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ITERATION OF THE FUNCTION $c\exp[az+b/z]$

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ABSTRACT. We will study iteration of the function $c\exp[az+b/z]$. In this note, we investigate orbits of the critical points, which are useful to find conditions for real parameters a, b and c such that the Julia set of $c\exp[az+b/z]$ coincides with $\mathbb{C}^*$.

PART I

1. INTRODUCTION

There are many investigations on iteration of entire functions (analytic self-mappings on $\mathbb{C}$), especially of exponential function ce^{az}. We shall study iteration of analytic functions on the punctured plane $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$.

We denote by E^* the set of analytic self-mappings on $\mathbb{C}^*$. It is known that $f\in E^*$ can be expressed as

$$f(z)=z^m\exp\left[g(z)+h(1/z)\right],$$

where m is an integer and g , h are entire functions. We call that f belongs to the Rådström class, when both g and h are non-constant. The most simple example of the function in this class is

$$f(z)=c\exp[az+b/z],\quad a,b,c\in\mathbb{R}\backslash\{0\},$$

iteration of which will be investigated in this note.

We use the standard terminology in the iteration theory. We denote by f^n the n-th iteration of the function f. *Fatou set* and *Julia set* of f are denoted by $F(f)$ and $J(f)$, respctively. It is known that the set of repelling periodic points of f is dense in $J(f)$ [6].

As seen below, the analytic map $w = c\exp[az + b/z], a, b, c \in \mathbb{R}\backslash\{0\}$ is reduced to one of the following three canonical forms:

$$W = \lambda \exp[\mu(Z + 1/Z)], \quad \text{if } ab > 0, \tag{1.1}$$

$$W = \lambda \exp[\mu(Z - 1/Z)], \quad \text{if } ab < 0,\ ac > 0, \tag{1.2}$$

$$W = \lambda \exp[-\mu(Z - 1/Z)], \quad \text{if } ab < 0,\ ac < 0, \tag{1.3}$$

where λ and μ are positive real numbers.

Case (1) : $ab > 0$

(1-1) When $a > 0, b > 0, c > 0$. Put $z = \sqrt{\frac{b}{a}}Z$, $w = \sqrt{\frac{b}{a}}W$, then we have (1.1) with $\lambda = c\sqrt{\frac{a}{b}} > 0$, $\mu = \sqrt{ab} > 0$.

(1-2) When $a > 0, b > 0, c < 0$. Put $z = -\sqrt{\frac{b}{a}} \cdot \frac{1}{Z}$, $w = -\sqrt{\frac{b}{a}} \cdot \frac{1}{W}$, then we have (1.1) with $\lambda = -\frac{1}{c}\sqrt{\frac{b}{a}} > 0$, $\mu = \sqrt{ab} > 0$.

(1-3) When $a < 0, b < 0, c > 0$. Put $z = \sqrt{\frac{b}{a}} \cdot \frac{1}{Z}$, $w = \sqrt{\frac{b}{a}} \cdot \frac{1}{W}$, then we have (1.1) with $\lambda = \frac{1}{c}\sqrt{\frac{b}{a}} > 0$, $\mu = \sqrt{ab} > 0$.

(1-4) When $a < 0, b < 0, c < 0$. Put $z = -\sqrt{\frac{b}{a}}Z$, $w = -\sqrt{\frac{b}{a}}W$, then we have (1.1) with $\lambda = -c\sqrt{\frac{a}{b}} > 0$, $\mu = \sqrt{ab} > 0$.

Case (2) : $ab < 0$, $ac > 0$.

(2-1) When $a > 0, b < 0, c > 0$. Put $z = \sqrt{\frac{-b}{a}}Z$, $w = \sqrt{\frac{-b}{a}}W$, then we have (1.2) with $\lambda = c\sqrt{\frac{a}{-b}} > 0$, $\mu = \sqrt{-ab} > 0$.

(2-2) When $a < 0, b > 0, c < 0$. Put $z = -\sqrt{\frac{b}{-a}}Z$, $w = -\sqrt{\frac{b}{-a}}W$, then we have (1.2) with $\lambda = -c\sqrt{\frac{-a}{b}} > 0$, $\mu = \sqrt{-ab} > 0$.

