $f^{n_1}(z_0) = z_0$ for an n_1 with $n_1 | n_0$) and $(f^{n_0})'(z_0) = 1$ and every point $z \in U$ satisfies $f^{n_0 k}(z) \to z_0$ as $k \to \infty$. The point z_0 is called *a parabolic periodic point* and the domain U is called *a parabolic basin.*

3. There exists a point $z_0 \in U$ with $f^{n_0}(z_0) = z_0$ and $(f^{n_0})'(z_0) = e^{2\pi i\theta}$ $(\theta \in \mathbb{R} \setminus \mathbb{Q})$ and $f^{n_0}|U$ is conjugate to an irrational rotation of a unit disk.

The domain U is called *a Siegel disk.*

4. For every $z \in U$, $f^{n_0 k}(z) \to \infty$ as $k \to \infty$. The domain U is called *a Baker domain.*

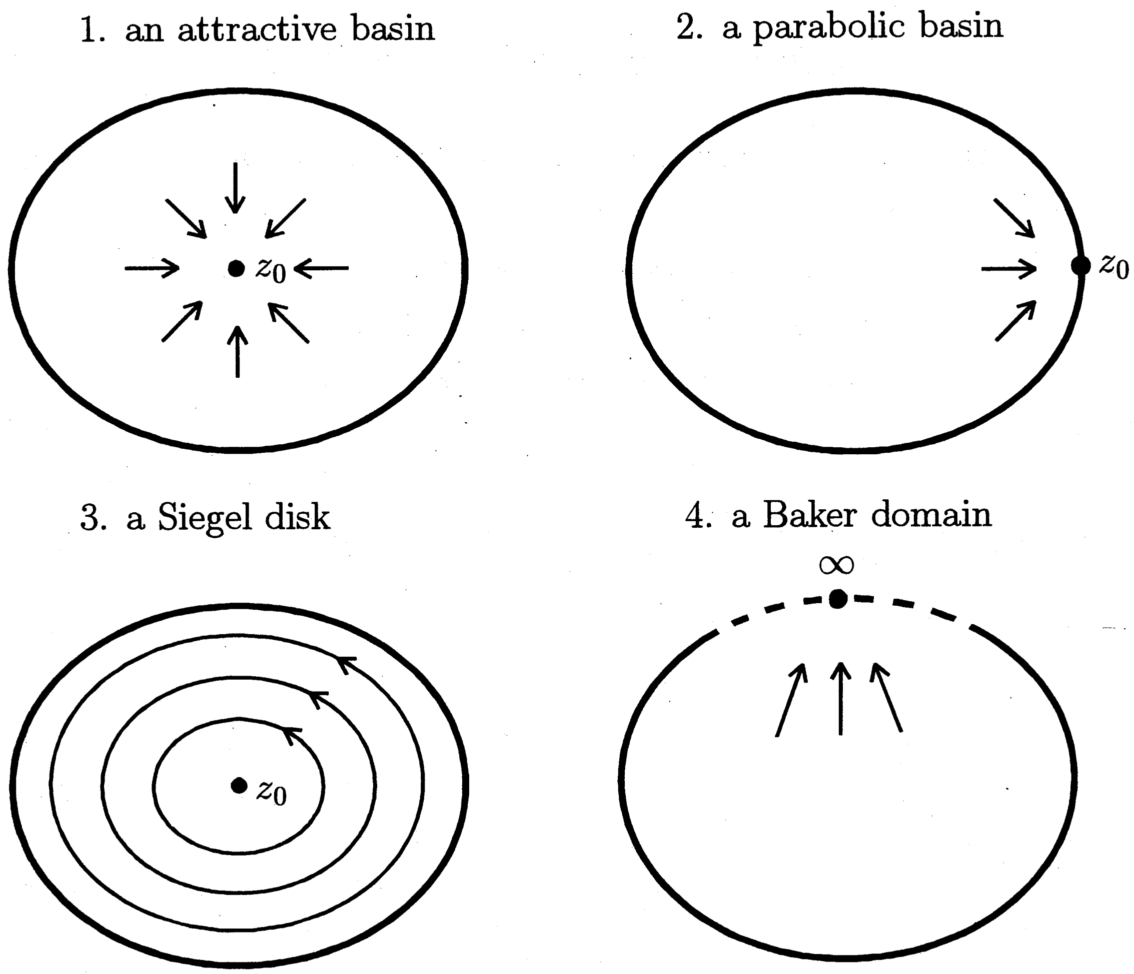

Figure 1. Invariant Fatou components

The natural number n_0 is called the *period* of a component U. Figure 1 shows these periodic Fatou components schematically in the case that its period n_0 is equal to one. In particular in this case, U is called *an invariant component.* By definition Baker domains are unbounded but attractive basins, parabolic bains and even Siegel disks can be unbounded as follows:

Example 1. Consider the exponential family $E_\lambda(z) := \lambda e^z$.