Case (3) : $ab < 0$, $ac < 0$.

(3-1) When $a < 0, b > 0, c > 0$. Put $z = \sqrt{\frac{b}{-a}}Z$, $w = \sqrt{\frac{b}{-a}}W$, then we have (1.3) with $\lambda = c\sqrt{\frac{-a}{b}} > 0$, $\mu = \sqrt{-ab} > 0$.

(3-2) When $a > 0, b < 0, c < 0$. Put $z = -\sqrt{\frac{-b}{a}}Z$, $w = -\sqrt{\frac{-b}{a}}W$, then we have (1.3) with $\lambda = -c\sqrt{\frac{a}{-b}} > 0$, $\mu = \sqrt{-ab} > 0$.

Put

$$f_1(z) = \lambda \exp\left[\mu(z + 1/z)\right], \tag{1.4}$$

$$f_2(z) = \lambda \exp\left[\mu(z - 1/z)\right], \tag{1.5}$$

$$f_3(z) = \lambda \exp\left[-\mu(z - 1/z)\right], \tag{1.6}$$

where $\lambda > 0, \mu > 0$. Note that f_1 has critical points at ± 1, f_2 and f_3 have at $\pm i$.

We say that the function $f \in E^*$ belongs to the class S_q^* if there exist $\alpha_1, \dots, \alpha_q \in \mathbb{C}^*$ such that

$$f : \mathbb{C}^* \backslash f^{-1}(\{\alpha_1, \dots, \alpha_q\}) \to \mathbb{C}^* \backslash \{\alpha_1, \dots, \alpha_q\}$$

is a covering map. $\alpha_1, \dots, \alpha_q$ are called finite singularities of f^{-1} and we write sing $(f^{-1}) = \{\alpha_1, \dots, \alpha_q\}$. The union of all the classes S_q^* is denoted by S^*. Obviously all of $f_1(z)$, $f_2(z)$ and $f_3(z)$ belong to S^*.

Theorem 1. *Let f be a function in the class S^*, and $s \in$ sing (f^{-1}). The sequence $\{f^n(s)\}$ is said to satisfy the condition* (S) *provided that one of the following conditions is satisfied:*

(S1) $|f^n(s)| \to 0$,

(S2) $|f^n(s)| \to \infty$,

(S3) *s is a preperiodic, not a periodic point of f.*

Suppose that for every $s \in$ sing (f^{-1}) the sequence $\{f^n(s)\}$ satisfy the condition (S). *Then $J(f) = \mathbb{C}^*$.*

In the case of entire functions, this theorem was given by Baker [1] and Devaney [4]. This theorem is a generalization of the results of them.

As seen by Theorem 1, it is important to observe the orbits of critical points. We shall discribe the orbits of the critical points of

$$f_1(z) = \lambda \exp\left[\mu(z + 1/z)\right], \quad \lambda > 0, \quad \mu > 0.$$

We write simply f for f_1.

Consider the funcion c_1 defind by

$$c_1 = c_1(\mu) = \frac{1}{2\mu}\left(\sqrt{1 + 4\mu^2} - 1\right) \exp\left[\sqrt{1 + 4\mu^2} - 2\mu\right]. \tag{1.7}$$

This funcion varies from 0 to 1 as μ varies from 0 to ∞. Note that the equation $f(z) = z$ has at least one positive solution if and only if $\lambda e^{2\mu} \cdot c_1(\mu) \leqq 1$. Indeed, $f(z)/z = \lambda \exp[\mu(z + 1/z)]/z$ has the minimum value $\lambda e^{2\mu} \cdot c_1(\mu)$ at

$z = (1+\sqrt{1+4\mu^2})/2\mu > 1$. If $\lambda e^{2\mu} < 1/c_1$, the equation $f(z) = z$ has two positive solutions A_l, A_u such that

$$A_l < \frac{1+\sqrt{1+4\mu^2}}{2\mu} < A_u.$$

Further

$$\begin{aligned} 1 < A_l \qquad & \text{if } 1 < \lambda e^{2\mu}, \\ A_l = 1 \qquad & \text{if } \lambda e^{2\mu} = 1, \\ A_l < 1 < A_u \qquad & \text{if } 0 < \lambda e^{2\mu} < 1, \end{aligned}$$

because f has critical point at 1 and $f(1) = \lambda e^{2\mu}$. Firstly we observe the orbit of 1 by iteration of f. Let $a_n = f^n(1)$, then $a_1 = \lambda e^{2\mu}$ and

$$(1.8) \quad a_n = \lambda \exp\left[\mu\left(a_{n-1} + \frac{1}{a_{n-1}}\right)\right] = a_1 \exp\left[\mu\left(a_{n-1} + \frac{1}{a_{n-1}} - 2\right)\right].$$

Thus for any $n \in \mathbb{N}$,

$$(1.9) \qquad a_n \geqq a_1.$$

We consider $a_2 = a_1 \exp[\mu(a_1 + 1/a_1 - 2)]$. Let

$$(1.10) \qquad K(x) = x \exp\left[\mu\left(x + \frac{1}{x} - 2\right)\right],$$

then K has a minimum at $c_2 = (-1+\sqrt{1+4\mu^2})/2\mu < 1$, and decreases in $0 < x < c_2$, increases in $c_2 < x$. If c_0 is the solution of the equation

$$(1.11) \qquad c_0^2 \exp\left[\mu\left(c_0 + \frac{1}{c_0} - 2\right)\right] = 1, \quad 0 < c_0 < 1.$$

then $K(c_0) = A_u$ and $c_0 = 1/A_u$. So we shall prove existence of the solution of the equation (1.11). Let

$$K_0(x) = x^2 \exp\left[\mu\left(x + \frac{1}{x} - 2\right)\right], \quad 0 < x \leqq 1.$$

Then K_0 has a minimum at $c = (-1+\sqrt{1+\mu^2})/\mu < 1$, and decreases in $0 < x < c$, increases in $c < x \leqq 1$. Since $\lim_{x\to 0} K_0(x) = \infty$ and $K_0(1) = 1$, there exist only one $0 < c_0 < 1$ such that $K_0(c_0) = 1$. We note that

$$(1.12) \qquad K_0(x) > 1, \quad \text{that is} \quad x \exp\left[\mu\left(x + \frac{1}{x} - 2\right)\right] > \frac{1}{x}$$

if $0 < x < c_0$.

Theorem 2. *Let c_1 and c_0 be as above. Then we have the following:*

(1) *If* $\lambda e^{2\mu} > 1/c_1$, *then* $f^n(1) \to \infty$ *as* $n \to \infty$.
(2) *If* $1/c_1 \geqq \lambda e^{2\mu} \geqq c_0$, *then* $\{f^n(1)\}$ *is bounded from* 0 *and* ∞.
(3) *If* $c_0 > \lambda e^{2\mu} > 0$, *then* $f^n(1) \to \infty$ *as* $n \to \infty$.

Next we observe the orbit of -1. Let $b_n = f^n(-1)$. Then $b_1 = \lambda e^{-2\mu}$ and

$$b_n = \lambda \exp\left[\mu\left(b_{n-1} + \frac{1}{b_{n-1}}\right)\right] = b_1 \exp\left[\mu\left(b_{n-1} + \frac{1}{b_{n-1}} + 2\right)\right].$$

Define two functions L_0 and F by

$$L_0(x) = x \exp\left[\mu\left(x + \frac{1}{x} + 2\right)\right],$$

and

$$F(x) = \frac{\sqrt{4x^2+1}-1}{2x} \exp\left[\sqrt{4x^2+1} + 2x\right].$$

It is then easy to see that, $L_0(x) \geqq F(\mu)$ for any $x > 0$. Define two functions L_1 and G by

$$L_1(x) = x^2 \exp\left[\mu\left(x + \frac{1}{x} + 2\right)\right],$$

and

$$G(x) = \left(\frac{\sqrt{x^2+1}-1}{x}\right)^2 \exp\left[2\sqrt{x^2+1} + 2x\right].$$

We easily get that $L_1(x) \geqq G(\mu)$, for any $x > 0$. Note that $F(\mu), G(\mu)$ increase from 0 to ∞ in $\mu > 0$.