(1) If E_λ has an attracting fixed point, then its basin is always unbounded.

(2) If $\lambda = \frac{1}{e}$, then it is easy to see that it has an unbounded parabolic basin.

(3) If there is a Siegel disk on which E_λ is conjugate to a irrational rotation $z \longmapsto e^{2\pi i\theta}z$ and θ satisfies the Diophantine condition, then it is unbounded ([H]).

So throughout this paper we assume that f has an unbounded periodic Fatou component U with period n_0.

Then when is $\partial U \subset \mathbb{C}$ (or $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$) locally connected? For this problem we have the following result:

Theorem A. If U is either

(i) an attractive basin, (ii) a parabolic basin, (iii) a Siegel disk, or

(iv) a Baker domain on which $f^{n_0}|U$ is a d to 1 mapping ($2 \leq d < \infty$),

then $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is not locally connected. Also $\partial U \subset \mathbb{C}$ is not locally connected.

The local connectivity of ∂U is intimately related to the local connectivity of J_f by the following proposition:

Proposition 2. ([W]) A compact set $K \subset \widehat{\mathbb{C}}$ is locally connected if and only if the following two conditions are satisfied:

1. The boundary of each connected component of K^c (:= complement of K) is locally connected.
2. For any $\varepsilon > 0$ the number of connected components of K^c with diameter (with respect to the spherical metric) greater than ε is finite.

From this proposition and Theorem A we can prove the following result:

Theorem B. Assume that a transcendental entire function f has an unbounded periodic Fatou component U with period n_0. If U is either

(i) an attractive basin, (ii) a parabolic basin, (iii) a Siegel disk, or

(iv) a Baker domain on which $f^{n_0}|U$ is a d to 1 mapping ($1 \leq d < \infty$),

then $J_f \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is not locally connected. Also $J_f \subset \mathbb{C}$ is not locally connected.

2 Outline of the proof of Theorem A

In what follows we shall assume that $n_0 = 1$, that is, U is an invariant component for simplicity. In general cases similar arguements are valid if we consider f^{n_0} instead of f.

Since U is an unbounded component, it is simply connected ([**EL**]). So let $\varphi : \mathbb{D}(:= \{|z| < 1\}) \to U$ be a Riemann map of U. Then the following theorem is well known:

Theorem 3 (Carathéodory). Let $U \subset \widehat{\mathbb{C}}$ be a simply connected domain.
(1) There is one to one correspondence between $\partial\mathbb{D}$ and the set of prime ends: $e^{i\theta} \mapsto$ a prime end $P(e^{i\theta})$ of U.

(2) Let $I(P(e^{i\theta}))$ be the impression of a prime end $P(e^{i\theta})$. Then the following three conditions are equivalent:

1. The Riemann map $\varphi : \mathbb{D} \to U$ extends to a continuous map $\overline{\varphi} : \overline{\mathbb{D}} := \{|z| \leq 1\} \to \overline{U}$.
2. ∂U is locally connected.
3. For any prime end $P(e^{i\theta})$ the impression $I(P(e^{i\theta}))$ is reduced to a single point.

Remark 4. (1) For the definitions of the prime end, its impression and the proof of Theorem 3, see, for example, [**CL**].

(2) Since $U \subset \mathbb{C}$ is unbounded in our case, we should write $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ in the above theorem.

We also use the following result:

Theorem 5 ([BaW]). Let f and U be as above. Suppose that U is not a Baker domain then every impression $I(P(e^{i\theta}))$ of a prime end $P(e^{i\theta})$ of U contains the point ∞.

First let us consider the case (i), (ii) and (iii). Suppose that $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is locally connected. Then by Theorem 3, the Riemann map φ extends to a continuous map $\overline{\varphi}$ and moreover by Theorem 5 we have $\overline{\varphi}|\partial\mathbb{D} \equiv \infty$, which contradicts the following fact:

Proposition 6 ([CL]). For almost every point $e^{i\theta} \in \partial\mathbb{D}$ the radial limit $\lim_{r\nearrow 1}\varphi(re^{i\theta})$ exists and is nonconstant. Moreover for each $p \in \partial U$ the capacity of the set

$$\{e^{i\theta} \mid \lim_{r\nearrow 1}\varphi(re^{i\theta}) = p\} \subset \partial\mathbb{D}$$

is equal to zero.

This completes the proof for the case (i), (ii) and (iii).