Let μ_1, μ_2 be positive numbers such that $F(\mu_1) = 1$, $G(\mu_2) = 1$, respectively. Note that $0 < \mu_1 < \mu_2 < 1$. Then we obtain the following.

Theorem 3. *Let* c_1, c_0 *be as in (1.7),(1.11).*

(1) *Suppose that* $\mu > \mu_2$. *Then, for any* $\lambda > 0$, $f^n(-1) \to \infty$ *as* $n \to \infty$.
(2) *Suppose that* $0 < \mu \leqq \mu_2$. *If* $\lambda e^{2\mu} > 1/c_1$ *or* $0 < \lambda e^{2\mu} < c_0$, $f^n(-1) \to \infty$ *as* $n \to \infty$.

Theorem 4. *Let* c_1, c_0 *be as in (1.7),(1.11), and*

$$c_* = c_*(\mu) = \frac{\sqrt{1+4\mu^2}-1}{2\mu} \exp\left[2\mu - \sqrt{1+4\mu^2}\right].$$

(1) *If* $\lambda e^{2\mu} > 1/c_1$, *then* $J(f) = \mathbb{C}^*$.
(2) *If* $c_* \leqq \lambda e^{2\mu} \leqq 1/c_1$, *then* $J(f) \neq \mathbb{C}^*$.
(3) *If* $0 < \lambda e^{2\mu} \leqq c_0$, *then* $J(f) = \mathbb{C}^*$.

2. Proof of Theorem 1

We need the lemma below.

Lemma. *Suppose that $f \in S^*$. Let E be the set of points of the form $f^n(s)$, $s \in \text{sing}\,(f^{-1})$, $n = 1, 2, \dots$. Then any constant limit α of a sequence $\{f^{n_k}(z)\}$ in a component of $F(f)$ belongs to $L = \overline{E} \cup \{0\} \cup \{\infty\}$.*

Although this lemma was given by Baker [1, Theorem 2] for entire functions, his proof also works for analytic function $f : \mathbb{C}^* \to \mathbb{C}^*$.

Suppose that $F(f)$ would be non-empty. By a result of Makienko [6, Lemma 4], f^{n_k} converges to a constant $\alpha \in \mathbb{C}^*$ in a component D of $F(f)$. Since $F(f)$ has no wandering component [6, Theorem 4], we may suppose that $f^p(D) \subset D$ for some p. Thus f^{n_k+p} also converges to α in D, and hence $\alpha = f^p(\alpha)$. The lemma above implies $\alpha \in L$, that is $\alpha = f^m(s)$ for some $m \in \mathbb{N}$ and $s \in \text{sing}\,(f^{-1})$. Moreover $D \backslash \{\alpha\}$ contains no finite singularities of f^{-1} by assumption. Thus D is attractive basin with super-attracting periodic point α. Thus $(f^p)'(\alpha) = 0$ hence $f'(\beta) = 0$ for $\beta = f^q(\alpha)$ with some $q, 0 \leqq q \leqq p-1$. Then $f(\beta) \in \text{sing}\,(f^{-1})$ is periodic, which contradicts the assumption.

3. Proof of Theorem 2

In the sequel, we frequently use the facts that

(3.1) the function $x + \dfrac{1}{x}$ increases on $x \geqq 1$, and decreases on $0 < x < 1$,

and

$$x > \lambda \exp\left[\mu\left(x + \frac{1}{x}\right)\right] \quad \text{if and only if} \quad A_l < x < A_u. \tag{3.2}$$

Case (1) : $\lambda e^{2\mu} > 1/c_1$. By (3.1), $a_{n+1} > a_n$ for any $n \in \mathbb{N}$, and hence we obtain that $a_n \to \infty$ as $n \to \infty$, since the equation $f(z) = z$ has no positive solution.

Case (2) : $1/c_1 \geqq \lambda e^{2\mu} \geqq 1$. By $a_1 \geqq 1$ and (3.1), $a_2 < \lambda \exp[A_l + A_l] = A_l$, $a_1 < a_n < a_{n+1}$. Hence, by means of (3.2), it is shown inductively that $a_1 < a_n < a_{n+1} < A_l$, for any $n \in \mathbb{N}$.

Note that $K(c_2) = c_1 < 1$ and $K(1) = 1$. So we may take the value c_3 such that $0 < c_3 < c_2$ and,

$$K(c_3) = c_3 \exp\left[\mu\left(c_3 + \frac{1}{c_3} - 2\right)\right] = 1, \tag{3.3}$$

By $A_u > 1$ and (3.1), we have $c_0 < c_3$.

Case (3) : $1 > \lambda e^{2\mu} > c_3$. We then have $c_1 < K(a_1) \leqq 1$, and $a_1 < a_2 \leqq 1$, so that,

$$a_3 = \lambda \exp\left[\mu\left(a_2 + \frac{1}{a_2}\right)\right] < \lambda \exp\left[\mu\left(a_1 + \frac{1}{a_1}\right)\right] = a_2,$$

$$a_4 = \lambda \exp\left[\mu\left(a_3 + \frac{1}{a_3}\right)\right] > \lambda \exp\left[\mu\left(a_2 + \frac{1}{a_2}\right)\right] = a_3.$$

and further,

$$a_3 - a_1 = \lambda \exp\left[\mu\left(a_2 + \frac{1}{a_2}\right)\right] - \lambda e^{2\mu} \geqq \lambda e^{2\mu} - \lambda e^{2\mu} = 0,$$

$$a_4 - a_2 = \lambda \left\{\exp\left[\mu\left(a_3 + \frac{1}{a_3}\right)\right] - \exp\left[\mu(a_1 + \frac{1}{a_1})\right]\right\} \leqq 0.$$

Hence $a_1 \leqq a_3 \leqq a_4 \leqq a_2 \leqq 1$, and we obtain

$$a_{2m-1} \leqq a_{2m+1} \leqq a_{2m+2} \leqq a_{2m} \leqq 1,$$

by induction.

Case (4) : $c_3 > \lambda e^{2\mu} \geqq c_0$. Then $K'(a_1) < 0, K(a_1) > 1$. Since $K(c_3) = 1$ and $K(c_0) = A_u$, $A_u \geqq a_2 > 1 > a_1$. Suppose $a_1 < a_n \leqq A_u$, then we have

$$a_{n+1} = \lambda \exp\left[\mu\left(a_n + \frac{1}{a_n}\right)\right] \leqq \lambda \exp\left[\mu\left(A_u + \frac{1}{A_u}\right)\right] = A_u, \qquad \text{if } 1 \leqq a_n,$$

$$a_{n+1} = \lambda \exp\left[\mu\left(a_n + \frac{1}{a_n}\right)\right] \leqq \lambda \exp\left[\mu\left(a_1 + \frac{1}{a_1}\right)\right] = a_2 \leqq A_u, \quad \text{if } 1 > a_n,$$

by (3.1). Hence for every $n \in \mathbb{N}$, $a_1 < a_n \leqq A_u$.

Case (5) : $c_0 > \lambda e^{2\mu} > 0$. From $K'(a_1) < 0$ and $K(a_1) > 1$, we have $K(a_1) > K(c_0) = A_u$. Thus we obtain from (3.1) and (3.2),

$$a_3 = \lambda \exp\left[\mu\left(a_2 + \frac{1}{a_2}\right)\right] > a_2,$$

$$a_4 = \lambda \exp\left[\mu\left(a_3 + \frac{1}{a_3}\right)\right] > \lambda \exp\left[\mu\left(a_2 + \frac{1}{a_2}\right)\right] = a_3.$$

and $a_{n+1} > a_n > A_u$, by induction. Hence $a_n \to \infty$.

4. Proof of Theorem 3

Case (1) : $\mu > \mu_2$. $L_0(x)$ and $L_1(x)$ have the minimum values $F(\mu)$ and $G(\mu)$ at $x = (\sqrt{1+4\mu^2}-1)/2\mu$ and $x = (\sqrt{1+\mu^2}-1)/\mu$, respectively. Hence by definition of μ_1 and μ_2, for any $x > 0$,

$$(4.1)\qquad \begin{aligned} L_0(x) &= x\exp\left[\mu\left(x+\frac{1}{x}+2\right)\right] \geqq F(\mu) > 1, \\ L_1(x) &= x^2\exp\left[\mu\left(x+\frac{1}{x}+2\right)\right] \geqq G(\mu) > 1. \end{aligned}$$