In the case (iv), define

$$I_\infty := \{e^{i\theta} \mid I(P(e^{i\theta})) \ni \infty\} \subset \partial\mathbb{D}, \qquad V := \partial\mathbb{D} \setminus I_\infty.$$

Then since U is unbounded, we have $I_\infty \neq \emptyset$. It is easy to see that V is open in $\partial\mathbb{D}$ and $V \neq \partial\mathbb{D}$. Consider the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & U \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ \mathbb{D} & \xrightarrow[g:=\varphi^{-1}\circ f\circ\varphi]{} & \mathbb{D} \end{array}$$

By the assumption that $f|U$ is a d to 1 mapping $(2 \le d < \infty)$, $g := \varphi^{-1}\circ f\circ\varphi$ is a finite Blaschke product. It can be shown that $g(V) \subseteq V$. On the other hand we can consider the Julia set J_g and it is easy to see that $J_g \subset \partial\mathbb{D}$. Suppose that $V \cap J_g \neq \emptyset$. Then from an elementary property of Julia sets of rational maps, we have $g^n(V) = \partial\mathbb{D}$ for sufficiently large $n \in \mathbb{N}$ and since $g(V) \subseteq V$, it follows that $V = \partial\mathbb{D}$, which contradicts the fact that $V \neq \partial\mathbb{D}$. Consequently we have $V \cap J_g = \emptyset$, that is, $J_g \subset I_\infty$. Suppose here that $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is locally connected. Then from Theorem 3 φ has a continuous extension $\overline{\varphi}$ and we must have $\overline{\varphi} \equiv \infty$ on the set I_∞. In particular $\overline{\varphi} \equiv \infty$ on J_g. But on the contrary since the Hausdorff dimension of the Julia set of a rational map is always positive ([**Bea**, Theorem 10.3.1]), J_g has positive Hausdorff dimension. In particular its capacity is positive. Then it follows that the set

$$\{e^{i\theta} \mid \lim_{r\nearrow 1}\varphi(re^{i\theta}) = \infty\}$$

has positive capacity, which contradicts Proposition 6. This completes the proof for the case (iv).

The non-local connectivity of $\partial U \subset \mathbb{C}$ follows from the following proposition, since U is simply connected, $\partial U \cup \{\infty\}$ is closed and connected.

Proposition C. Let $K \subset \widehat{\mathbb{C}}$ be a closed connected subset and $p \in K$. If K is not locally connected, then $K \setminus \{p\}$ is also not locally connected.

We shall omit the proof of this proposition. □

Remark 7. It is known that the boundary of a Baker domain U on which f is 1 to 1 mapping (i.e. univalent) can be a Jordan curve (i.e. $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is a Jordan curve and $\partial U \subset \mathbb{C}$ is a Jordan arc). The function $f(z) := 2 - \log 2 + 2z - e^z$ is such an example ([**Ber**, Theorem 2]). In particular in this case both $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ and $\partial U \subset \mathbb{C}$ are locally connected. So we cannot drop the assumption $2 \leq d$ in Theorem A. It is also known that if $\partial U \cup \{\infty\}$ is a Jordan curve in $\widehat{\mathbb{C}}$, then $f|U$ is univalent ([**BaW**, Theorem 4]).

3 Proof of Theorem B

By definition $J_f \cup \{\infty\}$ is a compact subset of $\widehat{\mathbb{C}}$ so we can apply Proposition 2. In the case (i), (ii) and (iii), the set $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is not locally connected from Theorem A. So by Proposition 2 $J_f \cup \{\infty\}$ is not locally connected.

Next let us consider the case (iv). If $2 \leq d$, the proof is completely the same as the previous cases. If $d = 1$, take a point $w_0 \neq \infty \in \partial U \cup \{\infty\}$ and $z_0 \in U$. Then from an elementary property of complex dynamical systems there exist $n_k \in \mathbb{N}$ with $n_k \nearrow \infty$ and $z_{n_k} \in f^{-n_k}(z_0)$ with $z_{n_k} \to w_0$. Since $f|U$ is univalent we can take z_0, $\{z_{n_k}\}$ and w_0 satisfying $z_{n_k} \notin U$. Let U_{n_k} be the Fatou component containing z_{n_k}. Then it follows that U_{n_k} $(k = 1, 2, \ldots)$ are mutually disjoint and also we have $U_{n_k} \cap U = \emptyset$. Since $z_{n_k} \to w_0$, $z_{n_k} \in U_{n_k}$ and U_{n_k} is unbounded, it follows that the condition 2 in Proposition 2 is not satisfied. Hence again $J_f \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is not locally connected.

For the non-local connectivity of $J_f \subset \mathbb{C}$ itself, we can again apply Proposition C, since $J_f \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is compact and connected in this case ([**K**, Corollary 1]). This completes the proof. □

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