First assume that $b_1 = \lambda e^{-2\mu} \geqq 1$. Then

$$b_2 = \lambda\exp\left[\mu\left(b_1+\frac{1}{b_1}\right)\right] \geqq \lambda e^{2\mu} = a_1 > b_1 \geqq 1,$$

and it is shown by induction that

$$b_{n+1} = \lambda\exp\left[\mu\left(b_n+\frac{1}{b_n}\right)\right] > \lambda\exp\left[\mu\left(b_{n-1}+\frac{1}{b_{n-1}}\right)\right] = b_n,$$

$$b_{n+1} = \lambda\exp\left[\mu\left(b_n+\frac{1}{b_n}\right)\right] \geqq \lambda\exp\left[\mu\left(a_{n-1}+\frac{1}{a_{n-1}}\right)\right] = a_n,$$

for any $n \geqq 2$. Note that $\lambda e^{2\mu} > 1/c_1 \geqq e^{4\mu} > e^{4\mu}/F(\mu) = 1/c_1$, by assumption. Thus Theorem 2 (1), implies $b_n \geqq a_{n-1} \to \infty$.

Next assume that $b_1 = \lambda e^{-2\mu} < 1$. Then by (4.1)

$$(4.2)\qquad b_2 = \lambda\exp\left[\mu\left(b_1+\frac{1}{b_1}\right)\right] > \frac{1}{b_1} > 1.$$

and hence by (3.1)

$$b_3 = \lambda\exp\left[\mu\left(b_2+\frac{1}{b_2}\right)\right] > \lambda\exp\left[\mu\left(\frac{1}{b_1}+b_1\right)\right] = b_2.$$

Further we have $b_{n+1} > b_n$, for every $n \geqq 2$ by induction. Note that $1/b_1 > A_u$ because of (4.2) and (3.2). Therefore we obtain $b_n \to \infty$.

Case (2) : $\mu_2 \geqq \mu > \mu_1$. Suppose that $\lambda e^{2\mu} > 1/c_1$. Then

$$b_n = \lambda\exp\left[\mu\left(b_{n-1}+\frac{1}{b_{n-1}}\right)\right] \geqq \lambda e^{2\mu} > \frac{1}{c_1} > 1, \quad n \geqq 2.$$

If $\lambda e^{-2\mu} \geqq 1$, in exactly the same way as Case (1), we have

$$b_{n+1} > a_n,$$

for any $n \geqq 2$. By Theorem 2 (1), $b_n \to \infty$. If $\lambda e^{-2\mu} < 1$, by (3.1),

$$b_2 = \lambda \exp\left[\mu\left(b_1 + \frac{1}{b_1}\right)\right] > \lambda e^{2\mu} = a_1 > \frac{1}{c_1} > 1.$$

Hence

$$b_3 = \lambda \exp\left[\mu\left(b_2 + \frac{1}{b_2}\right)\right] > \lambda \exp\left[\mu\left(a_1 + \frac{1}{a_1}\right)\right] = a_2.$$

By induction, $b_n > a_{n-1}$ for any $n \geqq 2$. By Theorem 2 (1), we obtain $b_n \to \infty$.

Suppose that $\lambda e^{2\mu} < c_0$. Then by (1.12)

$$a_2 = \lambda e^{2\mu} \exp\left[\mu\left(\lambda e^{2\mu} + \frac{1}{\lambda e^{2\mu}} - 2\right)\right] > \frac{1}{\lambda e^{2\mu}}.$$

By $\lambda e^{-2\mu} < \lambda e^{2\mu} < c_0 < 1$ and (3.1),

$$1 < \lambda^2 \exp\left[\mu\left(\lambda e^{2\mu} + \frac{1}{\lambda e^{2\mu}} + 2\right)\right] < \lambda e^{2\mu} \cdot \lambda \exp\left[\mu\left(\lambda e^{-2\mu} + \frac{1}{\lambda e^{-2\mu}}\right)\right] = a_1 \cdot b_2.$$

Thus we have

$$a_2 = \lambda \exp\left[\mu\left(\frac{1}{a_1} + a_1\right)\right] < \lambda \exp\left[\mu\left(b_2 + \frac{1}{b_2}\right)\right] = b_3,$$

and hence $a_n < b_{n+1}$, for any $n \geqq 2$, by induction. Therefore $b_n \to \infty$ as $n \to \infty$, as above.

Case (3) : $\mu_1 \geqq \mu > 0$. If $\lambda e^{2\mu} > 1/c_1$, then

$$\lambda e^{-2\mu} > \frac{2\mu}{\sqrt{4\mu^2+1} - 1} \exp\left[-\sqrt{4\mu^2+1} - 2\mu\right] = \frac{1}{F(\mu)} \geqq \frac{1}{F(\mu_1)} = 1.$$

In exactly the same way as Case (1), we obtain $b_n \to \infty$.

If $\lambda e^{2\mu} < c_0$, we can prove that $b_n \to \infty$, in the same way as case (2).

5. Proof of Theorem 4

If $\lambda e^{2\mu} > 1/c_1$ or $0 < \lambda e^{2\mu} < c_0$, Theorem 2, Theorem 3 and Theorem 1 imply the assertion. So we consider the case $\lambda e^{2\mu} = c_0$. In this case, by definition of c_0, it is shown that $a_1 = c_0$, $a_2 = 1/c_0 = A_u$, thus we have $a_n = a_2$ for $n \geqq 3$, $b_1 < a_1 = c_0$, and $b_2 > A_l$. Therefore a_2 is a repelling fixed point, since $f'(1/c_0) = \mu(A_u - 1/A_u) > 1$. Moreover $b_n \to \infty$ as $n \to \infty$ by Theorem 3(2). Hence both $\{f^n(1)\}$ and $\{f^n(-1)\}$ satisfy condition (S), and we have $J(f) = \mathbb{C}^*$ by Theorem 1.

Next we shall prove (2). Let

$$e_1 = e_1(\mu) = \frac{\sqrt{1+4\mu}+1}{2\mu},$$

$$e_2 = e_2(\mu) = \frac{\sqrt{1+4\mu}-1}{2\mu}.$$

Case (1) : $\lambda e^{2\mu} = 1/c_1$. Then the equation $f(z) = z$ has only one positive solution $e_1(\mu)$. Choose a positive real value $1 < z_0 < e_1$ near e_1 and put $\rho = e_1 - z_0$. Because of conformality of f the disk $\Delta = \{|z - z_0| < \rho\}$ is mapped by f in a disk $\Delta' = \{|z - z_0'| < \rho'\}$, where $z_0 < z_0' < e_1$ and $e_1 - z_0' = \rho' < \rho$. Hence $\{f^n\}$ is normal in Δ.

Case (2) : $1/c_1 > \lambda e^{2\mu} \geqq 1$. Then $1 \leqq A_l < e_1$ and $f'(A_l) \in [0,1)$. Thus A_l is an attracting fixed point, and $J(f) \neq \mathbb{C}^*$.

Case (3) : $1 > \lambda e^{2\mu} > c_*$. Then $1 > A_l > e_2$ and $f'(A_l) \in (-1,0)$. Thus A_l is an attracting fixed point, and $J(f) \neq \mathbb{C}^*$.

Case (4) : $\lambda e^{2\mu} = c_*$. Then $f'(A_l) = -1$ and so $(f^2)'(A_l) = 1$. Choose a positive real value $z_0 < A_l$ near A_l and put $\rho = A_l - z_0$. Since $0 < (f^2)' < 1$ on (z_0, A_l), we can see that $\{f^{2n}\}$ is normal in the disk $\Delta = \{|z - z_0| < \rho\}$ in the same way as Case (1).

PART II

The present Part II has been inspired by comments of attendants in the assembly 9.24–9.27, 1996, at the RIMS.

6. The situation when λ is fixed

In Part I, we have investigated behaviors of the function $f_{\lambda\mu}(z)$ for varying λ with μ fixed. In the present Part II, we suppose that λ is fixed and will consider the problem for varying μ. We write for simplicity f for $f_{\lambda\mu}$.

We consider only the case $\lambda > 0$, $\mu > 0$. Other cases can be treated similarly.

Put

$$M_1(\mu) = \frac{1}{2\mu}\left(\sqrt{1+4\mu^2}-1\right)\exp\left[\sqrt{1+4\mu^2}\right]. \tag{6.1}$$

Obviously $M_1(\mu)$ increases from 0 to ∞ as μ varies from 0 to ∞. Let $\mu^* = \mu^*(\lambda)$ be a number such that

$$M_1(\mu^*) = 1/\lambda.$$

By Theorems 2 and 3 in Part I, we know that

(6.2) $\mu > \mu^*(\lambda)$ imples $a_n = f^n(1) \to \infty,\ b_n = f^n(-1) \to \infty,$ as $n \to \infty$.

Let x_1 be the fixpoint of $f(x)$, then $f'(x_1) = \mu(x_1 - 1/x_1)$. If $f'(x_1) = -1$, then

$$x_1 = \frac{\sqrt{1+4\mu^2}-1}{2\mu}.$$

The value of λ with this fixpoint x_1 must be

$$\lambda = \frac{1}{2\mu}\left(\sqrt{1+4\mu^2}-1\right)\exp\left[-\sqrt{1+4\mu^2}\right]. \tag{6.3}$$

We write the right hand side of (6.3) as $M_2(\mu)$. Then $M_2(\mu)$ attains the maximum $\ell^* = (\sqrt{2}-1)e^{-\sqrt{2}} = 0.100702249\ldots$ at $\mu = 1/2$. Thus if $\lambda > \ell^*$, then there is no fixpoint x_1 with $f'(x_1) = -1$, that is, $-1 < f'(x) < 1$ if $f(x) = x$, provided $0 < \mu < \mu^*(\lambda)$. Therefore we obtain the following theorem:

Theorem 5.

(1) *If $\mu > \mu^*(\lambda)$, then $J(f) = \mathbb{C}^*$.*
(2) *Suppose $\lambda \geqq \ell^* = 0.100702249\ldots$, where ℓ^* has been defined above. Then if $0 < \mu \leqq \mu^{(}\lambda)$ we have $J(f) \neq \mathbb{C}^*$.*
(3) *If $\lambda > \lambda^* = 0.0322903204227226\ldots$, where λ^* will be defined below, then the case (3) in Theorem 4 does not occur. That is, there is no c_0 such that $J(f) = \mathbb{C}^*$ for $0 < \lambda e^{2\mu} < c_0$.*

We have only to prove the case (3) in Theorem 5. Put

$$M_3(x,\mu) = (xe^{2\mu})^2 \exp\left[\mu(xe^{2\mu} + 1/xe^{2\mu} - 2)\right] = K_0(xe^{2\mu}),$$

where $K_0(x)$ is the function in (1.12) of Part I. Let $x(\mu)$ be the implicit function determined by

$$M_3(x(\mu),\mu) = 1, \quad 0 < x(\mu)e^{2\mu} < 1.$$

Then

$$\frac{dx}{d\mu} = -x\frac{(2\mu+1)xe^{2\mu} - (2\mu-1)/(xe^{2\mu}) + 2}{\mu\left(xe^{2\mu} - 1/(xe^{2\mu})\right) + 2}.$$

Thus $x(\mu)$ takes the local maximum value $x = e^{-2\mu}(2\mu-1)/(2\mu+1)$ where μ satisfies

$$M_3\left(e^{-2\mu}\frac{2\mu-1}{2\mu+1}, \mu\right) = \left(\frac{2\mu-1}{2\mu+1}\right)^2 \exp\left[\frac{4\mu}{4\mu^2-1}\right] = 1,$$

from which we get $\mu = 0.62783439300776\ldots$. Then $x(\mu) = e^{-2\mu}(2\mu-1)/(2\mu+1) = 0.0322903204227226\ldots$, which number we denote as λ^*. Therefore if $\lambda > \lambda^*$, then $\lambda e^{2\mu}$ can not take the value $c_0(\mu)$ in (1.11), for any μ. Hence, if $\lambda > \lambda^*$, then $f^n(1)$ and $f^n(-1)$ are bounded from 0 and ∞ if $\mu \leqq \mu^*(\lambda)$.

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